LETTERS

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A note on subharmonic instabilities

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When a fluid system is subject to time-periodic forcing, it is well known that it may exhibit both harmonic and subharmonic instabilities, the classic example being Faraday oscillations. When the forcing is confined to a periodic shearing motion, however, it has been observed that the subharmonic response is absent. The underlying mathematical feature that unifies these systems is a conjugate-translation symmetry [A. C. Or, J. Fluid Mech. **335**, 213 (1997)]. We show that any subharmonic solutions of periodically driven systems with conjugate-translation symmetry must have Floquet multipliers with multiplicity greater than one. The effect of this constraint is that subharmonic solutions are very difficult to locate within the system's parameter space and, more importantly, that phase locking cannot occur for such systems. © *1999 American Institute of Physics.* [S1070-6631(99)00412-2]

The study of time-periodic forces on fluid systems comprises a vast body of research. Two canonical examples include the Stokes layer and the Faraday instability, which represent two classes of problems where the fluid is forced by either a tangential or normal motion of the boundary, respectively. Davis[1] gives a broad review covering time-periodic flows and Miles[2] gives a review specific to the Faraday instability.

A linear stability analysis of these systems leads directly to differential equations with time-periodic coefficients. Such equations, referred to as Floquet systems, give rise to the well-known phenomenon of parametric resonance, where the frequency of a solution's oscillations can become locked at either the driving frequency (harmonic response) or half that frequency (subharmonic response) over distinct *regions* of parameter space. This parametric resonance is in contrast to normal resonance which occurs at discrete parameter values. The most widely studied equation of this type is Mathieu's equation:

$$\ddot{u}+(\delta+\epsilon\cos t)u=0, \tag{1}$$

which appears in the analysis of numerous physical examples, including the *Faraday instability* (Benjamin and Ursell[3]) where a container of liquid is oscillated in the vertical direction.

A number of authors have noted a lack of a subharmonic response in certain periodically driven fluid systems (Kelly and Hu;[4] Or and Kelly;[5] Schulze and Davis;[6] Or[7]), the unifying feature of these systems being a harmonically oscillating shear flow. Or[7] further clarifies the issue by noting that subharmonics are suppressed when, upon linearization about the basic flow and decomposition into time-periodic normal modes, the resulting PDE's have a conjugate-translation (CT) symmetry: if $\hat{\mathbf{u}}(z,t)$ is a solution of the system, so is $\hat{\mathbf{u}}^*(z,t+a/2)$, where a is the period of oscillation. Note that the advective term in the Navier–Stokes equations (or in an advection diffusion equation) is of precisely the right form for this type of symmetry to arise in that it is nonlinear (allowing the basic state to provide the periodicity) and first-order (so that it will have a pure imaginary Fourier transform).

Thus, it is most natural to consider this particular (CT) symmetry in the context of a fluid or other continuous system governed by such PDE's. This observation is, however, closely related to a result due to Swift and Weisenfeld[8] that demonstrates the suppression of period doubling in more general dynamical systems with real coefficients and symmetry under the operations of inversion and translation: if $\mathbf{u}(t)$ is a solution of the system, so is $-\mathbf{u}(t+a/2)$. More recently, Nicolaisen and Werner[9] use group theoretic arguments to identify additional symmetries leading to the suppression of period doubling.

This paper explains the lack of subharmonic resonance by showing that any subharmonic solutions of periodically driven systems with CT symmetry must have Floquet multipliers with multiplicity greater than one. To do this, we first assume that the PDE's that determine the eigenfunctions $\hat{\mathbf{u}}(z,t)$ are reduced to ODE's via an expansion method that removes the remaining spatial dependence. This expansion may be exact or a numerical approximation. When this is done, one will be left with a system of ODE's with CT symmetry:

$$\dot{\mathbf{u}}=[\mathbf{R}+i\mathbf{P}(t)]\mathbf{u}, \tag{2}$$

where $\mathbf{u}$ is now a large (in principle infinite) vector of unknowns, $\mathbf{R}$ is a real matrix and $\mathbf{P}(t)$ is real, time-periodic matrix with zero mean, least period a and satisfies $\mathbf{P}(t)=-\mathbf{P}(t+a/2)$. Again, a simple example is provided by the Mathieu Eq. (1) if δ is taken to be real, ϵ pure imaginary and one converts the equation to a system of two first-order equations.

The main result in Floquet theory is that ODE's with periodic coefficients have at least one solution of the form

$$\mathbf{u}(t)=e^{\sigma t}\mathbf{p}(t),$$

where $\mathbf{p}(t)$ is a periodic vector function with (least) period equal to that of the forcing and σ is a complex constant known as the *Floquet exponent*. Those solutions which are of this form are called *Floquet solutions* and possess the property that

$$\mathbf{u}(t+a)=\lambda\mathbf{u}(t), \tag{3}$$

where λ is referred to as the *Floquet multiplier*. The Floquet multipliers and exponents are related by $\lambda=e^{\sigma a}$. In the cases where the Floquet multipliers are real, the response will be either synchronous with the forcing (no negative multipliers) or subharmonic (at least one negative multiplier). These features of Floquet systems are well known and are frequently used to restrict one's search for neutral stability curves to the cases $\sigma_i=0$ (synchronous instability) or $\sigma_i=\pi/a$ (subharmonic instability). The CT symmetry described above implies the usual symmetry of Floquet systems: if $\mathbf{u}(t)$ is a solution, so is $\mathbf{u}(t+a)$, and therefore places a further restriction on the solution's behavior.

Or[7] suggests that CT symmetry is incompatible with a subharmonic response, and sketches a proof that analyzes the structure of a solution's Fourier coefficients. The proof relies on the assumption that solutions to (2) have the property

$$\mathbf{u}^*(t+a/2)=k\mathbf{u}(t), \tag{4}$$

where k is said to be a complex constant. This assumption is apparently made in analogy to (3), but is too restrictive. Indeed, if this were true, one could immediately show that the Floquet multipliers λ would be real and non-negative since iterating (4) once gives $u(t+a)=kk^*u(t)$. There is no reason for this relationship to hold for arbitrary solutions, however, and solutions with complex multipliers, for example, are easy to find [see Fig. 1(b)]. [Note that even (3) applies only to the Floquet solutions—solutions formed from linear combinations of these solutions do not possess this property.]

What one can show in a simple and precise way is that any purely negative Floquet multiplier of a system with CT symmetry must have (geometric) multiplicity greater than one. The implication for subharmonic instability is that if a particular system should happen to have an isolated subharmonic solution, it will not persist when the system parameters are perturbed in the way that solutions to Mathieu's equation do. In other words, there is no phase locking for $\sigma_i=\pi/a$ [compare Figs. 1(a) and 1(b)].

To see that negative Floquet multipliers of CT-invariant equations must have multiplicity greater than one, we begin by observing that Eq. (2) has a fundamental matrix of linearly independent solutions $\mathbf{\Phi}(t)$ satisfying

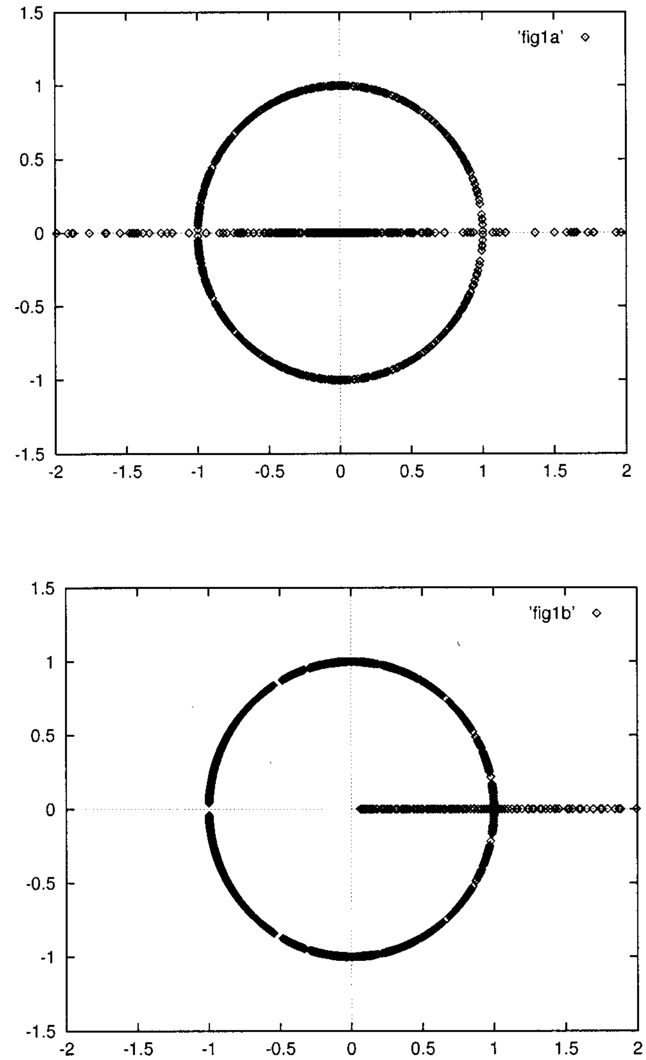

FIG. 1. (a) The locations within the complex λ-plane of the two Floquet multipliers for Mathieu's Eq. (1) using a range of real values for the coefficients δ and ϵ. (b) A similar plot for the CT-invariant version of Mathieu's equation, where δ is real and ϵ is pure imaginary. In case (a), two negative multipliers can assume separate values, allowing an infinite range of λ values as δ and ϵ are varied; in case (b) any negative multipliers must be degenerate, restricting the range of λ values to a single point.

$$\mathbf{\Phi}(0)=\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

The CT symmetry implies that if $\mathbf{\Phi}(t)$ is a fundamental matrix then so are $\mathbf{\Phi}(t+a)$ and $\mathbf{\Phi}^*(t+a/2)$, so that there exist complex matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\mathbf{\Phi}(t+a)=\mathbf{\Phi}(t)\mathbf{A}, \tag{5}$$

$$\mathbf{\Phi}^*(t+a/2)=\mathbf{\Phi}(t)\mathbf{B}, \tag{6}$$

where $\mathbf{A}=\mathbf{\Phi}(a)$ and $\mathbf{B}=\mathbf{\Phi}(a/2)$.

An arbitrary solution $\mathbf{u}(t)$ may be expressed as a linear combination of the solutions in $\mathbf{\Phi}(t)$:

$$\mathbf{u}(t)=\mathbf{\Phi}(t)\mathbf{x}.$$

Substituting this expression into (3) we see that there will be a set of linearly independent Floquet solutions with Floquet multipliers λ that satisfy the eigenvalue relationship

$$\mathbf{A}\mathbf{x}=\lambda\mathbf{x}. \tag{7}$$

Combining Eqs. (5) and (6) we see that

$$\mathbf{A}=\mathbf{B}\mathbf{B}^*. \tag{8}$$

The form of (8) immediately implies that the determinant of $\mathbf{A}$ must be real and positive since it is the product of complex conjugates and that any complex eigenvalues of $\mathbf{A}$ come in complex conjugate pairs, despite the fact that $\mathbf{A}$ is, generally, complex itself. It is also clear that there are an even number of negative eigenvalues, but it is somewhat harder to see that the negative eigenvalues must have multiplicity greater than one.

One way to show the degeneracy of negative eigenvalues for the matrix $\mathbf{A}$ is to use the following identity:

$$\mathbf{B}^*\mathbf{A}=\mathbf{B}^*\mathbf{B}\mathbf{B}^*=\mathbf{A}^*\mathbf{B}^*. \tag{9}$$

Multiplying (7) by $\mathbf{B}^*$ and using (9) we have

$$[\mathbf{A}^*-\lambda\mathbf{I}]\mathbf{B}^*\mathbf{x}=0,$$

which implies that $\mathbf{B}^*\mathbf{x}$ is an eigenvector of $\mathbf{A}^*$ with eigenvalue λ. If λ is a real eigenvalue we also have

$$\mathbf{A}^*\mathbf{x}^*=\lambda\mathbf{x}^*. \tag{10}$$

Thus, if λ has multiplicity one and λ is real, $\mathbf{B}^*\mathbf{x}$ and $\mathbf{x}^*$ must be related by

$$\mathbf{B}^*\mathbf{x}=\rho\mathbf{x}^*.$$

Finally, combining this relationship with its complex-conjugate we have

$$\mathbf{A}\mathbf{x}=\mathbf{B}\mathbf{B}^*\mathbf{x}=\rho\rho^*\mathbf{x}=\lambda\mathbf{x},$$

so that $\lambda\geqslant 0$.

Figures 1(a) and 1(b) illustrate this result using the Mathieu equation with real coefficients and the modified version with pure imaginary ϵ. These especially simple second-order equations give rise to two Floquet multipliers that must satisfy the additional constraint $\lambda_1\lambda_2=1$ (since the Wronskian evaluates to unity). Thus the eigenvalues distribute themselves over the unit circle in the complex plane and along the real axis. In the case of imaginary ϵ the negative multipliers are missing.

The argument above leaves open the possibility that there might sometimes be degenerate negative multipliers. This could happen, for example, if we had

$$\mathbf{B}=\begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{2}{5} & 0 \end{bmatrix},$$

a matrix that corresponds to the Mathieu equation when $\epsilon=0$, $\delta=\frac{25}{4}$ and the period is 2π. This is a trivial instance of CT symmetry having no periodic coefficient, but can be seen to give rise to a repeated multiplier $\lambda=-1$. A numerical solution for nonzero values of ϵ continues to indicate pairs of eigenvalues equal to -1 (see Fig. 2).

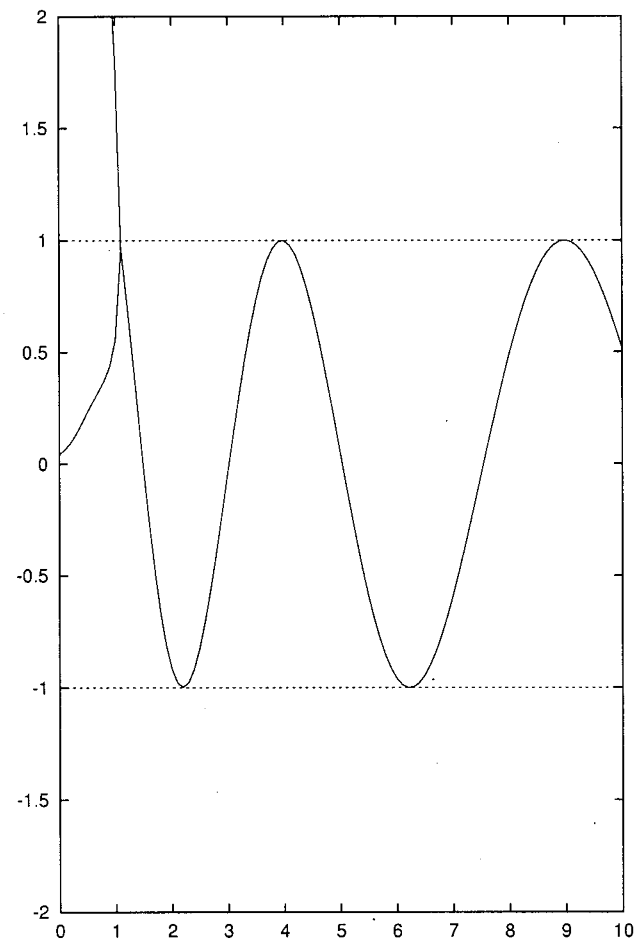

FIG. 2. A plot of the real part of λ for one of the two Floquet multipliers of the CT-invariant version of Mathieu's equation (ϵ pure imaginary) when $\epsilon=i$ and δ assumes a range of values (solid curve). Notice that the real part of the eigenvalue appears to be -1 for two values of δ. For this equation the eigenvalues must be complex conjugates with product unity, indicating that λ has no imaginary part and multiplicity two at these points.

In the end, the possibility of degenerate negative multipliers is only interesting in that the proof given above cannot rule it out; it has no practical implication for phase locking. Barring further symmetries, any such solutions will be isolated in parameter space as they are in Fig. 1(b). In contrast, pairs of positive multipliers are free to assume separate values, and synchronous solutions are relatively easy to find. More importantly, these synchronous solutions persist for small perturbations to the parameters, so that phase-locking can occur.

In summary, this paper explains that equations with CT symmetry are incompatible with subharmonic phase locking. This symmetry arises naturally from a consideration of harmonically oscillating shear flows. This result follows from the fact that any negative Floquet multipliers for equations with this symmetry must be degenerate and the reasonable assumption that such a degeneracy will not be maintained as the system parameters are varied.

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