

### 3.7 Chain Rule

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**Theorem 3.7.1** (Chain Rule).

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

**Example 3.7.1.** What is  $\frac{d}{dx} (\cos(x))^2$ ?

$$f(x) = x^2 \quad \rightarrow \quad f'(x) = 2x$$

$$g(x) = \cos x \quad g'(x) = -\sin x \quad f'(\cos x) = 2 \cos x$$

$$\frac{d}{dx} (\cos x)^2 = 2(\cos x)(-\sin x) = -2 \cos x \sin x$$

#### Common Uses of the Chain Rule

$$1. \frac{d}{dx} (g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

$$f(x) = x^n, g(x) = x$$

$$\text{Example 3.7.2. } y = (x^6 + 2)^{10} \quad n = 10, g(x) = x^6 + 2$$

$$y' = 10(x^6 + 2)^9 (6x^5) = 60x^5(x^6 + 2)^9$$

$$2. \frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x) = g'(x) e^{g(x)}$$

$$f(x) = e^x$$

$$\text{Example 3.7.3. } y = e^{x^3 + 7x}$$

$$y' = (3x^2 + 7)e^{x^3 + 7x}$$

$$\frac{d}{dx} e^{\pi} = \left( \frac{d}{dx} \pi \right) e^{\pi} = 0 \cdot e^{\pi} = 0$$

$$3. \frac{d}{dx} f(kx) = f'(kx) \cdot k = kf'(kx)$$

$$g(x) = kx$$

Example 3.7.4.  $y = \sin(\pi x)$        $f(x) = \sin x, k = \pi$

$$y' = \pi \sin(\pi x)$$

Example 3.7.5. Why is the chain rule true?

Let  $F(x) = f(g(x))$ . Now, let's use the limit definition to take the derivative.

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \\ &= \left( \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \cdot g'(x) \\ &= \left( \lim_{u \rightarrow g(x)} \frac{f(u) - f(g(x))}{u - g(x)} \right) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

Example 3.7.6.  $y = \cos(e^{3x+1})$

$$\begin{aligned} \text{Outside: } \cos x &\xrightarrow{\text{der.}} -\sin x \\ \text{inside: } e^{3x+1} &\xrightarrow{\text{der.}} 3e^{3x+1} \end{aligned}$$

$$y' = -\sin(e^{3x+1})(3e^{3x+1})$$

Example 3.7.7.  $g(\theta) = \frac{\sqrt{5\theta^2 + 3\theta}}{\theta + 1}$        $\frac{d}{d\theta} \sqrt{5\theta^2 + 3\theta} = \frac{1}{2}(5\theta^2 + 3\theta)^{-1/2}(10\theta + 3)$

$$g'(\theta) = \frac{\frac{1}{2}(5\theta^2 + 3\theta)^{-1/2}(10\theta + 3)(\theta + 1) - (5\theta^2 + 3\theta)^{1/2}}{(\theta + 1)^2}$$

Example 3.7.8.  $f(y) = \sec(ye^{-y})$

$$f'(y) = \sec(ye^{-y}) \tan(ye^{-y}) (e^{-y} - ye^{-y}) + (e^{-y})y$$

Outside:  $\sec(x) \xrightarrow{\text{der.}} \sec(x)\tan(x)$   
 Inside:  $ye^{-y} \xrightarrow{} e^{-y}$

Example 3.7.9.  $y = \frac{3x}{x^2+1} \longrightarrow y = 3x(x^2+1)^{-1}$

$$y' = \frac{3(x^2+1) - (3x)(2x)}{(x^2+1)^2} \quad y' = 3(x^2+1)^{-1} - (x^2+1)^{-2}(2x)(3x)$$

**Example 3.7.10.**  $h(x) = (x^2 - 1) \cos(2x)$

$$h'(x) = (2x)\cos(2x) - 2(x^2 - 1)\sin(2x)$$

**Example 3.7.11.**  $s = \tan^2(3t) = (\tan(3t))^2$

$$\begin{aligned} s' &= 2\tan(3t) \cdot \sec^2(3t) \cdot 3 \\ &= 6\tan(3t)\sec^2(3t) \end{aligned}$$

**Example 3.7.12.**  $y = \sqrt{1 + \sin^2 x^3} = (1 + \sin^2 x^3)^{1/2}$

$$\begin{aligned} y' &= \frac{1}{2}(1 + \sin^2 x^3)^{-1/2} (2\sin(x^3)\cos(x^3))(3x^2) \\ &= \frac{3x^2 \sin(x^3)\cos(x^3)}{\sqrt{1 + \sin^2 x^3}} \end{aligned}$$

**Example 3.7.13.**  $f(a) = (3a^2 + a + 1)^{-2}$

$$f'(a) = -2(3a^2 + a + 1)^{-3}(6a + 1)$$

Example 3.7.14.  $y = \sqrt{(t^2 - \cot t + 2)^3} = (t^2 - \cot t + 2)^{3/2}$

$$y' = \frac{3}{2}(t^2 - \cot t + 2)^{1/2} (2t + \csc^2 t)$$

Example 3.7.15.  $y = \frac{\sin(1+x)}{1+\sin x}$

$$y' = \frac{\cos(1+x)(1+\sin x) - \sin(1+x)\cos x}{(1+\sin x)^2}$$

$$= \sin(1+x)(1+\sin x)^{-1}$$

$$\Rightarrow y' = \cos(1+x)(1+\sin x)^{-1} - \sin(1+x)\cos x(1+\sin x)^{-2}$$

Example 3.7.16.  $a(b) = e^{e^b} + e^{b-12}$

$$a'(b) = e^b e^{e^b} + e^{b-12}$$

$$\frac{d}{db} e^{e^b} = e^b e^{e^b}$$

$$\frac{d}{db} e^{g(x)} = g'(x) e^{g(x)}$$

$$g(x) = e^b$$

Example 3.7.17.  $y = \sin(\cos(\sin x))$

$$\begin{aligned} y' &= \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x \\ &= -\cos x \sin(\sin x) \cos(\cos(\sin x)) \end{aligned}$$