

3.6 Trigonometric Functions

Rules so far...

- $\frac{d}{dx} c = 0$

- $\frac{d}{dx} x^n = nx^{n-1}$

- $\frac{d}{dx} e^x = e^x$

- $\frac{d}{dx} (cf(x)) = C f'(x)$

- $\frac{d}{dx} (f(x) \pm g(x)) = f'(x) \pm g'(x)$

- $\frac{d}{dx} (f(x) \cdot g(x)) = f'(x)g(x) + g'(x)f(x)$

- $\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$

Theorem 3.6.1 (New Rules). Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x)$$

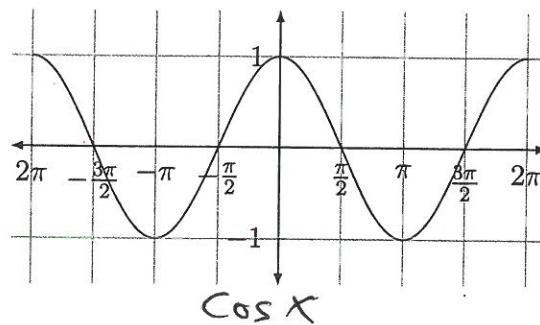
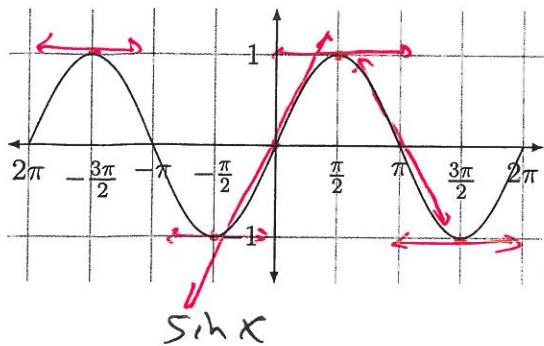
$$\frac{d}{dx} \cot(x) = -\csc^2(x)$$

$$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$$

Why is $\frac{d}{dx} \sin(x) = \cos(x)$?

Graphically:



Algebraically:

$$\begin{aligned}\frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2 \sin(\frac{h}{2}) \cos(\frac{2x+h}{2})}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{\sin(\frac{h}{2})} \cos(\frac{2x+h}{2})}{\frac{h}{2}} \\ &= \cos x\end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$

$$\sin(u) - \sin(v) = 2 \sin\left(\frac{u-v}{2}\right) \cos\left(\frac{u+v}{2}\right) \quad u = x+h \quad u-v = h \\ v = x \quad u+v = 2x+h$$

Example 3.6.1. Verify $\frac{d}{dx} \cot(x) = -\csc^2(x)$.

$$\begin{aligned}\frac{d}{dx} \cot(x) &= \frac{d}{dx} \frac{\cos x}{\sin x} = \frac{(-\sin x)\sin x - (\cos x)\cos x}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \\ &= -\csc^2 x\end{aligned}$$

Example 3.6.2. Compute the derivative of $f(\theta) = \theta \tan(\theta)$.

$$\begin{aligned} f'(\theta) &= (1)\tan\theta + (\sec^2\theta)\theta \\ &= \tan\theta + \theta \sec^2\theta \end{aligned}$$

Example 3.6.3. Find $g'(x)$ when $g(x) = \frac{x}{e^x \sin(x)}$.

$$\begin{aligned} g'(x) &= \frac{(1)(e^x \sin x) - (e^x \cos x + e^x \sin x)x}{(e^x \sin x)^2} & \frac{d}{dx} e^x \sin x \\ &= \frac{e^x \sin x - x e^x \cos x - x e^x \sin x}{(e^x)^2 \sin^2 x} & = e^x \cos x + e^x \sin x \\ &= \frac{\sin x - x \cos x - x \sin x}{e^x \sin^2 x} \end{aligned}$$

$$0 \cdot \sec(t) + 9 \underline{\sec(t) \tan(t)}$$

Example 3.6.4. Find $\frac{d}{dt}(9 \sec(t) - \cos^2(t)) = 9 \sec(t) \tan(t) + 2 \sin(t) \cos(t)$

$$\begin{aligned} \frac{d}{dt} \cos^2(t) &= \frac{d}{dt} \cos(t) \cos(t) \\ &= -\sin(t) \cos(t) - \sin(t) \cos(t) \\ &= -2 \sin(t) \cos(t) \end{aligned}$$

Example 3.6.5. Find the equation of the tangent line to $y = \frac{\sin(t)}{1 + \cos(t)}$ at $t = \frac{\pi}{3}$.

$$\begin{aligned}
 y' &= \frac{\cos(t)(1 + \cos(t)) - (-\sin(t))\sin(t)}{(1 + \cos(t))^2} = \frac{\cos(t) + \cos^2(t) + \sin^2(t)}{(1 + \cos(t))^2} \\
 &= \frac{\cos(t) + 1}{(1 + \cos(t))^2} = \frac{1}{1 + \cos t} \\
 y\left(\frac{\pi}{3}\right) &= \frac{\sin(\pi/3)}{1 + \cos(\pi/3)} = \frac{\frac{\sqrt{3}}{2}}{1 + \frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}} \\
 y'\left(\frac{\pi}{3}\right) &= \frac{1}{1 + \cos(\pi/3)} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}
 \end{aligned}$$

$$y - \frac{1}{\sqrt{3}} = \frac{2}{3}(x - \pi/3)$$

Example 3.6.6. $f(x) = \sin(x)$. What is $f^{(199)}(x)$?