

2.6 Squeeze Theorem

Tools for finding limits so far:

1. Continuity

2. Algebra

New tools:

$$3. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$4. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$\begin{aligned} 0 &= 0 \cdot 1 = \left(\lim_{x \rightarrow 0} \sin x \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x} = \left(\lim_{x \rightarrow 0} 1 + \cos x \right) \left(\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \right) \\ &= 2 \cdot \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \end{aligned}$$

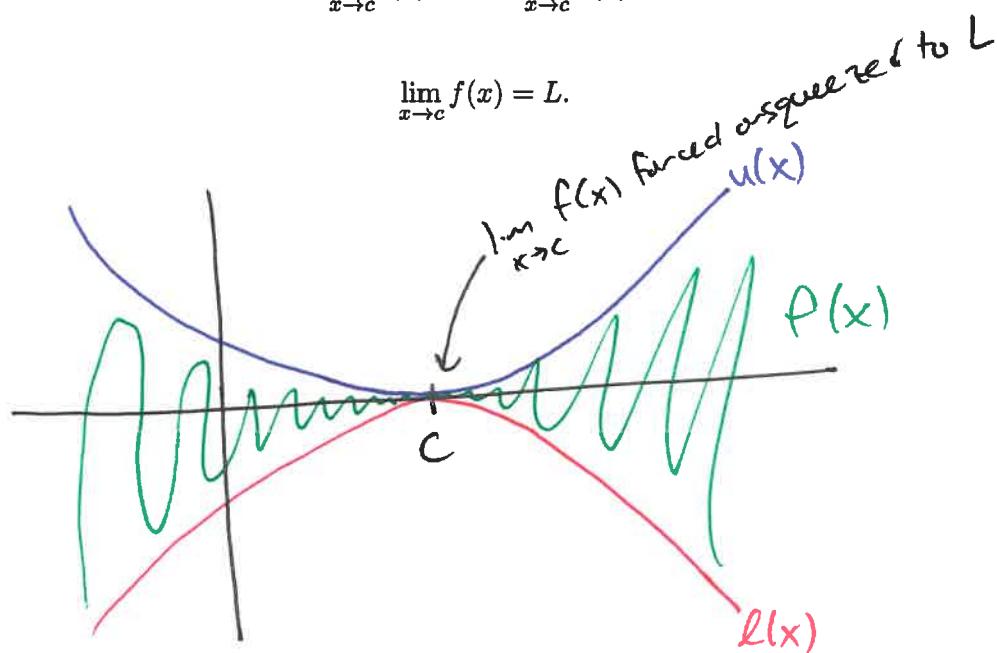
$$\text{Example 2.6.1. } \lim_{x \rightarrow 0} \frac{\sin(4x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x} \cdot 4 = 4 \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x} = 4 \cdot 1 = 4$$

5. Squeeze Theorem: If $l(x) \leq f(x) \leq u(x)$ and

$$\lim_{x \rightarrow c} l(x) = L = \lim_{x \rightarrow c} u(x),$$

then

$$\lim_{x \rightarrow c} f(x) = L.$$



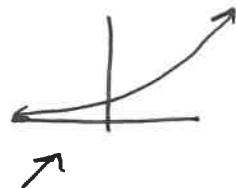
Example 2.6.2. $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$

$$\begin{aligned} -1 &\leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad (x \neq 0) \\ \Rightarrow -x^2 &\leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2 \quad (x \neq 0) \end{aligned}$$

Since $\lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2$ and $-x^2 \leq f(x) \leq x^2$, by

the squeeze theorem $\lim_{x \rightarrow 0} f(x) = 0$ also.

Example 2.6.3. $\lim_{x \rightarrow 0^+} \sqrt{x} e^{\cos(\pi/x)}$

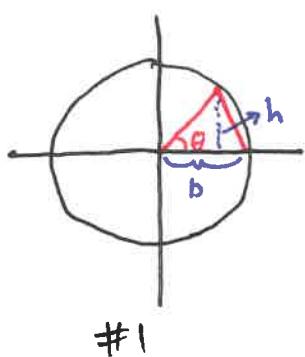


$$\begin{aligned} -1 &\leq \cos\left(\frac{\pi}{x}\right) \leq 1 \quad (x \neq 0) \\ \Rightarrow e^{-1} &\leq e^{\cos\left(\frac{\pi}{x}\right)} \leq e \quad (x \neq 0) \text{ since } e^x \text{ increasing function} \\ \Rightarrow \sqrt{x} e^{-1} &\leq \sqrt{x} e^{\cos\left(\frac{\pi}{x}\right)} \leq \sqrt{x} e \end{aligned}$$

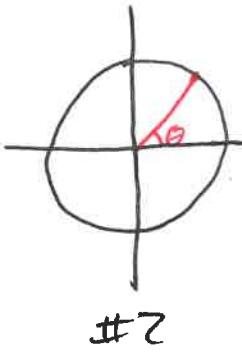
Since $\lim_{x \rightarrow 0^+} \sqrt{x} e^{-1} = 0 = \lim_{x \rightarrow 0^+} \sqrt{x} e$ and $\sqrt{x} e^{-1} \leq f(x) \leq \sqrt{x} e$,

by the squeeze theorem $\lim_{x \rightarrow 0^+} f(x) = 0$.

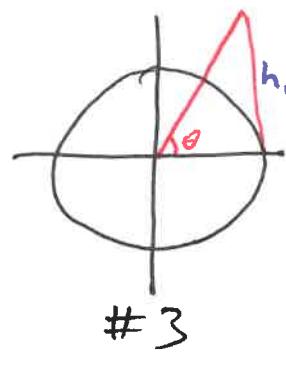
Example 2.6.4. Proof of $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.



$$\begin{aligned} b &= 1 \\ h &= \sin \theta \end{aligned}$$



#2



#3

$$h_1 = \tan \theta$$

[3 plots of circle w/
radius 1]

$$\begin{aligned} \text{Area}(\#1) &= \frac{1}{2}bh \\ &= \frac{1}{2}(1)(\sin \theta) \\ &= \frac{\sin \theta}{2} \end{aligned}$$

$$\begin{aligned} \text{Area}(\#2) &= \frac{\theta}{2} \\ &= \frac{1}{2}(1)(\theta) \\ &= \frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} \text{Area}(\#3) &= \frac{1}{2}bh_1 \\ &= \frac{1}{2}(1)(\tan \theta) \\ &= \frac{\tan \theta}{2} \end{aligned}$$

$$\text{Area}(\#1) \leq \text{Area}(\#2)$$

$$\text{Area}(\#2) \leq \text{Area}(\#3)$$

$$\Rightarrow \frac{\sin \theta}{2} \leq \frac{\theta}{2}$$

$$\Rightarrow \frac{\theta}{2} \leq \frac{\tan \theta}{2} = \frac{\sin \theta}{2 \cos \theta}$$

$$\Rightarrow \frac{\sin \theta}{\theta} \leq 1$$

$$\Rightarrow \cos \theta \leq \frac{\sin \theta}{\theta}$$

$$\lim_{\theta \rightarrow 0} 1 = 1 = \lim_{\theta \rightarrow 0} \cos \theta \quad \text{and} \quad \cos \theta \leq \frac{\sin \theta}{\theta} \leq 1,$$

by Squeeze theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\begin{aligned}
 \text{Example 2.6.5. } \lim_{t \rightarrow 0} \frac{\tan(4t)}{t \sec(t)} &= \lim_{t \rightarrow 0} \frac{\frac{\sin(4t)}{\cos(4t)}}{\frac{t}{\cos(t)}} \\
 &= \lim_{t \rightarrow 0} \frac{\sin(4t)}{\cos(4t)} \cdot \frac{\cos(t)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\sin(4t)}{t} \cdot \frac{\cos(t)}{\cos(4t)} \\
 &= \left(\lim_{t \rightarrow 0} \frac{\sin(4t)}{t} \right) \left(\lim_{t \rightarrow 0} \frac{\cos(t)}{\cos(4t)} \right) \\
 &= \left(4 \lim_{t \rightarrow 0} \frac{\sin(4t)}{4t} \right) \left(\frac{\cos(0)}{\cos(0)} \right) \\
 &= (4 \cdot 1)(1) \\
 &= 4
 \end{aligned}$$

$$\text{Example 2.6.6. } \lim_{h \rightarrow 0} \frac{1 - \cos(2h)}{h} \cdot \frac{2}{2}$$

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0$$

$$= \lim_{h \rightarrow 0} 2 \left(\frac{1 - \cos(2h)}{2h} \right)$$

$$= 2 \cdot 0$$

$$= 0$$

$$\begin{aligned} \text{Example 2.6.7. } & \lim_{z \rightarrow 0} \frac{\sin(\frac{z}{3})}{\sin(ez)} \cdot \frac{e}{e} \cdot \frac{\frac{1}{3}}{\frac{1}{3}} \cdot \frac{\frac{1}{z}}{\frac{1}{z}} \\ &= \lim_{z \rightarrow 0} \frac{\sin(\frac{z}{3})}{\frac{z}{3}} \cdot \frac{ez}{\sin(ez)} \cdot \frac{1}{\frac{1}{z}} \\ &= \frac{1}{3} \\ &= \frac{1}{3e} \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$\begin{aligned}
 \text{Example 2.6.8. } & \lim_{x \rightarrow 0} \frac{\sin(2x)}{\sin(3x)} \cdot \frac{\frac{2}{2} \cdot \frac{3}{3} \cdot \frac{x}{x}}{} \\
 &= \lim_{x \rightarrow 0} \left(\frac{3x}{\sin(3x)} \cdot \frac{\sin(2x)}{2x} \cdot \frac{2}{3} \right) \\
 &= \left(\lim_{x \rightarrow 0} \frac{3x}{\sin(3x)} \right) \left(\lim_{x \rightarrow 0} \frac{\sin(2x)}{2x} \right) \frac{2}{3} \\
 &= \frac{2}{3} \\
 &\quad \lim_{t \rightarrow 0} \frac{1 - \cos t}{\sin^2 t}
 \end{aligned}$$

$$\begin{aligned}
 \text{Example 2.6.9. } & \lim_{t \rightarrow 0} \frac{\sin^2(t)(1 - \cos t)}{4t^3} = \lim_{t \rightarrow 0} \frac{\sin^2 t}{4t^2} \cdot \frac{1 - \cos t}{t} \\
 &= \frac{1}{4} \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right) \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right) \left(\lim_{t \rightarrow 0} \frac{1 - \cos t}{t} \right) \\
 &= \frac{1}{4} \cdot 1 \cdot 1 \cdot 0 \\
 &= 0
 \end{aligned}$$