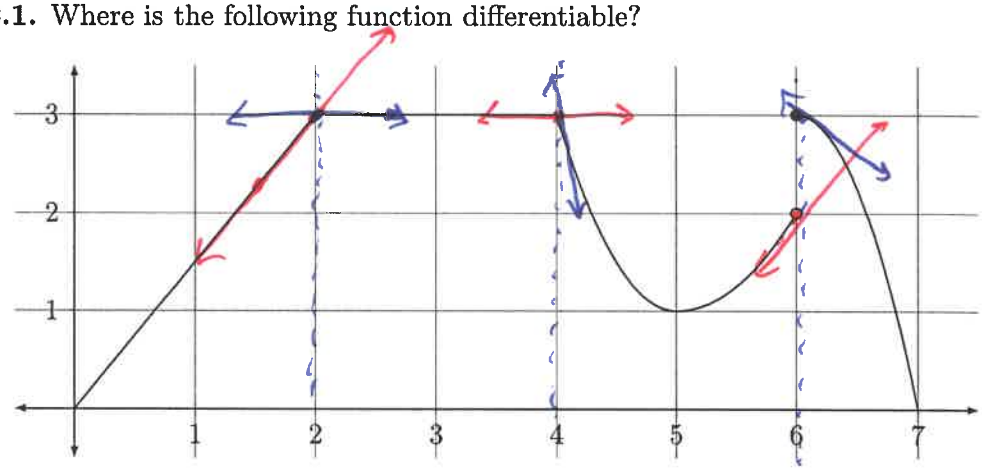
3.2 Derivative as a Function

Recall:

$$f'(a) = \lim_{h\to 0} \frac{f(h+a)-f(a)}{h} = \lim_{t\to a} \frac{f(t)-f(a)}{t-a}$$

Definition 3.2.1. When the above limit exists, we say $f(x)$ is differentiable at $x = a$. When the limit does not exist, we say $f(x)$ is non-differentiable at $x = a$.

Example 3.2.1. Where is the following function differentiable?



at $x=2$ $f(x)$ is non-differentiable

at $x=4$ $f(x)$ is non-differentiable

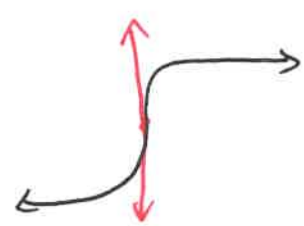
at $x=6$ $f(x)$ is non-differentiable

3 ways a function may not be differentiable

(1) function is discontinuous

(2) corner or cusp

(3) vertical tangent line



Notation 3.2.1. $f'(x)$, y', $\dfrac{df}{dx}$, $\dfrac{dy}{dx}$

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} \qquad \frac{d}{dx}(f(x)) = \frac{df}{dx}$$

Example 3.2.2. $f(x) = x^2$. What is $f'(x)$?

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x+h = 2x$$

Example 3.2.3. $f(x) = x^3$. What is $f'(x)$?

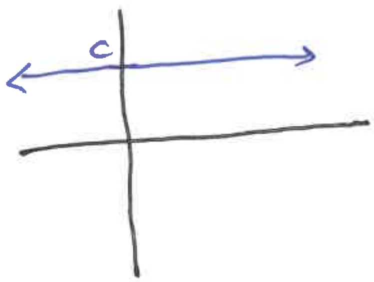
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2+3xh+h^2$$

$$= 3x^2$$

Example 3.2.4. $f(x) = c$. What is $f'(x)$?



$$f'(x) = 0$$

Theorem 3.2.1 (Derivative Rules).

(a) Constant Rule: $\frac{d}{dx}c = 0$

$$\frac{d}{dx}5 = 0, \quad \frac{d}{dx}\pi = 0, \quad \frac{d}{dx}e^6 = 0$$

(b) Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$

$$\frac{d}{dx}x^4 = 4x^3, \quad \frac{d}{dx}x^{\sqrt{17}} = \sqrt{17}\,x^{\sqrt{17}-1}$$

(c) Sum/Difference Rule: $\frac{d}{dx}(f \pm g) = f' \pm g'$

$$\frac{d}{dx}(x^2 + x^3 + x^4) = 2x + 3x^2 + 4x^3$$

(d) Constant Multiple Rule: $\frac{d}{dx}(cf) = cf'(x)$

$$\frac{d}{dx}(3x^2) = 3\left(\frac{d}{dx}x^2\right) = 3(2x) = 6x$$

(e) Exponential Rule: $\frac{d}{dx}e^x = e^x$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$\frac{d}{dx}(6e^x) = 6e^x$$

Example 3.2.5. $f(x) = 2x^2 - 5x + 1 + x^{\sqrt{17}}$. What is $f'(x)$?

$$f'(x) = 4x - 5 + 0 + \sqrt{17}\, x^{\sqrt{17}-1} = 4x - 5 + \sqrt{17}\, x^{\sqrt{17}-1}$$

Example 3.2.6. $f(x) = \dfrac{3}{x} + \sqrt{x} - \dfrac{1}{2}e^x$. What is $f'(x)$?

$$= 3x^{-1} + x^{\frac{1}{2}} - \frac{1}{2}e^x$$

$$f'(x) = -3x^{-2} + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}e^x$$

Example 3.2.7. $f(x) = 3e^{x-6} + (6x-5)^2$. What is $f'(x)$?

$$= 3e^x e^{-6} + 36x^2 - 60x + 25$$

$$f'(x) = 3e^{-6}e^x + 72x - 60$$

Example 3.2.8. $f(x) = \dfrac{2x^2 + 9x^{\frac{1}{2}}}{2x^2}$. What is $f'(x)$?

$$\frac{9}{2}x^{\frac{1}{2}-2} = \frac{9}{2}x^{\frac{1}{2}}x^{-2}$$

$$= \frac{2x^2}{2x^2} + \frac{9x^{1/2}}{2x^2} = 1 + \frac{9}{2}x^{-3/2}$$

$$f'(x) = \left(-\frac{3}{2}\right)\left(\frac{9}{2}\right)x^{-5/2} = -\frac{27}{4}x^{-5/2}$$