3.3 Product and Quotient Rules

Theorem 3.3.1 (Product Rule).

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$$

Caution: $\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$

Ex: $\frac{d}{dx}x^2 = 2x \neq 1$ $\qquad \frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x)$

Example 3.3.1. What is the derivative of $\underbrace{(x^2+1)}_{f}\underbrace{(x^3-2x)}_{g}$?

$f = x^2+1$ $\qquad g = x^3-2x$

$f' = 2x$ $\qquad g' = 3x^2-2$

$$\frac{d}{dx}(fg) = (2x)(x^3-2x) + (3x^2-2)(x^2+1)$$

Theorem 3.3.2 (Quotient Rule).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$\frac{d}{dx}x = 1 \neq$

$x = \frac{x}{1}$ $\qquad \frac{\frac{d}{dx}x}{\frac{d}{dx}1} = \frac{1}{0}$

Example 3.3.2. What is $\frac{d}{dx}\left(\frac{x^2+3x}{x^5+7}\right)$?

$f = x^2+3x$ $\qquad g = x^5+7$

$f' = 2x+3$ $\qquad g' = 5x^4$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{(2x+3)(x^5+7) - (5x^4)(x^2+3x)}{(x^5+7)^2}$$

$$= \frac{2x^6+14x+3x^5+21 - (5x^6+15x^5)}{(x^5+7)^2}$$

$$= \frac{-3x^6+14x-12x^5+21}{(x^5+7)^2} = \frac{-3x^5(x+4)+7(2x+3)}{(x^5+7)^2}$$

Example 3.3.3. Use the information in the table to answer the following questions.

$f(1)$	$f'(1)$	$g(1)$	$g'(1)$
4	-3	7	2

What is $(fg)'(1)$?

$(fg)' = f'g + fg'$

$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = (-3)(7) + (4)(2) = -13$

What is $\left(\frac{f}{g}\right)'(1)$?

$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - g'(1)f(1)}{(g(1))^2} = \frac{(-3)(7) - (2)(4)}{7^2}$

$= \frac{-29}{49}$

If $F(x) = \overbrace{\frac{1}{2}x^2}^{\text{I}}\overbrace{f(x)}^{\text{II}}$, what is $F'(1)$? $\quad \text{I}' = x \qquad \text{II}' = f'(x)$

$F' = \text{I}'\,\text{II} + \text{II}'\,\text{I} = x f(x) + f'(x)\left(\frac{1}{2}x^2\right)$

$F'(1) = f(1) + f'(1)\left(\frac{1}{2}\right) = 4 + -3\left(\frac{1}{2}\right) = 4 - \frac{3}{2} = \frac{5}{2}$

If $H(x) = \frac{x \to \text{I}}{\underbrace{f(x)g(x)}_{\text{II}}}$, what is $H'(1)$? $\quad \text{I} = x \qquad \text{I}' = 1$

$\text{II} = f(x)g(x)$

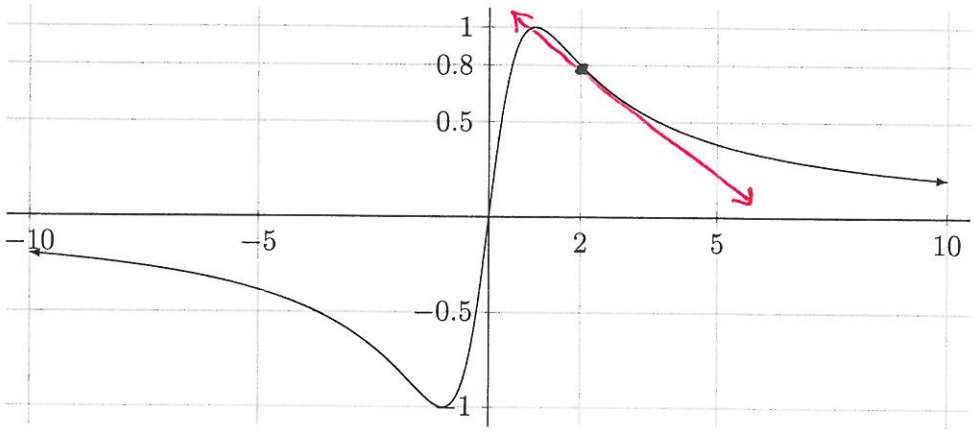
$\text{II}' = f'(x)g(x) + g'(x)f(x)$

$H'(x) = \frac{\text{I}'\,\text{II} - \text{II}'\,\text{I}}{(\text{II})^2}$

$= \frac{f(x)g(x) - (f'(x)g(x) + g'(x)f(x))\,x}{(f(x)g(x))^2}$

$H'(1) = \frac{4 \cdot 7 - (-13)}{(4 \cdot 7)^2} = \frac{41}{784}$

Example 3.3.4. Find the tangent line to $f(x) = \dfrac{2x}{x^2+1}$ at $x = 2$.



$$f'(x) = \frac{2(x^2+1) - (2x)(2x)}{(x^2+1)^2} = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2}$$

$$f'(2) = \frac{2(5) - 16}{5^2} = \frac{-6}{25}$$

$$f(2) = \frac{4}{5}$$

$$y - \frac{4}{5} = -\frac{6}{25}(x-2)$$

Example 3.3.5. $\frac{d}{dx}\left(\overbrace{5x^4}^{f}\overbrace{e^x}^{g}\right) = 20x^3e^x + 5x^4e^x$

Example 3.3.6. $\frac{d}{dx}\left(\frac{6e^x}{\sqrt{x}}\right) = \underline{6e^x\sqrt{x} - \left(\frac{1}{2}x^{-1/2}\right)(6e^x)}$

$\sqrt{x} = x^{1/2}$ $= \underline{\frac{6e^x\sqrt{x} - 3e^x/\sqrt{x}}{x}} = \underline{\frac{3e^x\left(\frac{2x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)}{x}}$

$= \frac{3e^x(2x-1)}{x^{3/2}}$

Example 3.3.7. $\frac{d}{dx}\left(\overbrace{x^3e^x}^{f}\overbrace{(8x^4 - 25x + 10)}^{g}\right) = (3x^2e^x + x^3e^x)(8x^4 - 25x + 10)$
$+ (32x^3 - 25)(x^3e^x)$

$f = x^3e^x$ $\quad g = 8x^4 - 25x + 10$

$f' = 3x^2e^x + x^3e^x$ $\quad g' = 32x^3 - 25$