

3.7 Chain Rule

Theorem 3.7.1 (Chain Rule).

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

Example 3.7.1. What is $\frac{d}{dx} (\cos(x))^2$?

$$\begin{aligned} f(x) &= x^2 & g(x) &= \cos x \\ f'(x) &= 2x & g'(x) &= -\sin x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \cos^2 x &= 2 \cos x (-\sin x) \\ &= -2 \cos x \sin x \end{aligned}$$

Common Uses of the Chain Rule

$$1. \frac{d}{dx} (g(x))^n = n(g(x))^{n-1} g'(x) \quad \frac{d}{dx} x^n = n x^{n-1} \cdot 1$$

$$f(x) = x^n, \quad h(x) = g(x)$$

Example 3.7.2. $y = (x^6 + 2)^{10}$

$$y' = 10(x^6 + 2)^9 (6x^5) = 60x^5(x^6 + 2)^9$$

$$2. \frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x) = g'(x)e^{g(x)}$$

$$f(x) = e^x, \quad g(x)$$

Example 3.7.3. $y = e^{x^3 + 7x}$

$$y' = e^{x^3 + 7x} (3x^2 + 7) = (3x^2 + 7)e^{x^3 + 7x}$$

$$3. \frac{d}{dx} f(kx) = k f'(kx)$$

$$f'(x), g(x) = kx$$

Example 3.7.4. $y = \sin(\pi x)$

$$y'(x) = \pi \cos(\pi x)$$

Example 3.7.5. Why is the chain rule true?

Let $F(x) = f(g(x))$. Now, let's use the limit definition to take the derivative.

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x)}$$

$$= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$\lim_{h \rightarrow 0} g(x+h) = g(x) \quad = \left(\lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \cdot g'(x)$$

$$u = g(x+h) \quad = \left(\lim_{u \rightarrow g(x)} \frac{f(u) - f(g(x))}{u - g(x)} \right) \cdot g'(x)$$

$$\lim_{h \rightarrow 0} u = g(x)$$

$$u \rightarrow g(x) \quad = f'(g(x)) \cdot g'(x)$$

Example 3.7.6. $y = \cos(e^{3x+1})$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$y' = -\sin(e^{3x+1})(3e^{3x+1})$$

$$g(x) = e^{3x+1}$$

$$g'(x) = 3e^{3x+1}$$

$$y' = -\sin(e^{3x+1})(3e^{3x+1})$$

Example 3.7.7. $g(\theta) = \frac{\sqrt{5\theta^2 + 3\theta}}{\theta + 1}$

$$\text{top} = (5\theta^2 + 3\theta)^{1/2}$$

$$g'(\theta) = \frac{\text{top}' \text{bottom} - \text{bottom}' \text{top}}{(\text{bottom})^2}$$

$$\text{top}' = \frac{1}{2}(5\theta^2 + 3\theta)^{-1/2}(10\theta + 3)$$

$$\text{bottom} = \theta + 1$$

$$= \frac{\frac{1}{2}(5\theta^2 + 3\theta)^{-1/2}(10\theta + 3)(\theta + 1) - (5\theta^2 + 3\theta)^{1/2}}{(\theta + 1)^2}$$

$$\text{bottom}' = 1$$

Example 3.7.8. $f(y) = \sec(ye^{-y})$

$$f'(y) = \sec(ye^{-y}) \tan(ye^{-y}) (-ye^{-y} + e^{-y})$$

$$\text{Out} = \sec(x) \quad I_1 = ye^{-y}$$

$$\begin{aligned}\text{Out}' &= \sec(x)\tan(x) & I_1' &= e^{-y} + (-e^{-y})y \\ &= e^{-y} - ye^{-y}\end{aligned}$$

Example 3.7.9. $y = \frac{3x}{x^2+1} = 3x(x^2+1)^{-1}$

$$\begin{aligned}y' &= \frac{3(x^2+1) - (2x)(3x)}{(x^2+1)^2} \\ &= \frac{3x^2 + 3 - 6x^2}{(x^2+1)^2} \\ &= \frac{-3x^2 + 3}{(x^2+1)^2} = -3 \frac{x^2 - 1}{(x^2+1)^2}\end{aligned}$$

$$y' = 3(x^2+1)^{-1} - (x^2+1)^{-2}(2x)(3x)$$

Example 3.7.10. $h(x) = (x^2 - 1) \cos(2x)$

$$\begin{aligned} h'(x) &= 2x \cos(2x) + (-\sin(2x))(2)(x^2 - 1) \\ &= 2x \cos(2x) - 2 \sin(2x)(x^2 - 1) \end{aligned}$$

Example 3.7.11. $s = \tan^2(3t) = (\tan(3t))^2$

$$\begin{aligned} s' &= 2(\tan(3t))(\sec^2(3t))(3) \\ &= 6 \tan(3t) \sec^2(3t) \end{aligned}$$

$$\begin{aligned} h_1(h_2(h_3(h_4(x)))) &= h_1(h_2(h_3(x^3))) = h_1(h_2(\sin x^3)) \\ &= h_1(1 + \sin^2 x^3) = \sqrt{1 + \sin^2 x^3} \end{aligned}$$

Example 3.7.12. $y = \sqrt{1 + \sin^2 x^3} = (1 + \sin^2 x^3)^{1/2}$

$$h_1 = x^{1/2}$$

$$\begin{aligned} y' &= \frac{1}{2}(1 + \sin^2 x^3)^{-1/2} (6x^2 \sin x^3 \cos x^3) \quad h_2 = \cancel{x^2} + 1 \\ &= \frac{3x^2 \sin x^3 \cos x^3}{\sqrt{1 + \sin^2 x^3}} \quad h_3 = \sin x \\ &\quad h_4 = x^3 \end{aligned}$$

$$\frac{d}{dx}(1 + \underbrace{\sin^2(x^3)}) = 2 \sin(x^3) (\cos x^3) (3x^2)$$

Example 3.7.13. $f(a) = (3a^2 + a + 1)^{-2}$

$$f \quad \text{out}(a) = a^{-2} \quad \text{in}(a) = 3a^2 + a + 1$$

$$f'(a) = -2(3a^2 + a + 1)^{-3} (6a + 1) \quad \text{out}' = -2a^{-3} \quad \text{in}' = 6a + 1$$

$$\text{Example 3.7.14. } y = \sqrt{(t^2 - \cot t + 2)^3} = (t^2 - \cot t + 2)^{3/2}$$

$$y' = \frac{3}{2} (t^2 - \cot t + 2)^{1/2} (2t + \csc^2 t)$$

$$\text{Example 3.7.15. } y = \frac{\sin(1+x)}{1+\sin x}$$

$$\begin{aligned} y' &= \frac{\cos(1+x)(1+\sin x) - \sin(1+x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{(1+\sin x)\cos(1+x) - \cos x \sin(1+x)}{(1+\sin x)^2} \end{aligned}$$

$$\text{Example 3.7.16. } a(b) = e^{e^b} + e^{b-12}$$

$$a'(b) = e^b e^{e^b} + e^{b-12}$$

$$\begin{aligned} e^{e^b} &= e^{g(b)} \\ &\downarrow \\ g'(b) e^{g(b)} & \\ e^b e^b & \end{aligned}$$

$$\text{Example 3.7.17. } y = \sin(\cos(\sin x))$$

$$\begin{aligned} y' &= \boxed{\cos(\cos(\sin x))} \left(-\boxed{\sin(\sin x)} \right) \boxed{\cos x} \\ &= -\cos x \sin(\sin x) \cos(\cos(\sin x)) \end{aligned}$$