

3.4 Rates of Change

Example 3.4.1. The population of Knoxville in 2010 was 558,696. Let $P(t)$ be the population in year t , measured in millions.

$$P(2010) = 0.558696$$

What is the meaning of $P'(2010)$?

rate of change of population in $t = 2010$

Estimate $P(2011)$, given $P'(2010) = 0.03$.

$$\begin{aligned} \text{Idea: } P'(t) &= \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h} \approx \frac{P(t+1) - P(t)}{1} \\ &= P(t+1) - P(t) \end{aligned}$$

$$P(t+1) \approx P(t) + P'(t)$$

$$P(2011) \approx P(2010) + P'(2010)$$

$$= 0.558696 + 0.03$$

$$= 0.588696 \text{ (pop. in millions)}$$

Example 3.4.2. The cost (in dollars) for a company to produce a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

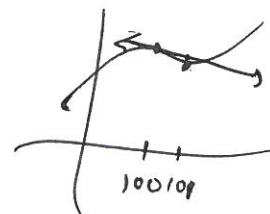
The marginal cost at producing level x is the cost of producing the $(x+1)$ st item.

$$C(x+1) - C(x) \approx C'(x)$$

What is the marginal cost at production level 100?

$$C(101) - C(100) = 2611.0702 - 2600 = 11.0702$$

$$C'(x) = 3 + 0.02x + 0.0006x^2, \quad C'(100) = 11$$



What is $C'(100)$ and how can we interpret this?

rate of change of the cost of making ¹⁰⁰ jeans
~~at 100 jeans~~

$C'(100) \approx$ marginal cost at 100

(that is, $C'(100)$ approximates
 the cost of producing the
^{pair of} 101st jeans)

Example 3.4.3. A tank holds 5000 gallons of water and a full tank can be emptied in 40 minutes when water drains from the bottom. Torricelli's Law tells us that the volume V in the tank after t minutes is

$$V(t) = 5000 \left(1 - \frac{t}{40}\right)^2 = 5000 \left(1 - \frac{t}{20} + \frac{t^2}{1600}\right)$$

Find the rate at which the water is draining out of the tank after 5, 10, 20, and 40 minutes. When is water draining out the fastest? the slowest?

$$V'(t) = 5000 \left(\frac{t}{800} - \frac{1}{20} \right)$$

$$V'(20) = -125$$

$$V'(5) = -218.75 \text{ fastest}$$

$$V'(10) = -187.5$$

$$V'(40) = 0 \text{ slowest}$$

Example 3.4.4. A ball is thrown upward with an initial velocity of 80 feet per second. What is the maximum height of the ball? (Use the formula $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$, where $g = 32 \frac{\text{ft}}{\text{sec}^2}$.)

↑ initial pos.
0
↑ initial vel.

$$s(t) = 80t - 16t^2$$

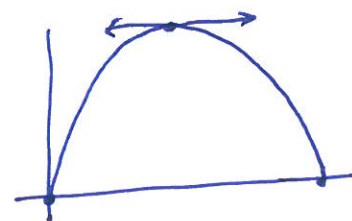
$$s'(t) = 80 - 32t$$

max height occurs when $s'(t) = 0$

$$80 - 32t = 0$$

$$\Rightarrow t = \frac{80}{32} = 2.5$$

$$s(2.5) = \text{max height} = 100 \text{ ft}$$



Example 3.4.5. When the brightness of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula developed is

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}.$$

What is the sensitivity, which is defined to be the rate of change of the reaction?

$$\begin{aligned} \text{Sensitivity} &= R' \\ &= \frac{(9.6x^{-0.6})(1 + 4x^{0.4}) - (1.6x^{-0.6})(40 + 24x^{0.4})}{(1 + 4x^{0.4})^2} \end{aligned}$$