2.4 Continuity

Idea: Continuous functions allow us to easily compute limits.

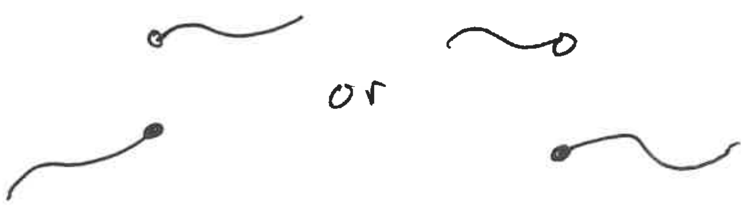
Definition 2.4.1. We say that $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Three Things Occur:

- $\lim_{x\to c} f(x)$ exists
- $f(c)$ exist/defined
- $\lim_{x\to c} f(x) = f(c)$

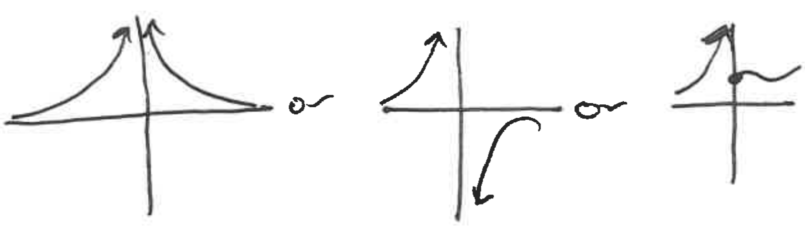
Types of Discontinuities:

1. Jump



$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

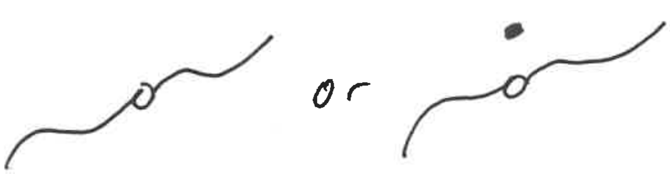
2. Infinite



$\lim_{x\to a^-} f(x) = \pm\infty$

or $\lim_{x\to a^+} f(x) = \pm\infty$

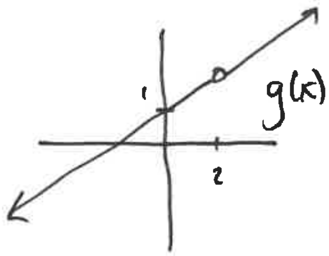
3. Removable



$\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$

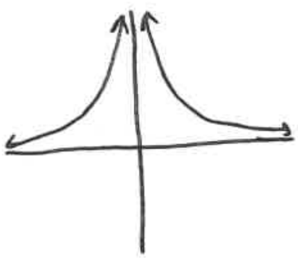
Classify the discontinuities of the following functions:

Example 2.4.1. $g(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x+1)(x-2)}{x-2} \overset{x \neq 2}{=} x + 1$



removable discontinuity at $x = 2$.

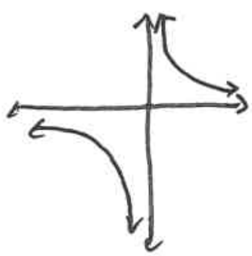
Example 2.4.2. $h(x) = \dfrac{1}{x^2}$



infinite discontinuity at $x = 0$

$\lim_{x \to 0} \frac{1}{x^2} = \infty$

Example 2.4.3. $f(x) = \dfrac{1}{x}$



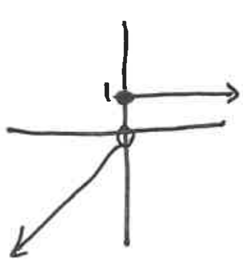
$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = \infty$

infinite discontinuity at $x = 0$

$\lim_{x \to 0} \frac{1}{x}$ DNE

Example 2.4.4. $a(x) = \begin{cases} x & x < 0 \\ 1 & x \geq 0 \end{cases}$



$\lim_{x \to 0^-} a(x) = \lim_{x \to 0^-} x = 0$

$\lim_{x \to 0^+} a(x) = \lim_{x \to 0^+} 1 = 1$

$\lim_{x \to 0} a(x)$ DNE

jump discontinuity at $x = 0$

Definition 2.4.2 (One-Sided Continuity).
We say that $f(x)$ is right-continuous if

$$\lim_{x \to a^+} f(x) = f(a)$$

We say that $f(x)$ is left-continuous if

$$\lim_{x \to a^-} f(x) = f(a)$$

Examples of Continuous Functions:

- Polynomials

  $x^2 + 3x + 1$

- Rational functions on their domains

  $\frac{f(x)}{g(x)}$, $\frac{x^2+3x+1}{x^2-2x-3}$ (domain: $g(x) \neq 0$)

- Trigonometric functions on their domains

  $\cos x, \sin x \to (-\infty, \infty)$, $\tan x = \frac{\sin x}{\cos x}$ (where $\cos x \neq 0$, $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$) $= \frac{\pi}{2} + \pi k$ k integer

- Root functions on their domains

  $\sqrt{x}$, $x^{1/3}$ domain $x^{1/n}$ if n even $(x \geq 0)$ if n odd $(x \in (-\infty, \infty))$

- Exponential functions

  b^x where $b > 0$

- Logarithmic functions on their domains

  $\ln(x)$, $x > 0$

Laws of Continuity: If $f(x)$ and $g(x)$ are continuous at $x = c$, then so are:

- $f \pm g$
- $f(x) \cdot g(x)$
- $\frac{f(x)}{g(x)}$, $g(x) \neq 0$
- $c f(x)$, c constant

Example 2.4.5. Where is $\sin x + x$ continuous?

$\sin x$: cont on $(-\infty, \infty)$ x: cont. on $(-\infty, \infty)$

Continuous on $(-\infty, \infty)$

Example 2.4.6. Where is $g(x) = \ln(9 - x^2)$ continuous?

$\ln(\cdot)$ cont when $\cdot > 0 \Longrightarrow$ need $9 - x^2 > 0$

$\Rightarrow 9 > x^2$

$\Rightarrow -3 < x < 3$

Compute the following limits:

cont. at $x = \pi$

Example 2.4.7. $\lim\limits_{x \to \pi} \dfrac{1}{\cos x} = \dfrac{1}{\cos \pi} = \dfrac{1}{-1} = -1$

$\dfrac{1}{\cos x}$ continuous when $\cos x \neq 0$

$\cos x = 0$ when $x = \pi/2 \ldots$

Example 2.4.8. $\lim\limits_{x \to 2} \left(\dfrac{7x + 2}{4 - x}\right)^{\frac{2}{3}} = \left(\dfrac{7 \cdot 2 + 2}{4 - 2}\right)^{2/3} = \left(\dfrac{16}{2}\right)^{2/3} = 8^{2/3} = 4$

not cont when $x = 4$

$\tan^{-1}(x) = \arctan(x)$

Example 2.4.9. $\lim\limits_{x \to 0} \tan^{-1}(e^x) = \tan^{-1}(e^0) = \tan^{-1}(1) = \dfrac{\pi}{4}$

cont on $(-\infty, \infty)$

cont on $(-\infty, \infty)$

cont on $(-\infty, \infty)$

Example 2.4.10. Determine where $f(x) = \underbrace{\sqrt{5x+3}}\,\underbrace{\sin(3x)}$ is continuous.

$\sqrt{5x+3}$ cont when $5x+3 \geq 0$

$5x \geq -3$

$x \geq -3/5$

$\sin(3x)$ cont. on $(-\infty, \infty)$

$f(x)$ cont. on $x \geq -3/5$ or $[-3/5, \infty)$

Example 2.4.11. Determine where $g(t) = (t+6)^{-\frac{1}{5}}$ is continuous.

$= \dfrac{1}{\sqrt[5]{t+6}}$

Cont on $(-\infty, -6) \cup (-6, \infty)$

OR $t \neq -6$

need $\sqrt[5]{t+6} \neq 0$

$\sqrt[5]{t+6} = 0$

$t+6 = 0$

$t = -6$

Example 2.4.12. Determine where $h(x) = \ln\left(\dfrac{3-\sqrt{x}}{9-x}\right)$ is continuous. $\sqrt{x} \to x \geq 0$

$$\frac{3-\sqrt{x}}{9-x} > 0$$

$$9 - x = 0 \Rightarrow x = 9$$

$$3 - \sqrt{x} > 0 \qquad 3 - \sqrt{x} < 0$$

$$3 > \sqrt{x} \qquad 3 < \sqrt{x}$$

$$\boxed{9 > x \geq 0} \qquad \boxed{9 < x}$$

$$\frac{3-\sqrt{x}}{9-x} \overset{x\neq 9}{=} \frac{3-\sqrt{x}}{(3+\sqrt{x})(3-\sqrt{x})} = \frac{1}{3+\sqrt{x}}$$

$x \geq 0$

$0 \leq x < 9$ or $9 < x$

$[0,9) \cup (9,\infty)$