

3.9 General Exponential and Logarithmic Functions

Theorem 3.9.1 (General Exponential and Logarithmic Derivatives).

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} b^x = b^x (\ln b) \quad (b > 0)$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

Logarithmic Differentiation:

$$f(x) = x^x \quad f'(x) = f(x) \cdot \frac{d}{dx} (\ln(f(x)))$$

$$\frac{d}{dx} x^x = x^x (\ln(x^x))' = x^x (\ln x + 1)$$

$$\frac{d}{dx} \ln(x^x) = \frac{d}{dx} (x \ln x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Example 3.9.1. $y = t \ln(t) - t$. What is y' ?

$$y' = \ln t + t \cdot \frac{1}{t} - 1 = \ln t + 1 - 1 = \ln t$$

Example 3.9.2. Find the derivative of 7^{4x-x^2} .

$$\begin{aligned}\frac{d}{dx} 7^{4x-x^2} &= 7^{4x-x^2} \cdot \ln 7 \cdot (4-2x) \\ &= (4-2x) 7^{4x-x^2} \ln 7\end{aligned}$$

Example 3.9.3. Find the derivative of $3^x \ln(1+x^2)$.

$$\begin{aligned}\frac{d}{dx} (3^x \ln(1+x^2)) &= \left(\frac{d}{dx} 3^x\right) \ln(1+x^2) + 3^x \left(\frac{d}{dx} \ln(1+x^2)\right) \\ &= (3^x \ln 3) \ln(1+x^2) + 3^x \left(\frac{2x}{1+x^2}\right) \\ &= 3^x \ln 3 \ln(1+x^2) + \frac{2x 3^x}{1+x^2}\end{aligned}$$

Example 3.9.4. Find the derivative of $\underbrace{(x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^5}_{f(x)}$.

$$f'(x) = f(x) \left(\ln f(x) \right)'$$

$$= f(x) \left(\ln \left((x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^5 \right) \right)'$$

$$= f(x) \left[\ln(x+1) + 2\ln(x+2) + 3\ln(x+3) + 4\ln(x+4) + 5\ln(x+5) \right]$$

$$= f(x) \left(\frac{1}{x+1} + 2 \cdot \frac{1}{x+2} + 3 \cdot \frac{1}{x+3} + 4 \cdot \frac{1}{x+4} + 5 \cdot \frac{1}{x+5} \right)$$

$$= (x+2)^2(x+3)^3(x+4)^4(x+5)^5 + 2(x+1)(x+2)(x+3)^3(x+4)^4(x+5)^5$$

$$+ 3(x+1)(x+2)^2(x+3)^2(x+4)^4(x+5)^5$$

$$+ 4(x+1)(x+2)^2(x+3)^3(x+4)^3(x+5)^5$$

$$+ 5(x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^4$$

Example 3.9.5. Find the derivative of $x^{\cos x}$.

$$\begin{aligned}\frac{d}{dx}(x^{\cos x}) &= (x^{\cos x}) (\ln(x^{\cos x}))' \\ &= x^{\cos x} (\cos x \ln x)' \\ &= x^{\cos x} \left(-\sin x \ln x + \frac{\cos x}{x} \right)\end{aligned}$$