

4.4 The Shape of a Graph

Definition 4.4.1. We say $f(x)$ is concave up at $x = c$ if $f''(c) > 0$.

We say $f(x)$ is concave down at $x = c$ if $f''(c) < 0$.

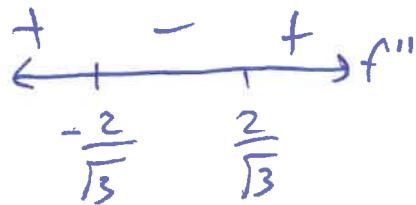
We say $(c, f(c))$ is an inflection point if $f(x)$ changes concavity at $x = c$.

Example 4.4.1. Let $f(x) = \frac{1}{4}x^4 - 2x^2$. Where is $f(x)$ concave up or down and what are the inflection points of $f(x)$?

$$f'(x) = x^3 - 4x$$

$$f''(x) = 3x^2 - 4$$

$$3x^2 - 4 = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \frac{2}{\sqrt{3}}$$



Concave up: $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$

Concave down: $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

IP: $(-\frac{2}{\sqrt{3}}, f(-\frac{2}{\sqrt{3}})), (\frac{2}{\sqrt{3}}, f(\frac{2}{\sqrt{3}}))$

Example 4.4.2. Let $y = (x^2 - 7)e^x$. Find intervals of concavity and inflection points.

$$y' = (2x)e^x + (x^2 - 7)e^x = (x^2 + 2x - 7)e^x$$

$$\begin{aligned} y'' &= (2x+2)e^x + (x^2+2x-7)e^x \\ &= e^x(x^2+4x-5) = e^x(x+5)(x-1) \end{aligned}$$

$$y'' = 0 \text{ when } x = -5, 1$$

$$\begin{array}{c} + - + + \\ \hline -5 \quad 1 \end{array} f''$$

concave up: $(-\infty, -5) \cup (1, \infty)$

" down: $(-5, 1)$

$$\text{IP: } (-5, (25-7)e^{-5}), (1, -6e)$$

Recall: First Derivative Test:

$$\begin{array}{c} \leftarrow + - \rightarrow \\ c \\ f' \end{array} \Rightarrow \text{local max at } x=c$$

$$\begin{array}{c} - + \rightarrow \\ c \\ f' \end{array} \Rightarrow \text{local min at } x=c$$

Theorem 4.4.1 (Second Derivative Test). If $x = c$ is a critical point of $f(x)$, then:

$$f''(c) < 0 \Rightarrow (c, f(c)) \text{ is a local max}$$

$$f''(c) > 0 \Rightarrow (c, f(c)) \text{ is a local min}$$

$$f''(c) = 0 \Rightarrow \text{nothing}$$

Example 4.4.3. Let $f(x) = \frac{1}{4}x^4 - 2x^2$. Find critical points and use the second derivative test to classify them.

$$\text{Then } f'(x) = x^3 - 4x$$

$$f''(x) = 3x^2 - 4$$

$$f''(x) = 0 = x(x^2 - 4) \Rightarrow x = \pm 2, 0$$

$$\begin{array}{c} + - + \\ \frac{-2}{\sqrt{3}} \quad \frac{2}{\sqrt{3}} \\ f'' \end{array}$$

$$f''(-2) > 0$$

$$f''(0) < 0$$

$$f''(2) > 0$$

$$\begin{array}{l} \Rightarrow \min \text{ at } x = -2, 2 \\ \text{2nd der.} \quad \max \text{ at } x = 0 \\ \text{test} \end{array}$$