Bergman-type reproducing kernels and invariant subspaces

(and complete Nevanlinna-Pick kernels also)

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Disclaimer: To keep matters simple I have modified many theorems and definitions a little. While I have tried to tell the truth, it should be pointed out that often in the original references a larger generality is achieved.

Spaces of analytic functions on the open unit disc $\mathbb{D}$:

$$f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$$

H^2, $\quad ||f||^2_{H^2} = \sum_{n\geq 0} |\widehat{f}(n)|^2$

$L^2_a = \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\}$,

$$||f||^2_{L^2_a} = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} = \sum_{n\geq 0} \frac{|\widehat{f}(n)|^2}{n+1}$$

$D = \{f \in Hol(\mathbb{D}) : f' \in L^2_a\}$,

$$||f||^2_D = ||f||^2_{H^2} + ||f'||^2_{L^2_a} = \sum_{n\geq 0} (n+1)|\widehat{f}(n)|^2$$

$$D \quad \subseteq \quad H^2 \quad \subseteq \quad L^2_a$$

$\zeta(z) = z$

$(M_\zeta, \mathcal{H}) \qquad M_\zeta f = \zeta f,\ f \in \mathcal{H}$.

$\mathcal{M} \in$ Lat$(M_\zeta, \mathcal{H})$, if $M_\zeta f \in \mathcal{M}$ for all $f \in \mathcal{M}$

If $S \subseteq \mathcal{H}$, then
$[S] =$ smallest $\mathcal{M} \in$ Lat$(M_\zeta, \mathcal{H})$ containing S.

ind $\mathcal{M} = \dim \mathcal{M} \ominus \zeta\mathcal{M} = \dim \mathcal{M} \cap (\zeta\mathcal{M})^\perp$

Note: ind $[S] \leq$ card S

G is the *extremal function* for $\mathcal{M}$, if G is the solution to

$$\sup\{Re\ g^{(k)}(0) : g \in \mathcal{M}, ||g|| = 1\},$$

where k is the smallest integer such that the sup is > 0.

If $\mathcal{M} \neq (0)$, then an extremal function G exists, is unique, and $G \in \mathcal{M} \ominus \zeta\mathcal{M}$

Thm 1 (H^2 - Beurling's theorem). *If $\mathcal{M} \in$ Lat(M_ζ, H^2),$\mathcal{M} \neq (0)$, then*

$$ind\mathcal{M} = 1.$$

If G is extremal for $\mathcal{M}$, then G is inner and $\mathcal{M} = [G] = GH^2$, so

$$\frac{\mathcal{M}}{G} = H^2 \quad isometrically,$$

i.e. $||\frac{f}{G}|| = ||f||$ for all $f \in \mathcal{M}$.

Thm 2 (D - Ri, Sundberg, Aleman). *If $\mathcal{M} \in$ Lat(M_ζ, D),$\mathcal{M} \neq (0)$, then*

$$ind\mathcal{M} = 1.$$

If G is extremal for $\mathcal{M}$, then $\mathcal{M} = [G]$.

If $\mathcal{M}$,$\mathcal{N} \in$ Lat(M_ζ, D),

$$(0) \neq \mathcal{N} \subseteq \mathcal{M} \subseteq D,$$

if $G_\mathcal{N}$,$G_\mathcal{M}$ are the extremal functions for $\mathcal{N}$ and $\mathcal{M}$, then one has the contractive inclusions

$$D \ \subseteq \ \frac{\mathcal{M}}{G_\mathcal{M}} \ \subseteq \ \frac{\mathcal{N}}{G_\mathcal{N}} \ \subseteq \ H^2,$$

i.e. $||G_\mathcal{M} f|| \leq ||f||$ *for all* $f \in D$,

$||\frac{G_\mathcal{N}}{G_\mathcal{M}} f|| \leq ||f||$ *for all* $f \in \mathcal{M}$, *and*

$||\frac{f}{G_\mathcal{N}}||_{H^2} \leq ||f||$ *for all* $f \in \mathcal{N}$, Fact (ABFP): In L_a^2 there are invariant subspaces of arbitrary index.

Thm 3. *Hedenmalm, ARS, H-Jacobsson-Shimorin*
If $\mathcal{M} \in$ *Lat*(M_ζ, D), $\mathcal{M} \neq (0)$, *then*

$$\mathcal{M} = [\mathcal{M} \ominus \zeta\mathcal{M}].$$

If $\mathcal{M}, \mathcal{N} \in$ *Lat*(M_ζ, L_a^2), *ind* $\mathcal{M} =$ *ind* $\mathcal{N} = 1$,

$$(0) \neq \mathcal{N} \subseteq \mathcal{M} \subseteq L_a^2,$$

if $G_\mathcal{N}, G_\mathcal{M}$ *are the extremal functions for* $\mathcal{N}$ *and* $\mathcal{M}$, *then one has the contractive inclusions*

$$H^2 \ \subseteq \ \frac{\mathcal{N}}{G_\mathcal{N}} \ \subseteq \ \frac{\mathcal{M}}{G_\mathcal{M}} \ \subseteq \ L_a^2,$$

i.e. $||G_\mathcal{N} f|| \leq ||f||_{H^2}$ *for all* $f \in H^2$,

$||\frac{f}{G_\mathcal{M}}|| \leq ||f||$ *for all* $f \in \mathcal{M}$, *and*

$||\frac{G_\mathcal{M}}{G_\mathcal{N}} f|| \leq ||f||$ *for all* $f \in \mathcal{N}$.

Cor 4. *If* $f \in L^2_a$, $f \neq 0$, *then* f *has an* L^2_a*-inner-outer factorization, i.e.*

$$f = FG,$$

where G *is extremal for* $[f]$ *and* $F \in L^2_a$ *is cyclic,* $[F] = L^2_a$.

$\mathcal{H} = \mathcal{H}(k)$ has reproducing kernel $k_\lambda(z)$:

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for all $f \in \mathcal{H}$.

$k_\lambda(z) >> 0$ positive definite, i.e.

$$\forall a_1, .., a_n \qquad \sum_{i,j} a_i \overline{a_j}\ k_{\lambda_i}(\lambda_j) \geq 0,$$

because $|| \sum_i a_i k_{\lambda_i} ||^2 = \sum_{i,j} a_i \overline{a_j} k_{\lambda_i}(\lambda_j)$.

If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}(k)$, then

$$k_\lambda(z) = \sum \overline{e_n(\lambda)} e_n(z).$$

<u>Fact:</u> If $u_\lambda(z) >> 0$ is analytic in z, then $\exists u_n \in Hol(\mathbb{D})$ such that

$$u_\lambda(z) = \sum \overline{u_n(\lambda)} u_n(z).$$

$$\mathcal{H}(k^1) \quad := \quad \frac{\mathcal{N}}{G_{\mathcal{N}}} \subseteq \frac{\mathcal{M}}{G_{\mathcal{M}}} \quad =: \quad \mathcal{H}(k^2)$$

<u>Abstract nonsense:</u>

Set $||\frac{f}{G_{\mathcal{M}}}||_{\mathcal{M}} = ||f||, f \in \mathcal{M}$, $||\frac{f}{G_{\mathcal{N}}}||_{\mathcal{N}} = ||f||, f \in \mathcal{N}$.

Then the contractive inequality

$$||\frac{G_{\mathcal{M}}}{G_{\mathcal{N}}} f|| \leq ||f|| \text{ for all} f \in \mathcal{N}$$

is equivalent to

$$||g||_{\mathcal{M}} \leq ||g||_{\mathcal{N}} \text{ for all } g \in \frac{\mathcal{N}}{G_{\mathcal{N}}}.$$

If $g \in \frac{\mathcal{N}}{G_{\mathcal{N}}}$, then $g = \frac{f}{G_{\mathcal{N}}}$ for some $f \in \mathcal{N}$, so

$$||g||_{\mathcal{M}} = ||\frac{f}{G_{\mathcal{N}}}||_{\mathcal{M}} = ||\frac{G_{\mathcal{M}}}{G_{\mathcal{N}}}\frac{f}{G_{\mathcal{M}}}||_{\mathcal{M}} = ||\frac{G_{\mathcal{M}}}{G_{\mathcal{N}}} f||$$

$$\leq ||f|| = ||\frac{f}{G_{\mathcal{N}}}||_{\mathcal{N}} = ||g||_{\mathcal{N}}$$

$\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}(k)$

$$\mathcal{H}(k^1) \quad := \quad \frac{\mathcal{N}}{G_{\mathcal{N}}} \subseteq \frac{\mathcal{M}}{G_{\mathcal{M}}} \quad =: \quad \mathcal{H}(k^2)$$

reproducing kernels : $\langle f, k_\lambda \rangle = f(\lambda)$

$\mathcal{H}(k) \qquad - \qquad k_\lambda$

$\mathcal{M} \qquad - \qquad P_{\mathcal{M}} k_\lambda$

$\frac{\mathcal{M}}{G_{\mathcal{M}}} \qquad - \qquad \frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G_{\mathcal{M}}(\lambda)} G_{\mathcal{M}}(z)}$

Thm 5 (Aronszajin). *$\mathcal{H}(k^1) \subseteq \mathcal{H}(k^2)$ contractively, if and only if*

$$k^1 << k^2,$$

i.e. $k^2 - k^1 >> 0$.

Proof: Define $T : \mathcal{H}(k^2) \to \mathcal{H}(k^1)$ by

$$T\left(\sum a_i k^2_{\lambda_i}\right) = \sum a_i k^1_{\lambda_i},$$

then T is a contraction if $k^1 << k^2$, and $T^* : \mathcal{H}(k^1) \to \mathcal{H}(k^2)$ is the inclusion map. ■

We shall get :

$$\frac{k^2}{k^1} >> 0.$$

Lemma 6. *If*

$$\frac{k^2}{k^1} >> 0$$

and if $k_0^1(z) = k_0^2(z) = 1$, *then*

$$k^2 - k^1 >> 0.$$

Proof: Since $\frac{k_0^2(z)}{k_0^1(z)} = 1$ we have

$$\frac{k^2}{k^1} - 1 >> 0,$$

hence by the Schur product theorem

$$k^2 - k^1 = k^1 \left(\frac{k^2}{k^1} - 1 \right) >> 0.$$

<u>Note:</u> In our situation $k_0^1(z) = k_0^2(z) = 1$ for all $z \in \mathbb{D}$:

Say $\mathcal{H}(k) := \frac{\mathcal{M}}{G_{\mathcal{M}}}$, where ind $\mathcal{M} = 1$ and $G_{\mathcal{M}} \in \mathcal{M} \ominus \zeta\mathcal{M}$, $||G_{\mathcal{M}}|| = 1$.

Then, for $f \in \mathcal{M}$ we have

$$\langle f, G_{\mathcal{M}} \rangle = \frac{f}{G_{\mathcal{M}}}(0)$$

(= a property of extremal functions), hence

$$\langle \frac{f}{G_{\mathcal{M}}}, 1 \rangle_{\mathcal{M}} = \frac{f}{G_{\mathcal{M}}}(0),$$

and so

$$\langle g, 1 \rangle_{\mathcal{M}} = g(0) \text{ for all } g \in \frac{\mathcal{M}}{G_{\mathcal{M}}}.$$

Thus

$$1 = k_0.$$

Interpretation: dilation theorem

Thm 7 (Agler). *If*

$$\frac{k^2}{k^1} >> 0,$$

then

$(M_\zeta, \mathcal{H}(k^2))$ *dilates to* $(M_\zeta^{(\infty)}, \bigoplus \mathcal{H}(k^1))$*,*

i.e. $\exists \mathcal{M} \in$ *Lat* $(M_\zeta^{(\infty)*}, \bigoplus \mathcal{H}(k^1))$ *such that*

$(M_\zeta^*, \mathcal{H}(k^2))$ *is u.e.* $(M_\zeta^{(\infty)*}, \bigoplus \mathcal{H}(k^1))|\mathcal{M}$.

proof omitted in talk

Proof: Write $u_\lambda(z) = \frac{k^2_\lambda(z)}{k^1_\lambda(z)}$, then

$u_\lambda(z) = \sum_n u_n(z)\overline{u_n(\lambda)}$, hence

$k^2_\lambda(z) = \sum_n u_n(z)\overline{u_n(\lambda)}k^1_\lambda(z)$.

Thus, if we set

$$M^*_{u_n} : \mathcal{H}(k^2) \to \mathcal{H}(k^1), M^*_{u_n}k^2_\lambda = \overline{u_n(\lambda)}k^1_\lambda,$$

one checks that each $M^*_{u_n}$ extends to be bounded and

$$\sum_n ||M^*_{u_n}f||^2_1 = ||f||^2_2,$$

whenever f is a finite linear combination of reproducing kernels. Thus, the map

$V : \mathcal{H}(k^2) \to \bigoplus \mathcal{H}(k^1) \quad Vf = \{M^*_{u_n}f\}_n$

is an isometry. Set $\mathcal{M} = ranV$, then the Theorem follows, because

$V(M^*_\zeta, \mathcal{H}(k^2)) = (M^{(\infty)*}_\zeta, \bigoplus \mathcal{H}(k^1))V.$

Defn 8. *Let k be a reproducing kernel for $\mathcal{H}(k)$ with $k_0(z) = 1$.*

(a) k is a complete NP kernel, if $\exists u >> 0$ such that

$$k_\lambda(z) = \frac{1}{1 - u_\lambda(z)}.$$

(b) k is a Bergman-type kernel, if $\exists u >> 0$ with $u_0(z) = 0$ and $\exists \rho \in H^\infty$, outer, $1 \leq |\rho|^2 \leq 2$, $\varphi = \zeta\rho$,

$$k_\lambda(z) = \frac{1}{1 - \overline{\varphi(\lambda)}\varphi(z)(1 - u_\lambda(z))},$$

and $||k_z|| \to \infty$ as $|z| \to 1$.

Examples:

H^2: $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ is both.

D: $k_\lambda(z) = \frac{1}{\overline{\lambda}z} \log \frac{1}{1-\overline{\lambda}z}$ is a complete NP kernel (proof omitted).

L^2_a: $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^2} = \frac{1}{1-2\overline{\lambda}z(1-\frac{1}{2}\overline{\lambda}z)}$ is a Bergman-type kernel, $\varphi(z) = \sqrt{2}z, u_\lambda(z) = \frac{1}{2}\overline{\lambda}z$.

$k_\lambda^\beta(z) = \frac{1}{(1-\overline{\lambda}z)^\beta}$, $1 \leq \beta \leq 2$ is a Bergman-type kernel

$$\frac{1}{k_\lambda^\beta(z)} = (1-\overline{\lambda}z)^\beta = 1 - \beta\overline{\lambda}z + \sum_{n\geq 2} c_n(\overline{\lambda}z)^n,$$

where $c_n \geq 0$ for all $n \geq 2$, since $1 \leq \beta \leq 2$.

Fact: If k is a Bergman-type kernel, then

$$H^2 \subseteq \mathcal{H}(k) \subseteq L^2_a$$

contractively.

Thm 9 (Shimorin). *If* $\mathcal{M} \in$ *Lat* (M_ζ, D), $\mathcal{M} \neq (0)$, *and if* G *is the extremal function for* $\mathcal{M}$, *then*

$$\frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)}G(z)}$$

is a complete NP kernel.

Thm 10 (McCullough-Ri). *If* $\mathcal{M} \in$ *Lat* (M_ζ, L^2_a), *ind* $\mathcal{M} = 1$, *and if* G *is the extremal function for* $\mathcal{M}$, *then*

$$\frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)}G(z)}$$

is a Bergman-type kernel.

Fact 1: $\exists$ a complete NP kernel k and invariant subspace $\mathcal{M}$ of $(M_\zeta, \mathcal{H}(k))$ such that $\frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)}G(z)}$ is **not** a complete NP kernel.

Fact 2: If k is a Bergman-type kernel, if $\mathcal{M}$ is a zero-based invariant subspace of $(M_\zeta, \mathcal{H}(k))$, then $\frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)}G(z)}$ is a Bergman-type kernel.

Operator identities:

$u_\lambda(z) >> 0$ so $u_\lambda(z) = \sum u_n(z)\overline{u_n(\lambda)}$.

k is an NP kernel, iff

$$(1 - \sum u_n(z)\overline{u_n(\lambda)})k_\lambda(z) = 1,$$

iff

$$(I - \sum M_{u_n}M_{u_n}^*)k_\lambda = k_\lambda(0) = Q_0 k_\lambda,$$

where $Q_0 f = f(0)$. Thus

$$I - \sum M_{u_n}M_{u_n}^* = Q_0,$$

or

$$\sum M_{u_n}M_{u_n}^* = I - Q_0 = P_0.$$

k is a Bergman-type kernel, iff

$$\left(1 - \overline{\varphi(\lambda)}\varphi(z)(1 - u_\lambda(z))\right) k_\lambda = k_\lambda(0),$$

so

$$\frac{k_\lambda - k_\lambda(0)}{\overline{\varphi(\lambda)}\varphi(z)} + u_\lambda(z) k_\lambda = k_\lambda$$

$$\frac{P_0 k_\lambda}{\overline{\varphi(\lambda)}\varphi(z)} + \sum u_n(z)\overline{u_n(\lambda)} k_\lambda = k_\lambda$$

$$M_{\frac{1}{\varphi}} P_0 M_{\frac{1}{\varphi}}^* + \sum M_{u_n} M_{u_n}^* = I$$

Thm 11 (McCullough-Trent). *If k is a complete NP kernel, and if $\mathcal{M}$ is a multiplier invariant subspace of $\mathcal{H}(k)$, then*

$$\frac{P_{\mathcal{M}} k_\lambda(z)}{k_\lambda(z)} >> 0.$$

Hence, if G is extremal for $\mathcal{M}$, then with

$$k^1_\lambda(z) = \frac{P_{\mathcal{M}} k_\lambda}{\overline{G(\lambda)} G(z)},$$

we have

$$\frac{k^1}{k} >> 0$$

and we obtain the contractive inclusion

$$\mathcal{H}(k) \subseteq \frac{\mathcal{M}}{G}.$$

Proof: Set $Q(A) = \sum M_{u_n} A M_{u_n}^*$, so

$$I - Q(I) = Q_0.$$

Also

$$P_{\mathcal{M}} Q(P_{\mathcal{M}}) P_{\mathcal{M}} = Q(P_{\mathcal{M}}),$$

because $Q(P_{\mathcal{M}}) P_{\mathcal{M}^\perp} = 0$, $P_{\mathcal{M}^\perp} Q(P_{\mathcal{M}}) = 0$.

Set

$$T = P_{\mathcal{M}} - Q(P_{\mathcal{M}})$$

$$= P_{\mathcal{M}}(I - Q(P_{\mathcal{M}}))P_{\mathcal{M}} = P_{\mathcal{M}}(I - Q(I - P_{\mathcal{M}^\perp}))P_{\mathcal{M}}$$

$$= P_{\mathcal{M}}(I - Q(I)) + Q(P_{\mathcal{M}^\perp}))P_{\mathcal{M}}$$

$$= P_{\mathcal{M}} Q_0 P_{\mathcal{M}} + P_{\mathcal{M}} Q(P_{\mathcal{M}^\perp}) P_{\mathcal{M}} \ \geq \ 0$$

Thus,

$$0 << \langle T k_\lambda, k_z \rangle = P_{\mathcal{M}} k_\lambda(z) - \sum \langle M_{u_n} P_{\mathcal{M}} M_{u_n}^* k_\lambda, k_z \rangle$$

$$= P_{\mathcal{M}} k_\lambda(z) - u_\lambda(z) P_{\mathcal{M}} k_\lambda(z)$$

$$= (1 - u_\lambda(z)) P_{\mathcal{M}} k_\lambda(z) = \frac{P_{\mathcal{M}} k_\lambda(z)}{k_\lambda(z)} \blacksquare$$

Thm 12 (McCullough-Ri). *If k is a Bergman-type kernel, if $\mathcal{M} \in$ Lat $(M_\zeta, \mathcal{H}(k))$, ind $\mathcal{M} = 1$, if G is the extremal function for $\mathcal{M}$, then*

$$\frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)} G(z)} = (1 - l_\lambda(z)) k_\lambda(z)$$

for some $l_\lambda(z) >> 0$.

Hence, if we set

$$k_\lambda^1(z) = \frac{P_{\mathcal{M}} k_\lambda(z)}{\overline{G(\lambda)} G(z)},$$

then we get

$$\frac{k_\lambda(z)}{k_\lambda^1(z)} = \frac{1}{1 - l_\lambda(z)} = \sum_{n \geq 0} l_\lambda(z)^n >> 0.$$

Thus we have the contractive inclusion

$$\frac{\mathcal{M}}{G} \subseteq \mathcal{H}(k).$$

Proof. Recall

$$M_{\frac{1}{\varphi}} P_0 M_{\frac{1}{\varphi}}^* + \sum M_{u_n} M_{u_n}^* = I.$$

Set

$$T = P_{\mathcal{M}} - (M_{\frac{1}{\varphi}} P_{\zeta \mathcal{M}} M_{\frac{1}{\varphi}}^* + \sum M_{u_n} P_{\mathcal{M}} M_{u_n}^*),$$

then as in the proof of the McCullough-Trent theorem one calculates that $T \geq 0$.

Thus

$$v_\lambda(z) = \langle T k_\lambda, k_z \rangle >> 0,$$

calculate some more, the theorem will follow with

$$l_\lambda(z) = \frac{\varphi(z)\overline{\varphi(\lambda)} v_\lambda(z)}{G(z)\overline{G(\lambda)}}.$$

Thm 13 (McCullough-Ri). *If k is a Bergman-type kernel, if*

$$\mathcal{M} \in \text{ Lat } (M_\zeta, \mathcal{H}(k)), \quad \mathcal{M} \neq (0),$$

if

$$\mathcal{C} = \mathcal{M} \ominus \zeta \mathcal{M},$$

then

$$M_\zeta | \mathcal{M} \text{ is u.e. to } M_\zeta$$

on a space of $\mathcal{C}$-valued analytic functions with operator-valued reproducing kernel of the type

$$\mathbf{K}_\lambda(z) = (I_{\mathcal{C}} - z\overline{\lambda} V(z) V(\lambda)^*) k_\lambda(z),$$

where V is a contractive analytic function

$$V : \mathbb{D} \to \mathcal{B}(\mathcal{E}, \mathcal{C})$$

for some auxiliary Hilbert space $\mathcal{E}$.

This implies that

$$H^2(\mathcal{C}) \subseteq \mathcal{H}(\mathbf{K}) \subseteq \mathcal{H}(k, \mathcal{C})$$

contractively.

Here $\mathcal{H}(k, \mathcal{C}) = \mathcal{H}(k) \otimes \mathcal{C}$, i.e. the $\mathcal{C}$-valued $\mathcal{H}(k)$-space.