

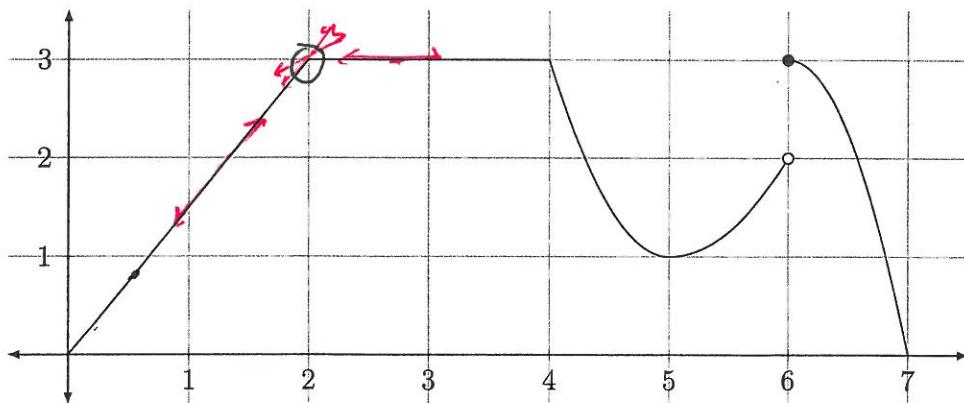
3.2 Derivative as a Function

Recall:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{t \rightarrow a} \frac{f(t) - f(a)}{t-a}$$

Definition 3.2.1. When the above limit exists, we say $f(x)$ is differentiable at $x = c$. When the limit does not exist, we say $f(x)$ is non-differentiable at $x = a$.

Example 3.2.1. Where is the following function differentiable?



at $x=2$ ~~the function~~ $f(x)$ is non-diff.

at $x=4$ $f(x)$ is non-diff

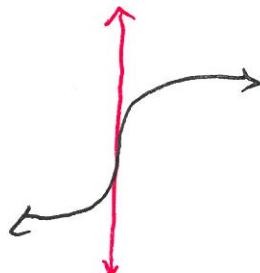
at $x=6$ $f(x)$ is non-diff.

3 ways a function may not be differentiable:

① corner or cusp

② fcn can be discontinuous

③ vertical tangent line



Notation 3.2.1. $f'(x)$, y' , $\frac{df}{dx}$, $\frac{dy}{dx}$

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} \quad \frac{d}{dx}(f(x)) = \frac{df}{dx}$$

Example 3.2.2. $f(x) = x^2$. What is $f'(x)$?

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x \end{aligned}$$

Example 3.2.3. $f(x) = x^3$. What is $f'(x)$?

$$\begin{aligned} \frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \end{aligned}$$

Example 3.2.4. $f(x) = c$. What is $f'(x)$?

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

$$= \lim_{h \rightarrow 0} 0$$

Theorem 3.2.1 (Derivative Rules).

(a) Constant Rule: $\frac{d}{dx} c = 0$

$$\frac{d}{dx} 5 = 0, \quad \frac{d}{dx} \pi = 0, \quad \frac{d}{dx} e^6 = 0$$

(b) Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$

$$\frac{d}{dx} x^4 = 4x^3, \quad \frac{d}{dx} x^{\sqrt{17}} = \sqrt{17} x^{\sqrt{17}-1}$$

(c) Sum/Difference Rule: $\frac{d}{dx} (f \pm g) = f' \pm g'$

$$\frac{d}{dx} (x^2 + x^3 - x^4) = 2x + 3x^2 - 4x^3$$

$$(cf)' = \lim_{h \rightarrow 0} \frac{(cf)(x+h) - (cf)(x)}{h} = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h}$$

(d) Constant Multiple Rule: $\frac{d}{dx} (cf) = cf'(x)$

$$2x. \quad \frac{d}{dx} (3x^2) = 3 \left(\frac{d}{dx} x^2 \right) = 3(2x) = 6x$$

$$= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = c f'(x)$$

(e) Exponential Rule: $\frac{d}{dx} e^x = e^x$

$$\frac{d}{dx} (6e^x) = 6e^x$$

Why?

$$\lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$$

Example 3.2.5. $f(x) = 2x^2 - 5x + 1 + x^{\sqrt{17}}$. What is $f'(x)$?

$$f'(x) = 4x - 5 + \sqrt{17} x^{\sqrt{17}-1}$$

Example 3.2.6. $f(x) = \frac{3}{x} + \sqrt{x} - \frac{1}{2}e^x$. What is $f'(x)$?

$$= 3x^{-1} + x^{1/2} - \frac{1}{2}e^x$$

$$f'(x) = -3x^{-2} + \frac{1}{2}x^{-1/2} - \frac{1}{2}e^x$$

Example 3.2.7. $f(x) = 3e^{x-6} + (6x-5)^2$. What is $f'(x)$?

$$= 3e^x e^{-6} + 36x^2 - 60x + 25$$

$$= 3e^{-6} e^x + 36x^2 - 60x + 25$$

$$f'(x) = 3e^{-6} e^x + 72x - 60$$

$$\frac{1}{2} (2x^3 + 9x^{1/2}) x^{-2}$$

Example 3.2.8. $f(x) = \frac{2x^2 + 9x^{1/2}}{2x^2}$. What is $f'(x)$?

$$= \frac{2x^2}{2x^2} + \frac{9x^{1/2}}{2x^2} = 1 + \frac{9}{2} x^{-3/2}$$

$$f'(x) = \frac{9}{2} \left(-\frac{3}{2} x^{-5/2} \right)$$

$$= -\frac{27}{4} x^{-5/2}$$