3.1 Definition of the Derivative

Definition 3.1.1. The derivative of $f(x)$ at $x = a$ is the limit (assuming it exists)

$$f'(a) = \lim_{h\to 0} \boxed{\frac{f(a+h) - f(a)}{h}} \quad \text{difference quotient}$$

Alternatively,

$$f'(a) = \lim_{t\to a} \frac{f(t) - f(a)}{t - a}. \rightarrow t = a + h \text{ as } \begin{matrix} t \to a \\ h \to 0 \end{matrix}$$

Example 3.1.1. Let $f(x) = 2x^2 - 5x + 1$ What is $f'(3)$?

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{2(3+h)^2 - 5(3+h) + 1 - 4}{h}$$

$$= \lim_{h\to 0} \frac{2(9+6h+h^2) - 15 - 5h + 1 - 4}{h}$$

$$= \lim_{h\to 0} \frac{18 + 12h + 2h^2 - 18 - 5h}{h}$$

$$= \lim_{h\to 0} \frac{7h + 2h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(7 + 2h)}{h}$$

$$= \lim_{h\to 0} 7 + 2h$$

$$= 7$$

Note: $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$. Graphically this means that $f'(a)$ is the slope of the tangent line.

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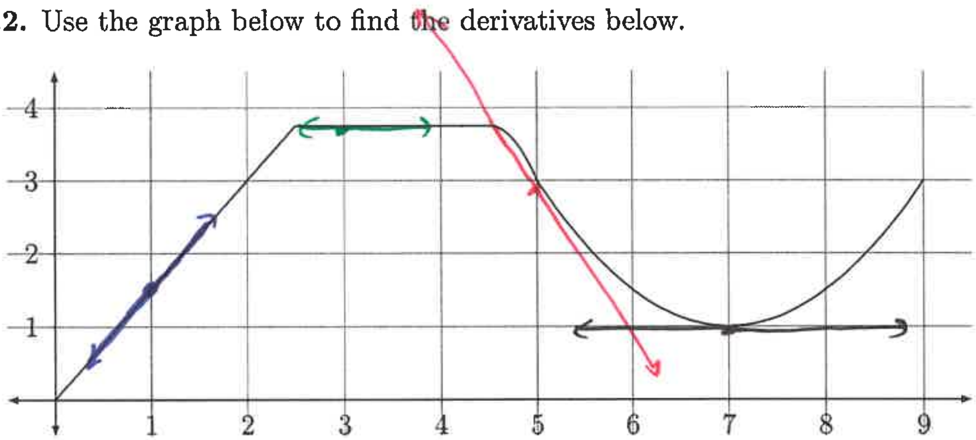
$\dfrac{f(t)-f(a)}{t-a}$: slope of secant line

$\lim_{t \to a} \dfrac{f(t)-f(a)}{t-a}$ = slope of tangent line

Example 3.1.2. Use the graph below to find the derivatives below.



$f'(1) = \frac{3}{2}$

$f'(3) = 0$

$f'(5) = -2$ (estimate)

$f'(7) = 0$

Example 3.1.3. What is the equation of the tangent line at $x = 3$ for the function $f(x) = 2x^2 - 5x + 1$?

Recall: $f'(3) = 7$ (from example 3.1.1)

eqn of line: $y = mx + b$

$$y - y_0 = m(x - x_0)$$

$$m = \frac{y - y_0}{x - x_0}$$

$f(3) = +4$

tangent line to $f(x)$ at $x = 3$ is

$$y - 4 = 7(x - 3)$$

$$\boxed{\begin{array}{c}\text{tangent line to } f(x) \text{ at } x = a \text{ is} \\ y - f(a) = f'(a)(x - a)\end{array}}$$

Example 3.1.4. Find the tangent line to $f(x) = x + \frac{1}{x}$ at $a = 1$ and $a = \frac{1}{2}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{a+h + \frac{1}{a+h} - \left(a + \frac{1}{a}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{h + \frac{a}{a(a+h)} - \frac{a+h}{a(a+h)}}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{a-(a+h)}{a(a+h)}}{h} = \lim_{h\to 0} \frac{h + \frac{-h}{a(a+h)}}{h}$$

$$= \lim_{h\to 0} \frac{h\left(1 + \frac{-1}{a(a+h)}\right)}{h} = \lim_{h\to 0} 1 + \frac{-1}{a(a+h)}$$

$$= 1 + \frac{-1}{a^2} = 1 - \frac{1}{a^2}$$

$$f'(1) = 1 - \frac{1}{1^2} = 0$$

$$f(1) = 1 + \frac{1}{1} = 2$$

$$y - 2 = 0(x-1)$$

$$\boxed{y = 2}$$

tangent line of f(x) at x=1

$$f'\left(\tfrac{1}{2}\right) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2} = 1 - 4 = -3$$

$$f\left(\tfrac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = 2.5$$

$$\boxed{y - 2.5 = -3\left(x - \tfrac{1}{2}\right)}$$

tangent line of f(x) at $x = \frac{1}{2}$