

Section 3.1: Definition of the Derivative

Definition of the Derivative: The derivative of $f(x)$ at $x = a$ is the limit (assuming it exists)

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{difference quotient} \quad (1)$$

Alternatively,

$$f'(a) = \lim_{t \rightarrow a} \frac{f(t) - f(a)}{t - a}. \quad (2)$$

$$t = a+h \quad \text{then} \quad t \rightarrow a \quad \Leftrightarrow \quad h \rightarrow 0$$

Example 1: Let $f(x) = 2x^2 - 5x + 1$. What is $f'(3)$?

$$18 - 15 + 1$$

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(3+h)^2 - 5(3+h) + 1 - 18}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(9+6h+h^2) - 15 - 5h - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{18+12h+2h^2 - 18 - 5h}{h}$$

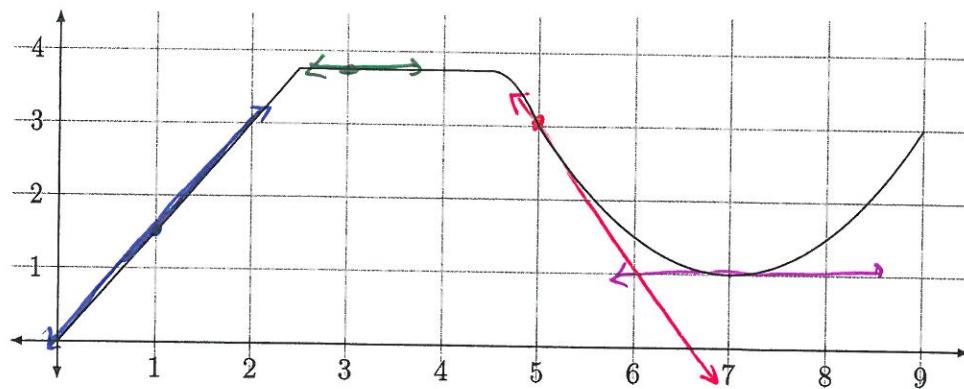
$$= \lim_{h \rightarrow 0} \frac{7h+2h^2}{h} = \lim_{h \rightarrow 0} \frac{h(7+2h)}{h} = \lim_{h \rightarrow 0} 7+2h$$

$$= 7$$

Note: $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$. Graphically this means that $f'(a)$ is the slope of the tangent line.

$$\begin{array}{ccc} \text{AROC} & \longrightarrow & \text{IROC} \\ \frac{f(t) - f(a)}{t - a}, \text{slope of secant line} & & \lim_{t \rightarrow a} \frac{f(t) - f(a)}{t - a}, \text{slope of tangent line} \end{array}$$

Example 2: Use the graph below to find the derivatives below.



$$f'(1) = \frac{3}{2}$$

$$f'(3) = 0$$

$$f'(5) = -2 \text{ (estimate)}$$

$$f'(7) = 0$$

Example 3: What is the equation of the tangent line at $x = 3$ for the function $f(x) = 2x^2 - 5x + 1$?

$$y = m(x - x_0) + y_0 \quad (\text{point slope})$$

$$x_0 = 3$$

$$m = f'(3) \Rightarrow \text{(from example 1)}$$

$$y_0 = f(3) = 4$$

$$y = 7(x - 3) + 4 \quad [\text{tangent line}]$$

tangent line to $f(x)$ at $x = a$

$$y = f'(a)(x - a) + f(a)$$

Example 4: Find the tangent line to $f(x) = x + \frac{1}{x}$ at $a = 1$ and $a = \frac{1}{2}$.

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{(a+h) + \frac{1}{a+h} - (a + \frac{1}{a})}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h + \frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{h + \frac{\frac{a}{a+h} - \frac{a+h}{a(a+h)}}{h}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h + \frac{a - (a+h)}{a(a+h)}}{h} = \lim_{h \rightarrow 0} \frac{h - \frac{h}{a(a+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h \left(1 - \frac{1}{a(a+h)}\right)}{h} = \lim_{h \rightarrow 0} \left(1 - \frac{1}{a(a+h)}\right) \\
 &= 1 - \frac{1}{a^2}
 \end{aligned}$$

$$x_0 = 1$$

$$f'(1) = 1 - \frac{1}{1^2} = 1 - 1 = 0$$

$$f(1) = 2$$

$$y = 0(x-1) + 2 = 2$$

tangent line at $a = 1$

$$x_0 = \frac{1}{2}$$

$$f'(\frac{1}{2}) = 1 - \frac{1}{(\frac{1}{2})^2} = 1 - 4 = -3$$

$$f(\frac{1}{2}) = \frac{5}{2} = 2.5$$

$$= \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + 2$$

$$y = -3(x - \frac{1}{2}) + \frac{5}{2}$$

tangent line at $a = \frac{1}{2}$