## 2.1 Limits, Rates of Change, and Tangent Lines

**Average Rate of Change:** The AROC of $f(x)$ on $[a, b]$ is

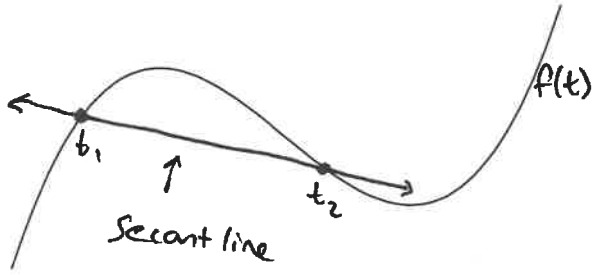
$$\frac{f(b) - f(a)}{b - a}$$

**Example 2.1.1.** An iPhone is thrown in the air from ground level with an initial velocity of 60 $\frac{\text{m}}{\text{s}}$. Its height at time $t$ is $h(t) = 60t - 4.9t^2\text{m}$. Compute the ~~stone's~~ iPhone's average velocity over the time interval $[1, 3]$.

$$\frac{h(3)-h(1)}{3-1} = \frac{60\cdot 3 - 4.9(3^2) - (60\cdot 1 - 4.9\cdot 1^2)}{2} = \frac{80.8}{2} = 40.4 \text{ m/s}$$

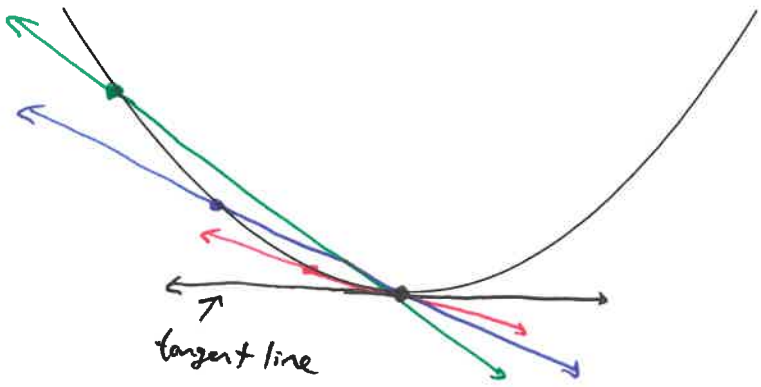
**Note 2.1.1.** If $p(t)$ is a position function, then if $s < t$

$$\frac{p(t) - p(s)}{t - s} = \text{average velocity}$$




Slope of secant line:

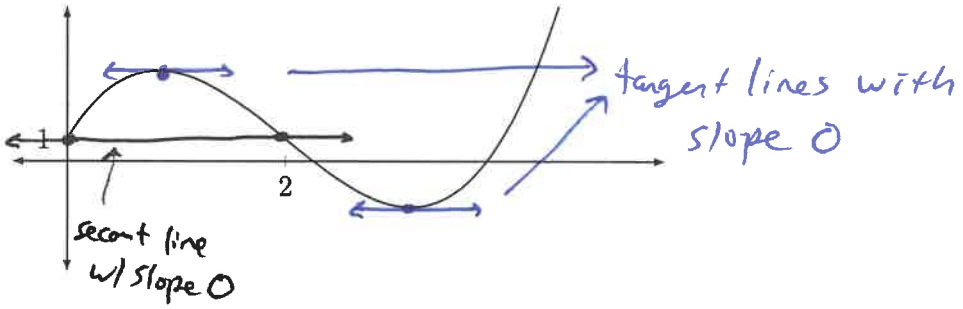
$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} = \text{AROC of } f(t) \text{ over } [t_1, t_2]$$




AROC ↔ slopes of secant lines approach slope of tangent line ↔ IROC

**Note 2.1.2.** Slope of tangent line (at $x_1$) is what the AROC over $[x_0, x_1]$ and $[x_1, x_0]$ "go towards" as $x_0$ "goes towards" $x_1$. We call the slope of the tangent line the instantaneous rate of change (IROC).

**Example 2.1.2.** Graph of $f(x) = x^3 - 6x^2 + 8x + 1$




(a) Using the graph, what is the AROC of $f(x)$ over $[0, 2]$?

Slope of secant line is 0, so AROC is 0. Check: $\frac{f(2)-f(0)}{2-0} = \frac{1-1}{2} = 0$

(b) Where is the IROC equal to 0?

look at graph

(c) Estimate the IROC of $f(x)$ at $x = 2$.

| Interval | AROC |
|---|---|
| $[1, 2]$ | $-3$ |
| $[1.9, 2]$ | $-3.99$ |
| $[1.99, 2]$ | $-3.9999$ |
| $[1.999, 2]$ | $-3.999999$ |

approach $-4$

| Interval | AROC |
|---|---|
| $[2, 3]$ | $-3$ |
| $[2, 2.1]$ | $-3.99$ |
| $[2, 2.01]$ | $-3.9999$ |
| $[2, 2.001]$ | $-3.999999$ |

approach $-4$

IROC of $f(x)$ at $x=2$ is about $-4$

**Example 2.1.3.** Let $f(x) = x^3 - 2x$.

(a) What is the AROC of $f(x)$ on $[0, 1]$?

$\frac{f(1)-f(0)}{1-0} = \frac{-1-0}{1} = -1$

(b) What is the AROC of $f(x)$ on $[1, 1.5]$?

$\frac{f(1.5)-f(1)}{1.5-1} = \frac{.375+1}{.5} = 2.75$

(c) Estimate the IROC of $f(x)$ at $x = 1$.

| Interval | AROC |
|---|---|
| $[0, 1]$ | $-1$ |
| $[.9, 1]$ | $0.71$ |
| $[.99, 1]$ | $0.9701$ |
| $[.999, 1]$ | $0.997001$ |

approach 1

| Interval | AROC |
|---|---|
| $[1, 2]$ | $5$ |
| $[1, 1.1]$ | $1.31$ |
| $[1, 1.01]$ | $1.0301$ |
| $[1, 1.001]$ | $1.003001$ |

approach 1

IROC of $f(x)$ at $x=1$ is about 1