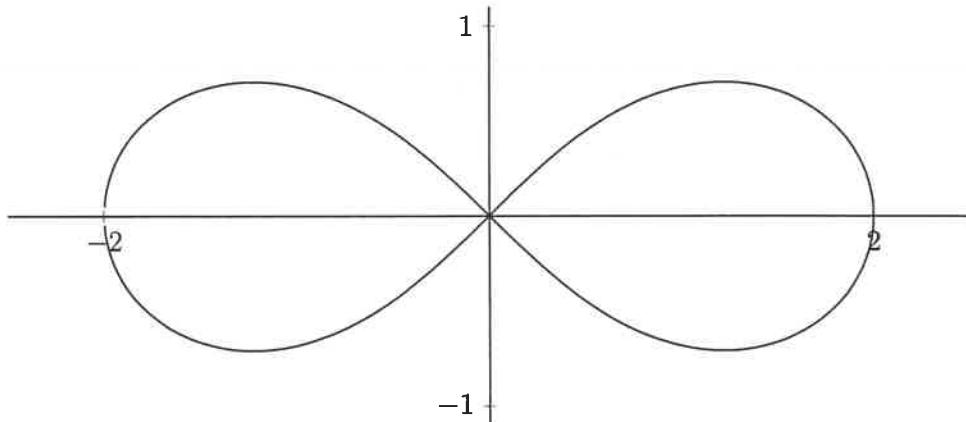


### 3.8 Implicit Differentiation

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**Definition 3.8.1.** If we can write  $y$  as a function of  $x$  (i.e.  $y = f(x)$ ), then we say  $y$  is given *explicitly*. If we cannot, we say that  $y$  is given *implicitly*.

**Example 3.8.1.**  $(x^2 + y^2)^2 = 4(x^2 - y^2)$



**Example 3.8.2.**  $y^4 - 2y = 4x^3 + x$ . Find  $\frac{dy}{dx} \approx y'$

$$\frac{d}{dx}(y^4 - 2y) = (4y^3 \cdot y') - (2y') = (4y^3 - 2)y'$$

$$\frac{d}{dx}(4x^3 + x) = 12x^2 + 1$$

$$12x^2 + 1 = (4y^3 - 2)\frac{dy}{dx}$$

$$\frac{12x^2 + 1}{4y^3 - 2} = \frac{dy}{dx}$$

**Example 3.8.3.** Find  $\frac{dy}{dx}$  when  $\tan(x^2y) = (x+y)^3$ .

$$\begin{aligned} \frac{d}{dx} \tan(x^2y) &= \sec^2(x^2y) (2xy + x^2y') \\ &= 2xy\sec^2(x^2y) + \sec^2(x^2y)x^2y' \\ \frac{d}{dx} (x+y)^3 &= 3(x+y)^2(1+y') = 3(x+y)^2 + 3(x+y)^2y' \\ 3(x+y)^2y' - \sec^2(x^2y)x^2y' &= 2xy\sec^2(x^2y) - 3(x+y)^2 \\ y' (3(x+y)^2 - \sec^2(x^2y)x^2) &= 2xy\sec^2(x^2y) - 3(x+y)^2 \\ y' &= \frac{2xy\sec^2(x^2y) - 3(x+y)^2}{3(x+y)^2 - \sec^2(x^2y)x^2} \end{aligned}$$

**Example 3.8.4.** Find the equation of the tangent line to  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$  at  $(1, 1)$ .

$$\begin{aligned} \frac{d}{dx} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) &= \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \frac{dy}{dx} \\ \frac{d}{dx} 2 &= 0 \end{aligned}$$

$$\text{at } (1, 1) : 0 = \frac{2}{3} + \frac{2}{3} \left. \frac{dy}{dx} \right|_{(1, 1)}$$

$$\left. \frac{dy}{dx} \right|_{(1, 1)} = -1$$

$$y - 1 = -1(x - 1) = -(x - 1)$$

$$y = -x + 2$$

**Example 3.8.5.**  $(x^2 + y^2)^2 = 4(x^2 - y^2)$ . Find coordinates of points where the tangent line is horizontal.

$$\frac{d}{dx} (x^2 + y^2)^2 = 2(x^2 + y^2)(2x + 2yy')$$

$$\cancel{\frac{d}{dx}} 4(x^2 - y^2) = 8x - 8y y'$$

$$\boxed{\text{horiz} \Rightarrow y' = 0} \quad 2(x^2 + y^2)(2x) = 8x \\ (x \neq 0) \quad 2(x^2 + y^2) = 4$$

$$x^2 + y^2 = 2$$

$$y^2 = 2 - x^2$$

$$\Rightarrow x = \pm \sqrt{3/2}$$

$$\Rightarrow y = \pm \sqrt{1/2}$$

$$(\sqrt{3/2}, \sqrt{1/2}), (-\sqrt{3/2}, \sqrt{1/2}), (\sqrt{3/2}, -\sqrt{1/2}), (-\sqrt{3/2}, -\sqrt{1/2})$$

**Example 3.8.6.** Which of the following are incorrect?

inc 1.  $\frac{d}{dx} x^2 y \rightarrow 2xy + x^2 y'$

cor 2.  $\frac{d}{dx} xe^y = e^y + xe^y y'$

inc 3.  $\frac{d}{dx} (x+y)^5 = 1 + 5y^4$

$\rightarrow 5(x+y)^4 (1 + \frac{dy}{dx})$

Example 3.8.7. Compute  $y''$  at the point  $(1, 1)$  on  $x^3 + y^3 = 3x + y - 2$ .

$$\begin{aligned}
 \frac{d}{dx}(x^3 + y^3) &= 3x^2 + 3y^2 y' \\
 \frac{d}{dx}(3x + y - 2) &= 3 + y' \quad \rightarrow \quad 3 + 3y'(1, 1) \\
 \hline
 \frac{d}{dx}(3x^2 + 3y^2 y') &= 6x + 3(2yy')y' + 3(y'')y^2 \\
 \frac{d}{dx}(3 + y') &= y'' \\
 y''(1, 1) &= 6 + 3(2 \cdot 1 \cdot y''(1, 1))y'(1, 1) + 3y''(1, 1) \\
 y''(1, 1) &= 6 + 3y''(1, 1) \Rightarrow 0 = 6 + 2y''(1, 1) \\
 \Rightarrow -6 &= 2y''(1, 1) \\
 \Rightarrow y''(1, 1) &= -3
 \end{aligned}$$

Theorem 3.8.1 (Derivatives of Inverse Trig Functions).

$$\begin{array}{lll}
 \frac{d}{dx} \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \cos^{-1}(x) &= \frac{-1}{\sqrt{1-x^2}} \\
 \frac{d}{dx} \tan^{-1}(x) &= \frac{1}{1+x^2} & \frac{d}{dx} \cot^{-1}(x) &= \frac{-1}{1+x^2} \\
 \frac{d}{dx} \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx} \csc^{-1}(x) &= \frac{-1}{|x|\sqrt{x^2-1}}
 \end{array}$$

**Example 3.8.8.**  $\frac{d}{dx} e^{\cos^{-1}(x)}$

$$\rightarrow = e^{\cos^{-1}(x)} \cdot \frac{-1}{\sqrt{1-x^2}}$$

**Example 3.8.9.**  $\frac{d}{dt} \sec^{-1}(5t^2)$

$$\frac{d}{dt} \sec^{-1}(t) = \frac{1}{|t| \sqrt{t^2 - 1}}$$

$$\rightarrow = \frac{1}{|5t^2| \sqrt{(5t^2)^2 - 1}} \cdot 10t$$

$$= \frac{10t}{5t^2 \sqrt{25t^4 - 1}}$$

$$= \frac{2}{t \sqrt{25t^4 - 1}}$$

**Example 3.8.10.** Why is  $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ ?

$$\frac{d}{d\theta} \sin^{-1}(\sin(\theta)) = (\sin^{-1})'(\sin \theta) \cdot \cos \theta$$

if

$$\frac{d}{d\theta} \theta = 1 \quad f(x) = \sin^{-1}(x)$$

$$(\sin^{-1})'(\sin \theta) = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1-\sin^2 \theta}}$$

$$f'(\sin \theta)$$

$$x = \sin \theta \implies f'(x) = \frac{1}{\sqrt{1-x^2}} = (\sin^{-1}(x))'$$