3.4 Rates of Change

Example 3.4.1. The population of Knoxville in 2010 was 558,696. Let $P(t)$ be the population in year t, measured in millions.

$P(2010) = 0.558696$

What is the meaning of $P'(2010)$?

population growth/decay in 2010

rate of change of population in $t = 2010$

Estimate $P(2011)$, given $P'(2010) = 0.03$.

Idea: $P'(t) = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h} \underset{h=1}{\approx} \frac{P(t+1) - P(t)}{1}$

$= P(t+1) - P(t)$

$$P(t+1) \approx P(t) + P'(t)$$

$$P(2011) \approx P(2010) + P'(2010)$$

$$= 0.558696 + 0.03$$

$$= 0.588696$$

Example 3.4.2. The cost (in dollars) for a company to produce a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

The marginal cost at producing level x is the cost of producing the $(x+1)$st item.

$C(x+1) - C(x) \approx C'(x)$

What is the marginal cost at production level 100?

$C(101) - C(100) = 2611.0702 - 2600 = 11.0702$

$C'(x) = 3 + 0.02x + 0.0006x^2 \qquad C'(100) = 11$

What is $C'(100)$ and how can we interpret this?

rate of change of the cost of making 100 jeans.

$C'(100) \approx$ marginal cost at 100

(that is, $C'(100)$ approximates the cost of producing the 101st pair of jeans)

Example 3.4.3. A tank holds 5000 gallons of water and a full tank can be emptied in 40 minutes when water drains from the bottom. Torricelli's Law tells us that the volume V in the tank after t minutes is

$$V(t) = 5000\left(1 - \frac{t}{40}\right)^2. = 5000\left(1 - \frac{t}{20} + \frac{t^2}{1600}\right)$$

Find the rate at which the water is draining out of the tank after 5, 10, 20, and 40 minutes. When is water draining out the fastest? the slowest?

$V'(t) = 5000\left(-\frac{1}{20} + \frac{t}{800}\right)$

$V'(5) = -218.75$ fastest

$V'(10) = -187.5$

$V'(20) = -125$

$V'(40) = 0$ slowest

Example 3.4.4. A ball is thrown upward with an initial velocity of 80 feet per second. What is the maximum height of the ball? (Use the formula $s(t) = s_o + v_o t - \frac{1}{2}gt^2$, where $g = 32\ \frac{\text{ft}}{\text{sec}^2}$.)

$s(t) = 0 + 80t - 16t^2$

$v(t) = s'(t) = -32t + 80$

max height when $v(t) = 0$

$0 = -32t + 80$

$t = 80/32 = 20/8 = 10/4 = 5/2$ sec

$s(5/2) = 100$ ft

Example 3.4.5. When the brightness of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula developed is

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}.$$

What is the sensitivity, which is defined to be the rate of change of the reaction?

Sensitivity $= R'$

$$= \frac{(1+4x^{0.4})(9.6x^{-0.6}) - (40+24x^{0.4})(1.6x^{-0.6})}{(1+4x^{0.4})^2}$$