3.10 Related Rates

Calculus I Goals:

1. Compute limits and derivatives.
2. Display conceptual and graphical understanding of limits and derivatives.
3. Apply the concepts of limits and derivatives to solve a variety of problems (related rates, curve-sketching, optimization, etc.).

What are "related rates" problems?

Problems involving quantities that are related, but change over time. We interested in their rates at which the quantities are changing.

Before beginning, let's think about and answer the following questions:

1. Imagine yourself blowing up a balloon. What are some things about the balloon that are changing as you blow it up?

   Surface Area, Volume, air pressure, radius

2. Now imagine yourself pouring coffee into a coffee mug. What quantities (describing the coffee in the mug) are changing as you pour? (Assume the mug is a cylinder.)

   Volume of coffee
   height of coffee

3. How would you answer the previous question if you were pouring into a cone-shaped cup instead of a mug?

   SA of coffee radius of coffee
   Volume of coffee
   Height of coffee

In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

Key: Implicit differentiation! (Recall implicit differentiation is an application of the chain rule.)

Start w/ eqn relating quantities

differentiate → eqn relating rates at which the quantities are changing

Recall that when we used implicit differentiation, we had to keep in mind that y was a function of x so that

$$\frac{d}{dx}(y^2) = 2yy' = 2y\frac{dy}{dx}$$

In related rates problems, our quantities will be functions of time t. Now, suppose we are considering a radius, r, that is changing with time. Then,

$$\frac{d}{dt}(r^2) = 2rr' = 2r\frac{dr}{dt}$$

Example 3.10.1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm? Note that for a sphere $V = \frac{4}{3}\pi r^3$.

(a) Identify and assign symbols to the quantities that are changing with time.

V = volume of balloon

r = radius of balloon

(b) Separate the given information into "general" (always holds) and "particular" (only holds at one particular time).

General

$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Particular

$r = 25 \text{ cm}$ $(d = 2r = 50)$

(c) Write an equation that relates the quantities that are changing with time.

$$V = \frac{4}{3}\pi r^3$$

(d) Differentiate both sides with respect to time to get an equation that relates their rates.

$$\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}} = \frac{12\pi r^2}{3}\frac{dr}{dt}.$$

(e) Organize the information given in the problem.

Know:

$\frac{dV}{dt} = 100$

$r = 25$

Want: $\left.\frac{dr}{dt}\right|_{r=25}$

(f) Plug in and solve.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$$

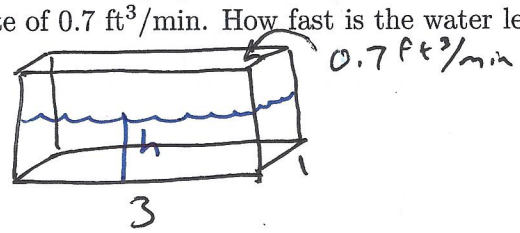
$$\left.\frac{dr}{dt}\right|_{r=25} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi} \frac{cm}{sec}$$

Steps for solving related rates problems:

1. Don't panic! Take your time and read the problem carefully.
2. Draw a picture.
3. Assign symbols to all quantities that are changing with time.
4. Separate the given information into "general" and "particular" (meaning information that only applies at a particular time). **Never plug in "particular" information until after you have taken the derivative.**
5. Find an equation that relates the quantities changing with time.
6. Take the derivative with respect to the underlying variable (using implicit differentiation).
7. Organize the given information (into what we know and what we are trying to find) and ensure that all units match properly.
8. Plug in and solve for the unknown. (Don't forget units!)

Example 3.10.2. Rachel is filling up a rectangular fish tank with dimensions 1 ft by 3 ft. The water is filling the tank at a rate of 0.7 ft^3/min. How fast is the water level rising?

① Picture



② Variables h = height of water in tank

V = volume of water in tank

③ Info (1):

General

tank: 1X3

rate of volume inc: 0.7 ft³/min

Particular

nothing

④ Eqn: $V = lwh = (3)(1)h = 3h$

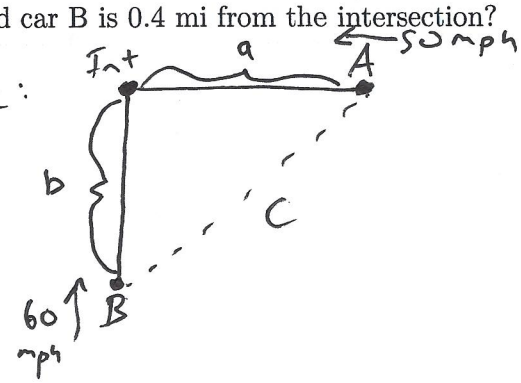
⑤ Derivative: $\frac{dV}{dt} = 3\frac{dh}{dt}$

⑥ Info (2): know: $\frac{dV}{dt} = 0.7 \frac{ft^3}{min}$ want: $\frac{dh}{dt}$

⑦ Plug in: $0.7 = 3\frac{dh}{dt} \Rightarrow \boxed{\frac{dh}{dt} = \frac{0.7}{3} \approx 0.25 \text{ ft/min}}$

Example 3.10.3. Car A is traveling west at 50 mph and car B is traveling north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

① Picture:



② Variables

a = distance from A to int

b = distance from B to int

c = distance b/n A + B

③ Info:

General

$\frac{da}{dt} = -50$ mph

$\frac{db}{dt} = -60$ mph

Particular

$a = 0.3$ mi

$b = 0.4$ mi

④ Eqn: $c^2 = a^2 + b^2$

⑤ Derivative: $2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} \Rightarrow cc' = aa' + bb'$

⑥ Info 2:

Know

$a = 0.3$ mi $\quad a' = -50$

$b = 0.4$ mi $\quad b' = -60$

Want

c'

$c = \sqrt{0.3^2 + 0.4^2} = 0.5$ mi

⑦ Plug in: $\overset{mi}{(0.5)}\, c' = \overset{mi}{(0.3)}\overset{mi/h}{(-50)} + \overset{mi}{(0.4)}\overset{mi/h}{(-60)}$

$\Rightarrow c' = -78$ mph

Formulas expected to know:

- Pythagorean Theorem
- Sine/cosine/tangent relationships in a right triangle (SOHCAHTOA)
- Formula for distance between two points (x_1, y_1) and (x_2, y_2): $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Easy volume formulas (such as $V = lwh$ for rectangular prisms)
- Easy area formulas (square, rectangle, circle)

Formulas that will be helpful when completing homework:

- Volume of a cone: $V = \frac{1}{3}\pi r^2 h$
- Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Example 3.10.4 (Concept Question). As gravel is being poured into a conical pile, its volume V changes with time. As a result, the height h and radius r also change with time. Knowing that at any particular moment $V = \frac{\pi}{3}r^2h$, the relationship between the changes in the volume, radius, and height with respect to time is

(a) $\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}\frac{dh}{dt}\right)$ no product rule

(b) $\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$

(c) $\frac{dV}{dr} = \frac{\pi}{3}\left(2rh + r^2\frac{dh}{dt}\right)$ ~~$\frac{dr}{dt}$ missing~~ $\frac{dh}{dr}$

(d) $\frac{dV}{dh} = \frac{\pi}{3}\left(r^2 + 2rh\frac{dr}{dh}\right)$

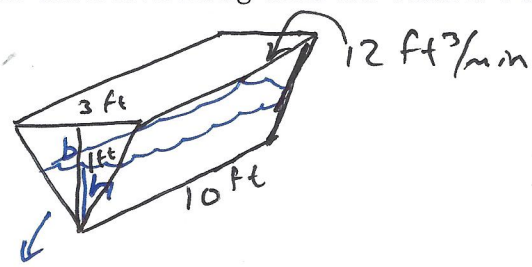
not wrt time

$V = \frac{\pi}{3}r^2h$

$\frac{dV}{dt} = \frac{\pi}{3}\left(2rr'h + r^2h'\right)$

$V = \frac{\pi}{3}$

Example 3.10.5. A trough is 10 ft long and its ends have the shape of an isosceles triangle that are 3 feet across at the top and have a height of 1 foot. If the trough is being filled at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 in deep?

① 

② h = height of water in trough

V = volume of water in trough

b = "base" of water in trough

③ Info 1:

General:

length of trough 10 ft

height of trough 1 ft

"base" of trough 3 ft

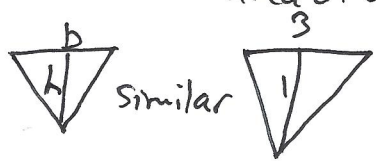
$\frac{dV}{dt} = 12 \text{ ft}^3/\text{min}$

Particular

$h = 6 \text{ in}$

④ Eqn: $V = \left(\frac{1}{2} bh\right)\ell = 5bh = 5(3h)h = 15h^2$

Area of cross-section



$\frac{b}{h} = \frac{3}{1} \Rightarrow b = 3h$

⑤ Derivative: $\frac{dV}{dt} = 30h\frac{dh}{dt}$

⑥ Info (2): know

$\frac{dV}{dt} = 12 \text{ ft}^3/\text{min}$

$h = 6 \text{ in} = 0.5 \text{ ft}$

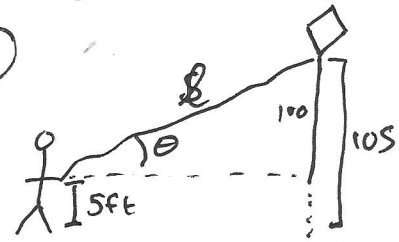
want:

$\frac{dh}{dt}$

⑦ Plug in $12 = (30)(0.5)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{15} = \frac{4}{5} = 0.8 \text{ ft/min}$

Example 3.10.6. A person flying a kite holds the string 5 ft above the ground, and the string is released at a rate of 2 ft/s as the kite moves horizontally at an altitude of 105 ft. Find the rate at which the angle the string makes to the horizontal is changing when 125 ft of string has been released.

(1)



(2) θ = angle string makes to the horizontal

ℓ = length of string

(3) Info 1: General

$\frac{ds}{dt} = 2$ ft/s

height b/n kite & horiz: 100

Particular

$\ell = 125$ ft

(4) Eqn: $\sin\theta = \frac{100}{\ell} = 100\ell^{-1}$

(5) Derivative: $\cos\theta \frac{d\theta}{dt} = -100\ell^{-2}\frac{d\ell}{dt}$

(6) Info 2: Know: $\ell = 125$ ft

$\frac{d\ell}{dt} = 2\frac{ft}{s}$

Want: $\frac{d\theta}{dt}$

Alternatively: when $\ell = 125$

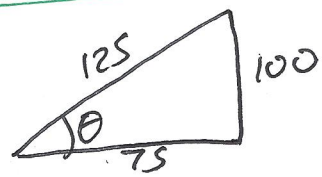
$\sin\theta = \frac{100}{125} = 0.8$

$\Rightarrow \theta = \sin^{-1}(0.8) \approx .9272$

$\Rightarrow \cos\theta = 0.6 = \frac{3}{5}$

$\cos\theta = \frac{adj}{hyp} = \frac{75}{125} = \frac{3}{5}$ radians

Kind of



$\sqrt{125^2 - 100^2} = 75$

(7) Plug in:

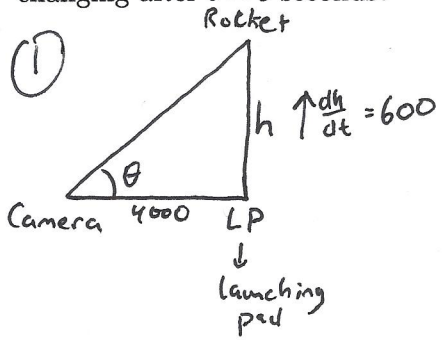
$\frac{3}{5}\frac{d\theta}{dt} = -100(125^{-2})(2)$

$\Rightarrow \frac{d\theta}{dt} = -0.0213$ rad/s

Example 3.10.7. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Suppose the rocket rises vertically at a constant rate of 600 ft/s.

(a) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing after $t = 5$ seconds?

(1)



(2) h = height of rocket

θ = angle of camera's elevation

(3) Info

General

Distance b/n camera + LP = 4000 ft

Rocket rising speed = 600 ft

Particular

$t = 5$ s

(4) Equation: $\tan\theta = \frac{opp}{adj} = \frac{h}{4000}$

(5) Derivative: $\sec^2\theta \frac{d\theta}{dt} = \frac{1}{4000}\frac{dh}{dt}$

(6) Info 2:

Know: $\frac{dh}{dt} = 600$ ft/s

Want: $\frac{d\theta}{dt}$

Need: $\sec^2\theta$

Alternatively: at $t=5$:

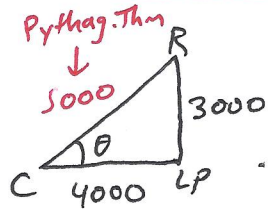
$\tan\theta = \frac{h}{4000} = \frac{3000}{4000} = \frac{3}{4}$

$\theta = \tan^{-1}\frac{3}{4} \approx 0.6435$

$\Rightarrow \sec^2\theta = \frac{1}{\cos^2\theta} \approx \frac{1}{\cos^2(0.6435)}$

$\Rightarrow \sec^2\theta = 1.5625 = \frac{25}{16}$

at $t=5$, $h = 600 \cdot 5 = 3000$ ft/s. We get

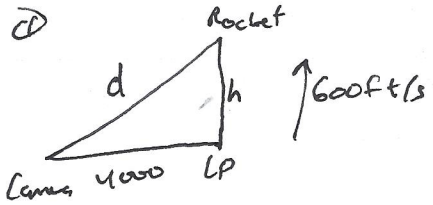


So, $\cos\theta = \frac{adj}{hyp} = \frac{4000}{5000} = \frac{4}{5} \Rightarrow \sec^2\theta = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$

(7) Plugin: $\frac{25}{16}\frac{d\theta}{dt} = \frac{600}{4000} \Rightarrow \frac{d\theta}{dt} = 0.096$ rad/s

(b) How fast is the distance from the television camera to the rocket changing after $t = 5$ seconds?

(1)



(2) h = height of rocket

d = distance b/n camera & rocket

(3) General: $\frac{dh}{dt} = 600$ ft/s

Particular: $t = 5$ s

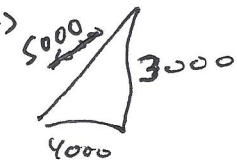
distance b/n camera & LP = 4000 ft

(4) Eqn: $4000^2 + h^2 = d^2$

(5) Derivative: $2h\frac{dh}{dt} = 2d\frac{dd}{dt} \Rightarrow h\frac{dh}{dt} = d\frac{dd}{dt}$

(6) Info: Know: $\frac{dh}{dt} = 600$ Want: $\frac{dd}{dt}$

$h = 600 \cdot 5 = 3000 \Rightarrow$



$d = 5000$

(7) $(3000)(600) = 5000\frac{dd}{dt}$

$\Rightarrow \frac{dd}{dt} = 360$ ft/s

Alternatively: We could use part (a) to get $\sin\theta = \frac{h}{d}$ and then $\cos\theta\,\theta' = \frac{h'd - d'h}{d^2}$ and we use θ', $\cos\theta$, h, and d from (a).