

### 3.3 Product and Quotient Rules

Theorem 3.3.1 (Product Rule).

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Caution:  $\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$

Ex:  $\frac{d}{dx} x^2 = 2x \neq 1 = \frac{d}{dx} x \cdot \frac{d}{dx} x^2 = x \cdot x$

Example 3.3.1. What is the derivative of  $\underbrace{(x^2+1)}_f \underbrace{(x^3-2x)}_g$ ?

$$\begin{aligned} \frac{d}{dx}(x^2+1)(x^3-2x) &= 2x(x^3-2x) + (x^2+1)(3x^2-2) \\ f' &= 2x \\ g' &= 3x^2-2 \end{aligned}$$

Theorem 3.3.2 (Quotient Rule).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Caution:  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$

$x = \frac{x}{1}$   $\frac{d}{dx} x = 1 \neq \frac{\frac{d}{dx} x}{\frac{d}{dx} 1} = \text{DNE}$

Example 3.3.2. What is  $\frac{d}{dx}\left(\frac{x^2+3x}{x^5+7}\right)$ ?

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2+3x}{x^5+7}\right) &= \frac{(2x+3)(x^5+7) - (5x^4)(x^2+3x)}{(x^5+7)^2} \\ f(x) &= x^2+3x & g(x) &= x^5+7 \\ f'(x) &= 2x+3 & g'(x) &= 5x^4 \end{aligned}$$

**Example 3.3.3.** Use the information in the table to answer the following questions.

$f(1)$	$f'(1)$	$g(1)$	$g'(1)$
4	-3	7	2

What is  $(fg)'(1)$ ?

$$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = (-3)(7) + (4)(2) = -13$$

What is  $\left(\frac{f}{g}\right)'(1)$ ?

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{(-3)(7) - (4)(2)}{49} = \frac{-29}{49}$$

If  $F(x) = \frac{1}{2}x^2 f(x)$ , what is  $F'(1)$ ?

$$F'(x) = \frac{1}{2}x^2 f'(x) + x f(x)$$

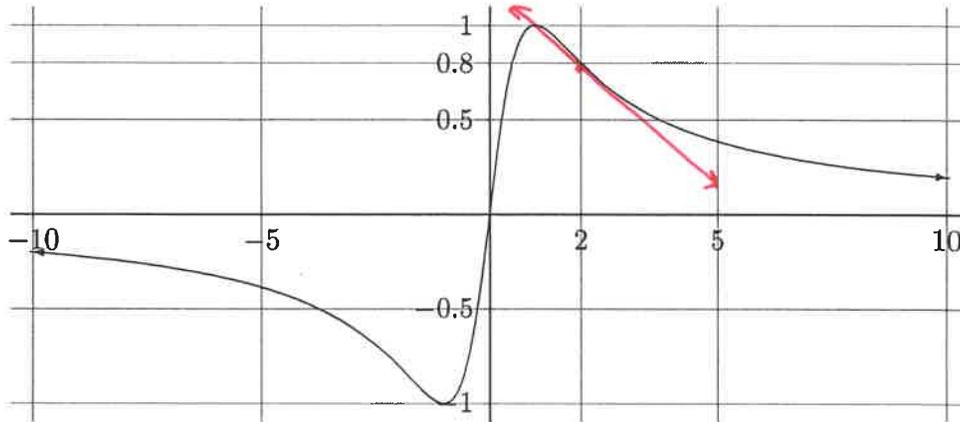
$$F'(1) = \frac{1}{2} \cdot 1 \cdot f'(1) + 1 \cdot f(1) = \frac{1}{2}(-3) + 4 = \frac{5}{2}$$

If  $H(x) = \frac{x}{f(x)g(x)}$ , what is  $H'(1)$ ?

$$H'(x) = \frac{f(x)g(x) - (f(x)g'(x) + f'(x)g(x))x}{(f(x)g(x))^2}$$

$$H'(1) = \frac{4 \cdot 7 - (4 \cdot 2 + (-3) \cdot 7) \cdot 1}{(4 \cdot 7)^2} = \frac{28 - (-13)}{784} = \frac{41}{784}$$

**Example 3.3.4.** Find the tangent line to  $f(x) = \frac{2x}{x^2 + 1}$  at  $x = 2$ .



$$f(2) = 0.8$$

$$f'(x) = \frac{2 \cdot (x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 2}{(x^2 + 1)^2}$$

$$f'(2) = \frac{-2(2^2) + 2}{(2^2 + 1)^2} = \frac{-8 + 2}{5^2} = \frac{-6}{25}$$

$$y - 0.8 = \frac{-6}{25}(x - 2)$$

Example 3.3.5.  $\frac{d}{dx}(5x^4e^x) = 20x^3e^x + 5x^4e^x$

Example 3.3.6.  $\frac{d}{dx}\left(\frac{6e^x}{\sqrt{x}}\right) = \frac{6e^x x^{1/2} - 6e^x (\frac{1}{2}x^{-1/2})}{(x)^2} = \frac{6e^x \sqrt{x} - 3e^x/\sqrt{x}}{1x}$

$$\begin{aligned}\frac{d}{dx}(6e^x x^{-1/2}) &= 6e^x x^{-1/2} + 6e^x(-\frac{1}{2}x^{-3/2}) \\ &= 6e^x x^{-1/2} - 3x^{-3/2}e^x\end{aligned}$$

Example 3.3.7.  $\frac{d}{dx}(x^3e^x(8x^4 - 25x + 10)) = (3e^x x^2 + e^x x^3)(8x^4 - 25x + 10) + (x^3e^x)(32x^3 - 25)$

$$x^3e^x(8x^4 - 25x + 10) = e^x(8x^7 - 25x^4 + 10x^3)$$

$$\begin{aligned}\frac{d}{dx} \uparrow &= e^x(8x^7 - 25x^4 + 10x^3) \\ &\quad + e^x(56x^6 - 100x^3 + 30x^2)\end{aligned}$$