Section 11.1: Parametric Equations

Motivation: We can describe a particle's motion by specifying its coordinates as a function of time t:

$$x = f(t) \qquad y = g(t). \tag{1}$$

In other words, at time t, the particle is located at the point

$$c(t) = (f(t), g(t)). \tag{2}$$

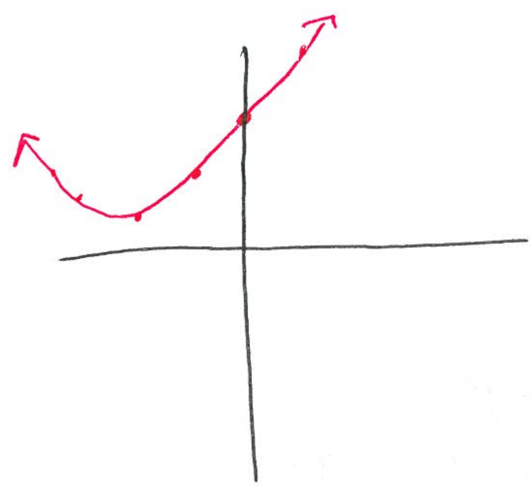
Definition: The equations in 1 are called **parametric equations**, and their graph is called a parametric curve. We refer to $c(t)$ as a parameterization with parameter t.

Note: Because x and y are functions of t, we often write $c(t) = (x(t), y(t))$ instead of $(f(t), g(t))$.

Example 1: Sketch the curve with parametric equations

$$x = 2t - 4 \qquad y = 3 + t^2 \tag{3}$$

t	x	y
1	-2	4
2	0	7
3	2	12
0	-4	3
-1	-6	4



Example 2: Describe the parametric curve

$$c(t) = (2t - 4, 3 + t^2) \tag{4}$$

in the form $y = f(x)$.

① Solve $\underbrace{2t-4 = x}$, $\cancel{3+t^2 = y}$ for t

$t = \frac{x+4}{2}$

② Plug in t into $3+t^2$

$y = 3 + t^2 = 3 + \left(\frac{x+4}{2}\right)^2 = 3 + \frac{(x+4)^2}{4}$

Example 3: A bullet follows the trajectory

$$c(t) = (80t, 200t - 4.9t^2). \tag{5}$$

What is the bullet's height at $t = 5$?

y = height

$= 200t - 4.9t^2$

$y(5) = 200 \cdot 5 - 4.9(5)^2 = 877.5$

Example 4: Verify that the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is parameterized by

$$c(t) = (a\cos t, b\sin t) \qquad -\pi \le t \le \pi \tag{6}$$

$x = a\cos t,\ y = b\sin t$

$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{a\cos t}{a}\right)^2 + \left(\frac{b\sin t}{b}\right)^2$

$= \cos^2 t + \sin^2 t$

$= 1.$ verified ✓

Theorem: **(Slope of the Tangent Line)** Let $c(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable. Assume that $x'(t)$ is continuous and $x'(t) \neq 0$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}. \tag{7}$$

Example 5: Let $c(t) = (t^2 + 1, t^3 - 4t)$. Find:

(a) An equation of the tangent line at $t = 3$.

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-4}{2t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=3} = \frac{3(3^2)-4}{2\cdot 3} = \frac{23}{6}$

$c(3) = (3^2+1, 3^3 - 4\cdot 3) = (10, 15)$

$\Rightarrow y - 15 = \frac{23}{6}(x - 10)$

slope $= 0 \Rightarrow \frac{dy}{dx} = 0$

(b) The points where the tangent is horizontal.

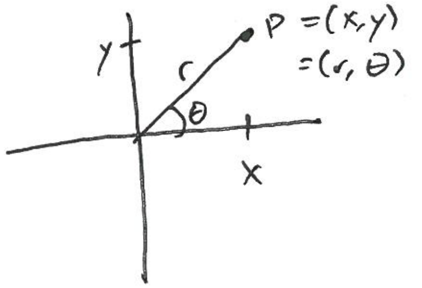
$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

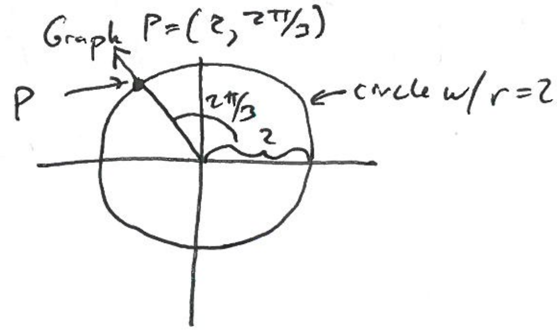
$0 = y'(t) = 3t^2 - 4 \Rightarrow t = \pm\frac{2}{\sqrt{3}}$ $\qquad x'(t) = 2t \Rightarrow x'\left(\pm\frac{2}{\sqrt{3}}\right) \neq 0$

$c\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{7}{3}, \frac{-16}{3\sqrt{3}}\right) \quad c\left(-\frac{2}{\sqrt{3}}\right) = \left(\frac{7}{3}, \frac{16}{3\sqrt{3}}\right)$

Section 11.3: Polar Coordinates

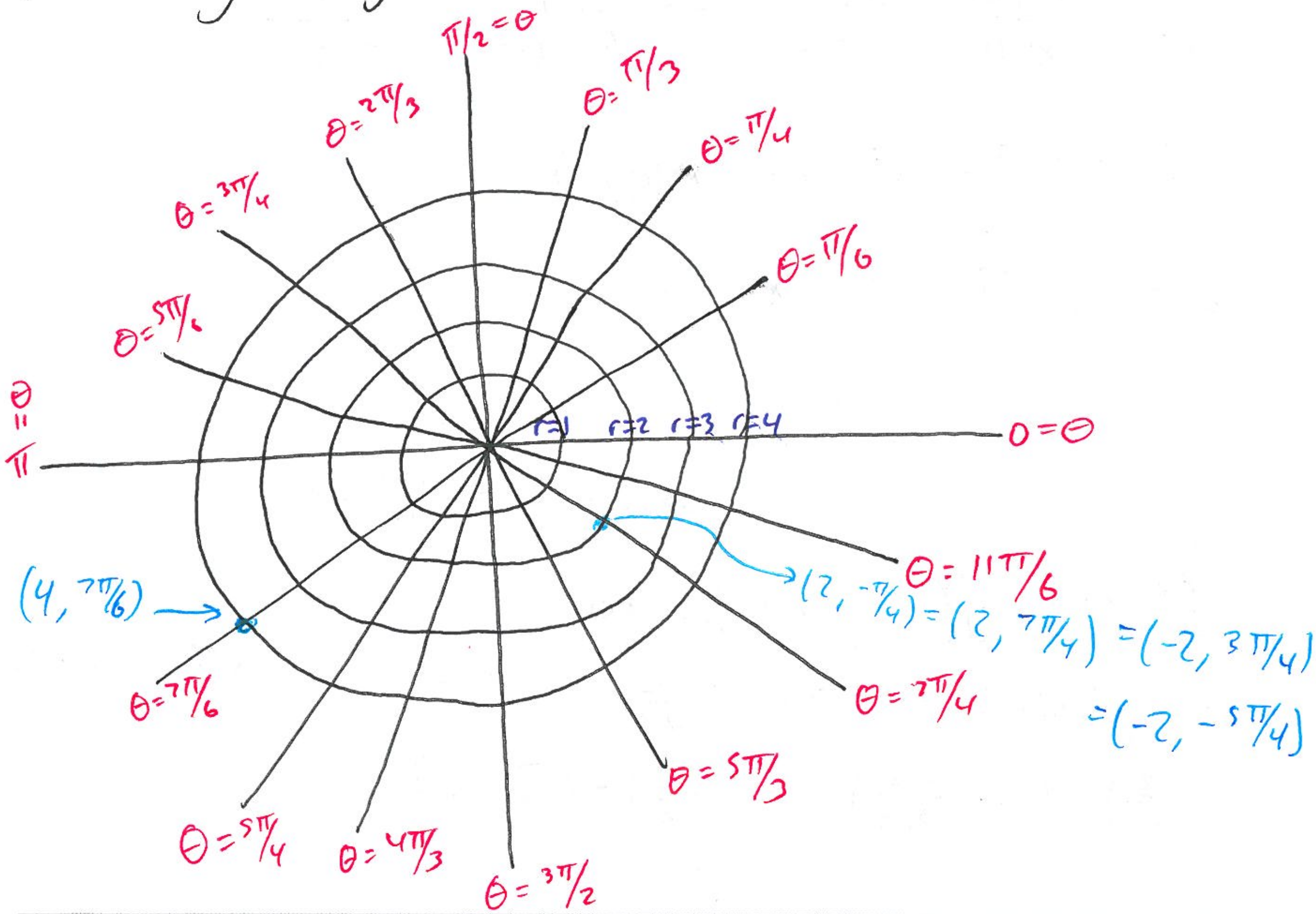
Definition: (Polar Coordinates) We label a point P by coordinates (r, θ), where r is the distance to the origin O and θ is the angle between $\overline{OP}$ and the positive x-axis. By convention, an angle is positive if the if the corresponding rotation is counterclockwise. We call r the **radial coordinate** and θ the **angular coordinate**.





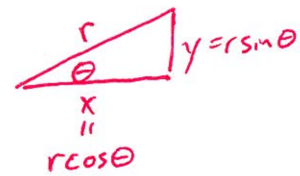
Circle centered at origin ⟷ r constant

line through origin ⟷ θ constant



Note: We have the following relationships between rectangular and polar coordinates: Polar to Rectangular:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
\tag{8}
$$



Rectangular to Polar:

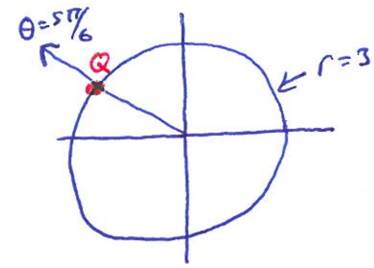
$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\tan\theta &= \frac{y}{x} \quad (x \neq 0)
\end{aligned}
\tag{9}
$$

Example 1: Find the rectangular coordinates of the point $Q = (3, \frac{5\pi}{6})$, which is in polar coordinates.

$$X = 3\cos 5\pi/6 = -\frac{3\sqrt{3}}{2}$$

$$y = 3\sin 5\pi/6 = 3/2$$

$$Q = \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

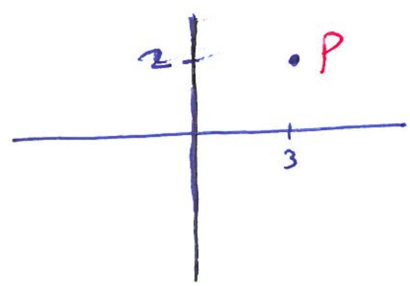


Example 2: Find the polar coordinates of the point $P = (3, 2)$, which is in rectangular coordinates.

$$r = \sqrt{3^2+2^2} = \sqrt{13}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588$$

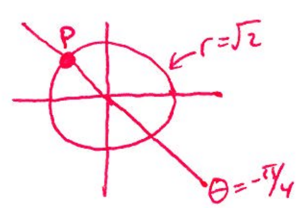
$$P = \left(\sqrt{13}, \tan^{-1}(2/3)\right)$$

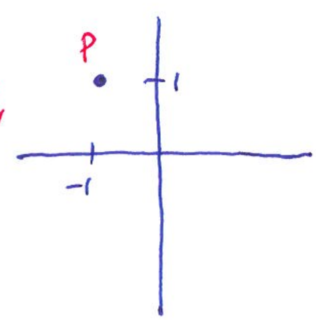


Example 3: Find two polar representations of $P = (-1, 1)$, one with $r > 0$ and one with $r < 0$.

$$r = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}(-1) = -\pi/4$$





$$P = \left(\sqrt{2}, 3\pi/4\right) = \left(-\sqrt{2}, -\pi/4\right)$$