Math 141 Worksheet 3

Name: Key

In this worksheet, we return to the concept of instantaneous rate of change. Now that you are adept at computing limits, you will be able to compute this without guesswork.

Recall our definition of the average rate of change of a function $f(x)$ over an interval $[a, t]$:

$$\text{Average rate of change of } f \text{ is } \frac{\Delta f}{\Delta x} = \frac{f(t) - f(a)}{t - a}.$$

The instantaneous rate of change of $f(x)$ at $x = a$ is the limit of the average rates of change as the interval shrinks to the point $x = a$. Now that we've studied limits, we can make this more precise:

$$\text{Instantaneous rate of change of } f \text{ at } a \text{ is } \lim_{t \to a} \frac{f(t) - f(a)}{t - a}.$$

This is good news! We no longer need to compute lots of average velocities, make a table of values, and then guess the instantaneous velocity. Instead, we can use our new skills at computing limits to determine the instantaneous rate of change.

1. Let $f(x) = x^2 + 3x$.

   (a) What is the average rate of change of $f(x)$ over the interval $[2, t]$?

   $$\frac{\Delta f}{\Delta x} = \frac{f(t) - f(2)}{t-2} = \frac{t^2 + 3t - 10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t+5$$

   (b) What limit do you need to compute to find the instantaneous rate of change of $f(x)$ at $x = 2$?

   $$\lim_{t \to 2} \frac{f(t) - f(2)}{t-2} = \lim_{t \to 2} t+5$$

   (c) Find the instantaneous rate of change of $f(x)$ at $x = 2$ (without any guesswork!)

   $$\lim_{t \to 2} t+5 = 7$$

2. Let $f(x) = 7x - 5$.

(a) Find the instantaneous rate of change of $f(x)$ at $x = 1$ by computing $\lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$.

$$\lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t \to 1} \frac{7t - 5 - 2}{t - 1} = \lim_{t \to 1} \frac{7t - 7}{t - 1}$$

$$= \lim_{t \to 1} \frac{7(t-1)}{t-1} = \lim_{t \to 1} 7 = 7$$

(b) Find the instantaneous rate of change of $f(x)$ at $x = 4$ by computing $\lim_{t \to 4} \frac{f(t) - f(4)}{t - 4}$.

$$\lim_{t \to 1} \frac{f(t) - f(4)}{t - 4} = \lim_{t \to 1} 7 = 7$$

(c) Explain your answers to (a) and (b) using our graphical interpretation of instantaneous rate of change. *Remember, explain with complete sentences.*

The function is a line, so the tangent line has the same slope.

3. Let $g(x) = \dfrac{1}{x}$.

(a) What is the average rate of change of $g(x)$ over the interval $[3, t]$?

$$\frac{g(3)-g(t)}{3-t} = \frac{\frac{1}{3}-\frac{1}{t}}{3-t} = \frac{\frac{t-3}{3t}}{3-t} = \frac{-1}{3t}$$

(b) Find the instantaneous rate of change of $g(x)$ at $x = 3$.

$$\lim_{t\to 3} \frac{g(3)-g(t)}{3-t} = \lim_{t\to 3} \frac{-1}{3t} = -\frac{1}{9}$$

4. Let $h(x) = \sqrt{x+1}$.

(a) What is the average rate of change of $h(x)$ over the interval $[2, t]$?

$$\frac{h(t)-h(2)}{t-2} = \frac{\sqrt{t+1}-\sqrt{3}}{t-2} \cdot \frac{\sqrt{t+1}+\sqrt{3}}{\sqrt{t+1}+\sqrt{3}} = \frac{t+1-3}{(t-2)(\sqrt{t+1}+\sqrt{3})}$$

$$= \frac{t-2}{(t-2)(\sqrt{t+1}+\sqrt{3})} = \frac{1}{\sqrt{t+1}+\sqrt{3}}$$

(b) Find the instantaneous rate of change of $h(x)$ at $x = 2$.

$$\lim_{t\to 2} \frac{h(t)-h(2)}{t-2} = \lim_{t\to 2} \frac{1}{\sqrt{t+1}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$