

## 2.5 Evaluating Limits Algebraically

**Definition 2.5.1.** The function  $f(x)$  has an indeterminate form at  $x = c$  means  $f(c)$  gives an expression of the type:

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty$$

Compute the following limits.

**Example 2.5.1.**  $\lim_{x \rightarrow 0} \frac{x^3 + 5x}{x^2 - x}$  "  $\frac{0}{0}$ "

$$= \lim_{x \rightarrow 0} \frac{x(x^2 + 5)}{x(x-1)} \quad \left( \frac{x}{x} = 1 \text{ when } x \neq 0 \right)$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + 5}{x-1}$$

$$= \frac{0+5}{0-1} = \frac{5}{-1} = -5$$

**Example 2.5.2.**  $\lim_{x \rightarrow 1} \frac{x^3 + 5x}{x^2 - x}$

$$= \lim_{x \rightarrow 1} \frac{x^2 + 5}{x-1}$$

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$$\lim_{x \rightarrow 1^-} \frac{x^2 + 5}{x-1} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{x^2 + 5}{x-1} = +\infty$$

**Example 2.5.3.**  $\lim_{x \rightarrow 0^+} \left( \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2+x}} \right) = \infty - \infty$

$$= \lim_{x \rightarrow 0^+} \left( \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}\sqrt{x+1}} \right)$$

$$= \lim_{x \rightarrow 0^+} \left( \frac{1}{\sqrt{x}} \cdot \frac{\sqrt{x+1}}{\sqrt{x+1}} - \frac{1}{\sqrt{x}\sqrt{x+1}} \right)$$

$$= \lim_{x \rightarrow 0^+} \left( \frac{\sqrt{x+1} - 1}{\sqrt{x}\sqrt{x+1}} \right) = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0^+} \left( \frac{\sqrt{x+1} - 1}{\sqrt{x}\sqrt{x+1}} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{\sqrt{x}\sqrt{x+1}(\sqrt{x+1} + 1)}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x+1}(\sqrt{x+1} + 1)}$$

$$= \frac{0}{\pi(\pi+1)} = \boxed{0}$$

Example 2.5.4.  $\lim_{x \rightarrow 8} \frac{\sqrt{x-4}-2}{x-8}$  " =  $\frac{0}{0}$ "

$$\begin{aligned} &= \lim_{x \rightarrow 8} \frac{x-4-4}{(x-8)(\sqrt{x-4}+2)} = \lim_{x \rightarrow 8} \frac{x-8}{(x-8)(\sqrt{x-4}+2)} \\ &= \lim_{x \rightarrow 8} \frac{1}{\sqrt{x-4}+2} \\ &= \frac{1}{\sqrt{4}+2} \\ &= \frac{1}{4} \end{aligned}$$

Example 2.5.5.  $\lim_{h \rightarrow 0} \frac{2(1+h)^2 - 2}{h}$  " =  $\frac{0}{0}$ "

$$= \lim_{h \rightarrow 0} \frac{2(1+2h+h^2) - 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2+4h+2h^2 - 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4h+2h^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(4+2h)}{h}$$

$$= \lim_{h \rightarrow 0} 4+2h = 4$$

**Example 2.5.6.**  $\lim_{x \rightarrow 5} \frac{\frac{1}{5} - \frac{1}{x}}{x - 5}$  " =  $\frac{0}{0}$ "

$$= \lim_{x \rightarrow 5} \frac{\frac{x}{5x} - \frac{5}{5x}}{x - 5}$$

$$= \lim_{x \rightarrow 5} \frac{\frac{x-5}{5x}}{x-5}$$

$$= \lim_{x \rightarrow 5} \frac{x-5}{5x} \cdot \frac{1}{x-5}$$

$$= \lim_{x \rightarrow 5} \frac{1}{5x}$$

$$= \frac{1}{25}$$