## 2.7 Limits at Infinity

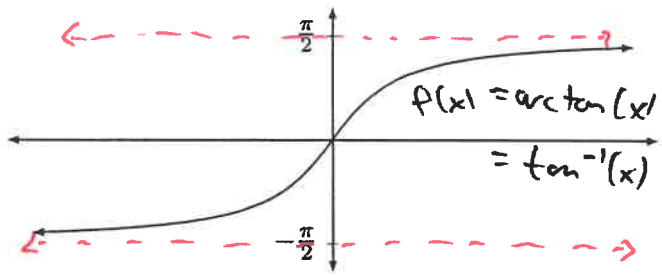
**Definition 2.7.1.**

$$\lim_{x\to\infty} f(x) = L$$

means $f(x)$ becomes arbitrarily close to $L$ when $x$ increases without bound.

$$\lim_{x\to-\infty} f(x) = L$$

means $f(x)$ becomes arbitrarily close to $L$ when $x$ decreases without bound.
In either case, $y = L$ is called a horizontal asymptote.



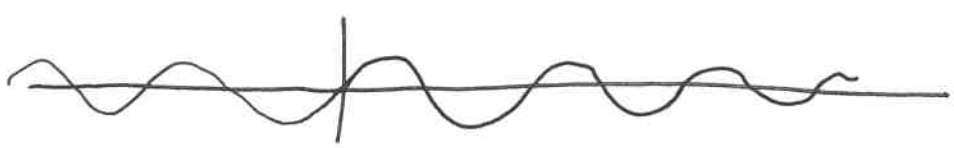

$\lim_{x\to\infty} f(x) = \frac{\pi}{2}$

$\lim_{x\to-\infty} f(x) = -\frac{\pi}{2}$

**Tools for Limits at Infinity:**

- $\lim_{x\to\infty} x^n = \infty$
- $\lim_{x\to\infty} \frac{1}{x^n} = 0$
- $\lim_{x\to-\infty} x^n = \begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$
- $\lim_{x\to-\infty} \frac{1}{x^n} = 0$

**Example 2.7.1.** Not every function has a limit at infinity. Consider $\sin x$.



$\lim_{x\to\infty} \sin(x)$

$= \lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$

**Example 2.7.2.** $\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{20}{x}}{4+\frac{9}{x^2}} = \frac{3}{4}$

**Example 2.7.3.** Find the horizontal asymptotes of $f(x) = \frac{\sqrt{36x^4+7}}{9x^2+4}$.

$$\lim_{x\to\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\sqrt{36+\frac{7}{x^4}}}{9+\frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}$$

$$\frac{1}{x^2}\sqrt{36x^4+7} = \sqrt{\frac{1}{x^4}(36x^4+7)} = \sqrt{36+\frac{7}{x^4}}$$

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\sqrt{36+\frac{7}{x^4}}}{9+\frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{36x^2+7}}{9x+4} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{36+\frac{7}{x^2}}}{9+\frac{4}{x}} = -\frac{6}{9} = -\frac{2}{3}$$

$$-\left(\frac{-1}{x}\right)\sqrt{36x^2+7} = -\sqrt{\left(\frac{1}{x^2}\right)(36x^2+7)} = -\sqrt{36+\frac{7}{x^2}}$$

**Example 2.7.4.** $\lim_{x\to\infty}\left(\ln(\sqrt{5x^2+2})-\ln(x)\right)$ "$= \infty - \infty$"

$$= \lim_{x\to\infty}\left(\frac{1}{2}\ln(5x^2+2) - \frac{1}{2}\ln x^2\right) = \lim_{x\to\infty} \ln\left(\frac{\sqrt{5x^2+2}}{x}\right)$$

$$= \lim_{x\to\infty} \frac{1}{2}\ln\left(\frac{5x^2+2}{x^2}\right)$$

$$= \frac{1}{2}\ln\left(\lim_{x\to\infty}\frac{5x^2+2}{x^2}\right)$$

$$= \frac{1}{2}\ln(5) = \ln(\sqrt{5})$$

$$\lim_{x\to\infty}\frac{5x^2+2}{x^2} = \lim_{x\to\infty}\frac{5x^2}{x^2} + \frac{2}{x^2} = \lim_{x\to\infty} 5 + \frac{2}{x^2} = 5$$

**Example 2.7.5.** $\lim_{x\to\infty}\left(\frac{x^2+7}{x^2+9}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \rightarrow \frac{\frac{x^2}{x^2}+\frac{7}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}}$

$$= \lim_{x\to\infty} \frac{1+\frac{7}{x^2}}{1+\frac{9}{x^2}} = \frac{1+0}{1+0} = 1$$

**Example 2.7.6.** $\lim_{x\to-\infty}\left(\frac{4x-6}{3x^3+7x+5}\right) \left(\frac{1/x^3}{1/x^3}\right)$

$$\lim_{x\to-\infty} \frac{\frac{4x}{x^3} - \frac{6}{x^3}}{\frac{3x^3}{x^3} + \frac{7x}{x^3} + \frac{5}{x^3}}$$

$$\lim_{x\to-\infty} \left(\frac{\cancel{\frac{4}{x^2}}^{\,0} - \cancel{\frac{6}{x^3}}^{\,0}}{3 + \cancel{\frac{7}{x^2}}^{\,0} + \cancel{\frac{5}{x^3}}^{\,0}}\right)$$

$$\boxed{\frac{0}{3}} = 0$$