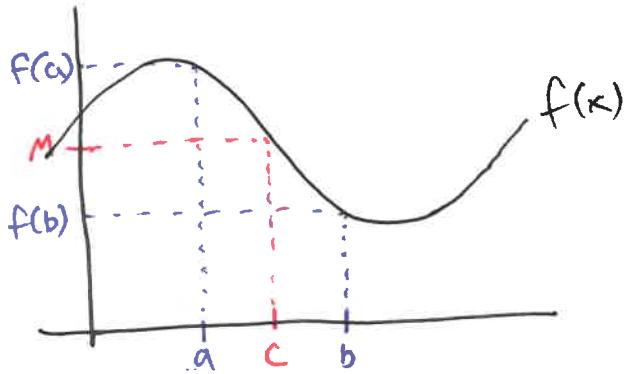
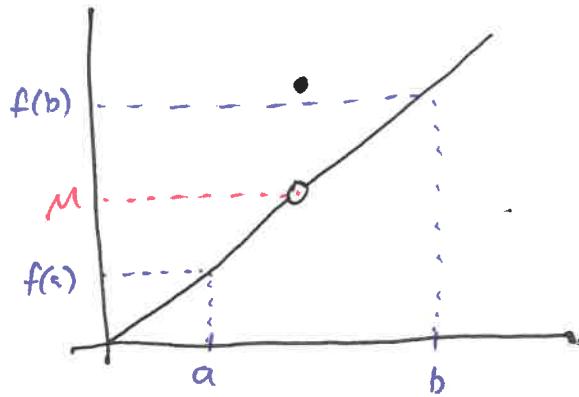
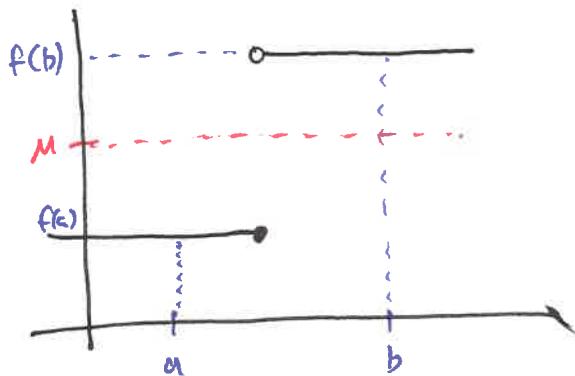


## 2.8 Intermediate Value Theorem

**Theorem 2.8.1** (Intermediate Value Theorem (IVT)). Suppose  $f(x)$  is continuous on the closed interval  $[a, b]$  and  $f(a) \neq f(b)$ . Then for every value  $M$  between  $f(a)$  and  $f(b)$ , there is at least one input number  $c$  in  $(a, b)$  so that  $f(c) = M$ .



continuous  
functions  
don't skip  
values



Example 2.8.1. Show that  $f(x) = \frac{x^2}{x^7 + 1}$  takes on the value 0.4.

$f(x)$  is continuous when  $x \neq -1$

$$f(0) = 0$$

$$f(1) = \frac{1}{2}$$

Since  $f$  is continuous on  $[0, 1]$ ,  $f(0) = 0 < 0.4 < \frac{1}{2} = f(1)$ , by the IVT there is a value  $c \in (0, 1)$  so that  
 $f(c) = 0.4$

Example 2.8.2. Find an interval of length  $\frac{1}{2}$  containing a root of  $f(x) = x^3 + 2x + 1$ .

$$\text{Root } \approx 0.5$$

$\hookrightarrow$  cont on  $(-\infty, \infty)$

$$f(-1) = -2$$

$$f(0) = 1$$

$$f(0.5) = -\frac{1}{8}$$

$\Rightarrow$  by IVT root of  $f$  in  $[-1, 0]$

$\Rightarrow$  by IVT root of  $f$  in  $[-\frac{1}{2}, 0]$

Since  $f$  is continuous on  $[-0.5, 0]$  and

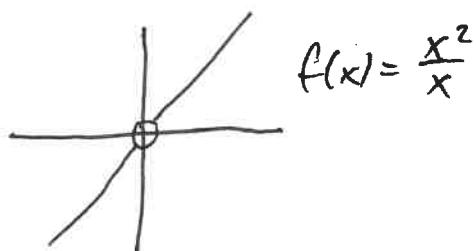
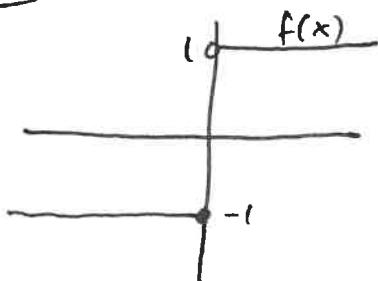
$f(0) = 1 > 0 > -\frac{1}{8} = f(-\frac{1}{2})$ , by the IVT there is a value  $c$  in  $(-0.5, 0)$  so that  $f(c) = 0$ .

( $\because$  there is a root of  $f$  in  $(-0.5, 0)$ )

**Example 2.8.3.** Suppose  $f(-1) = -1$  and  $f(1) = 1$ . Does the IVT imply that  $f(x)$  has a zero in the interval  $[-1, 1]$ ?

No, the function could have 0 as a missing value. Need  $f$  to be continuous to guarantee this.

Ex:



**Example 2.8.4.** Find an interval of length  $\frac{1}{16}$  containing the largest root of  $g(x) = x^3 - 8x - 1$ .

$$\begin{aligned}
 g(3) &= 2 && \text{root in } [2, 3] \\
 g(2) &= -9 && \text{root in } [2.75, 3] \\
 g(2.75) &< 0 && \text{root in } [2.8275, 3] \\
 g(2.8275) &< 0 && \text{root in } [2.8275, 2.9375] \\
 g(2.9375) &> 0 && \text{root in } [2.8275, 2.9375]
 \end{aligned}$$

↓  
cont on  $(-\infty, \infty)$

Example 2.8.5. Using the IVT, does  $\sqrt{x} + \sqrt{x+3} = 11$  have a solution?

$$f(x) = \sqrt{x} + \sqrt{x+3} \quad \text{cont when } x \geq 0$$

$$f(0) = \sqrt{3} < 11$$

$$f(64) = \sqrt{64} + \sqrt{67} = 8 + \sqrt{67} > 11$$

So, by IVT  $f(x) = 11$  for some  $x$