3.6 Trigonometric Functions

Rules so far... c constant

- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}(cf(x)) = cf'(x)$
- $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Theorem 3.6.1 (New Rules). Derivatives of Trigonometric Functions

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$

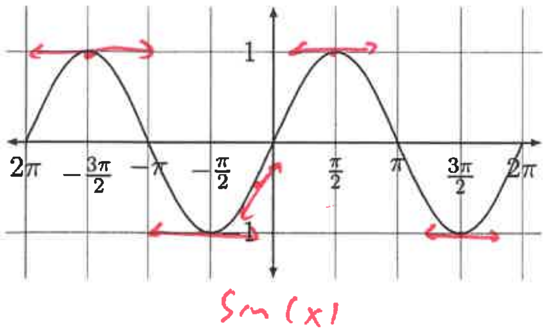
$\frac{d}{dx}\cot(x) = -\csc^2(x)$

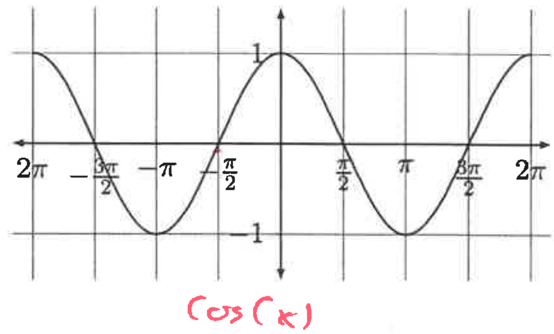
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Why is $\frac{d}{dx}\sin(x) = \cos(x)$?

Graphically:





Algebraically:

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$= \left(\lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)}{h}\right)\cos(x)$$

$$= \cos(x)$$

Example 3.6.1. Verify $\frac{d}{dx}\cot(x) = -\csc^2(x)$.

$$\frac{d}{dx}\cot(x) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

Example 3.6.2. Compute the derivative of $f(\theta) = \theta\tan(\theta)$.

$$f'(\theta) = (1)\tan(\theta) + \theta(\sec^2\theta)$$
$$= \tan\theta + \theta\sec^2\theta$$

Example 3.6.3. Find $g'(x)$ when $g(x) = \dfrac{x}{e^x\sin(x)}$. $\rightarrow e^x\sin x + e^x\cos x$

$$g'(x) = \frac{(1)e^x\sin(x) - (e^x\sin x + e^x\cos x)x}{(e^x\sin x)^2}$$
$$= \frac{e^x(\sin x - x\sin x - x\cos x)}{(e^x)^2\sin^2 x}$$
$$= \frac{\sin x - x\sin x - x\cos x}{e^x\sin^2 x}$$

Example 3.6.4. Find $\frac{d}{dt}\left(9\sec(t) - \cos^2(t)\right)$ $\rightarrow \cos t\cos t \xrightarrow{\text{der.}} (-\sin t)\cos t + (-\sin t)\cos t$

$$= -2\sin t\cos t$$

$$= 9\sec t\tan t + 2\sin t\cos t$$

Example 3.6.5. Find the equation of the tangent line to $y = \dfrac{\sin(t)}{1+\cos(t)}$ at $t = \frac{\pi}{3}$.

$$y\left(\frac{\pi}{3}\right) = \frac{\sin\frac{\pi}{3}}{1+\cos\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1+\frac{1}{2}} = \frac{\sqrt{3}/2}{3/2} = 1/\sqrt{3}$$

$$y' = \frac{\cos t\,(1+\cos t) - (-\sin t)(\sin t)}{(1+\cos t)^2}$$

$$= \frac{\cos t + \cos^2 t + \sin^2 t}{(1+\cos t)^2} = \frac{\cos t + 1}{(1+\cos t)^2}$$

$$= \frac{1}{1+\cos t}$$

$$y'(\pi/3) = \frac{1}{1+\cos\pi/3} = \frac{1}{1+1/2} = 2/3$$

$$\boxed{y - \frac{1}{\sqrt{3}} = \frac{2}{3}\left(x - \frac{\pi}{3}\right)}$$ tangent line

Example 3.6.6. $f(x) = \sin(x)$. What is $f^{(199)}(x)$?

$f(x) = \sin(x)$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(199)}(x) = -\cos x$