

Analytic contractions, nontangential limits, and the index of invariant subspaces

joint work by

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slides without figures are available at

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$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = \partial\mathbb{D},$$

$$(0) \neq \mathcal{H} \subseteq \text{Hol}(\mathbb{D}), \quad (M_z f)(z) = zf(z)$$

$\mathcal{M} \in \text{Lat}(M_z, \mathcal{H})$, if \mathcal{M} is an invariant subspace for M_z .

Examples: Hardy space

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \right\}$$

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \int_{\mathbb{T}} |f(z)|^2 \frac{|dz|}{2\pi}$$

Then:

- $\|z^n f\| = \|f\|$ for all $f \in H^2(\mathbb{D})$,
- every $f \in H^2(\mathbb{D})$ has nontangential limits a.e. on \mathbb{T} ,
- $\forall \mathcal{M} \in \text{Lat}(M_z, H^2), \mathcal{M} \neq (0)$ we have

$\mathcal{M} = \varphi H^2(\mathbb{D})$ for φ inner, and so

$$\dim \mathcal{M}/z\mathcal{M} = 1.$$

Bergman space

$$L_a^2 = \{f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi},$$

$dA = dx dy = \text{area measure.}$

Then:

- $\|z^n f\| \rightarrow 0$ for all $f \in L_a^2$,
- $\exists f \in L_a^2$ such that f has no nontangential limits on any set $E \subseteq \mathbb{T}$ of positive Lebesgue measure,
- $\forall n \in \{1, 2, \dots\} \cup \{\infty\} \exists \mathcal{M} \in \text{Lat}(M_z, L_a^2)$ such that

$$\dim \mathcal{M}/z\mathcal{M} = n.$$

Assume: $(0) \neq \mathcal{H} \subseteq \text{Hol}(\mathbb{D})$, reprod. kernel k_λ

$$(1) \quad \|zf\| \leq \|f\| \text{ for all } f \in \mathcal{H},$$

$$(2) \quad \sigma_e(M_z) = \partial\mathbb{D} \quad \&$$

$$(3) \quad \dim \mathcal{H}/z\mathcal{H} = 1$$

Def: If $\mathcal{M} \in \text{Lat}(M_z, \mathcal{H})$, then

$$\text{ind } \mathcal{M} = \dim \mathcal{M}/z\mathcal{M}$$

$$\text{ind } \mathcal{M} = -\text{ind } M_z|_{\mathcal{M}} = -\text{ind}(M_z - \lambda)|_{\mathcal{M}}$$

$$= \dim \mathcal{M}/(z - \lambda)\mathcal{M} \quad \text{for all } \lambda \in \mathbb{D}$$

Nontangential limits:

$$z \in \mathbb{T}, 0 < \alpha < 1$$

$$\Gamma_\alpha(z) =$$

If f is meromorphic in \mathbb{D} and $A \in \mathbb{C}$, then

$$\text{nt-}\lim_{\lambda \rightarrow z} f(\lambda) = A,$$

if, whenever $\{\lambda_n\} \rightarrow z$, $\{\lambda_n\} \subseteq \Gamma_\alpha(z)$ for some α , then $\{f(\lambda_n)\} \rightarrow A$.

Def: Let $E \subseteq \mathbb{T}$ be measurable, we say that \mathcal{H} admits nontangential limits on E , if the meromorphic function

$$g/f \text{ has nontangential limits a.e. on } E$$

for all $g \in \mathcal{H}$ and some $f \in \mathcal{H}, f \neq 0$.

Remark The definition is independent of f .

$g/h = \frac{g/f}{h/f}$, $h/f \neq 0$ for $h \neq 0$ follows from

Luzin-Privalov uniqueness theorem:

If k is a meromorphic function on \mathbb{D} that has 0 nontangential limit on $E \subseteq \mathbb{T}$ with $|E| > 0$, then $k = 0$.

Hence:

If $1 \in \mathcal{H}$, then \mathcal{H} admits nontangential limits on E , iff every function in \mathcal{H} has nontangential limits a.e. on E .

Example: If $\mathcal{H}_1 = H^2(\mathbb{D})$ and $\mathcal{H}_2 = fH^2(\mathbb{D})$, $\|fg\|_2 = \|g\|_{H^2}$ for some analytic function f , then

(M_z, \mathcal{H}_1) and (M_z, \mathcal{H}_2) are unitarily equivalent,

so they both satisfy (1), (2), and (3), and both admit nontangential limits on \mathbb{T} , no matter what f is.

Thm 1. Assume \mathcal{H} satisfies only (1), (2), (3)

A. $\forall f \in \mathcal{H} \quad \|z^n f\| \rightarrow 0 \text{ as } n \rightarrow \infty.$

(trivial) \Downarrow not \Uparrow (ARS)

B. $\exists f \in \mathcal{H}, f \neq 0 \quad \|z^n f\| \rightarrow 0 \text{ as } n \rightarrow \infty.$

(Khin.-Kol.) \Downarrow not \Uparrow (ARS)

C. \mathcal{H} does not admit nt-limits on any $\Delta \subseteq \mathbb{T}$,
 $|\Delta| > 0.$

(ARS) \Downarrow ? \Uparrow ?

D. $\exists M$ such that $\text{ind } M > 1.$

B \Rightarrow D follows from ABFP: $\mathbb{A}_1 \cap C_{00} = \mathbb{A}_{\mathbb{N}_0} \cap C_{00}.$

Thm 2. We have $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$, whenever \mathcal{H} satisfies (1), (2), (3) and either

- M_z is cyclic subnormal, i.e. $\mathcal{H} = P^2(\mu)$ for some μ , $\text{supp } \mu \subseteq \overline{\mathbb{D}}$, or

- $\exists c > 0 \quad \left\| \frac{z-\lambda}{1-\bar{\lambda}z} f \right\| \geq c \|f\|$ for all $f \in \mathcal{H}, \lambda \in \mathbb{D}.$

Theorem (Khinchin-Kolmogorov)

If $\sum_{n \geq 0} |a_n|^2 = \infty$, then for some choice of $\varepsilon_n \in \{-1, 1\}$ the function

$$g(z) = \sum_{n \geq 0} \varepsilon_n a_n z^n$$

has no nontangential limits on any $\Delta \subseteq \mathbb{T}$, $|\Delta| > 0$.

B \Rightarrow C:

If $\|z^n f\| \rightarrow 0$ for $f \in \mathcal{H}$, $f \neq 0$, then choose $\{n_k\}$ such that $\sum_{k \geq 0} \|z^{n_k} f\| < \infty$. Then

$$g(z) = \sum_{k \geq 0} \varepsilon_{n_k} z^{n_k} f(z) \in \mathcal{H}$$

and g/f has no nontangential limits.

C \Rightarrow D:

Case 1: The polynomials are dense in \mathcal{H} .

Use the hypothesis to show that there is a dominating sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{D}$ that is interpolating for \mathcal{H} .

Then $\mathcal{M} = \{f \in \mathcal{H} : f(\lambda_n) = 0 \text{ for all } n\}$ satisfies that $M_z^*|_{\mathcal{M}^\perp}$ is similar to a diagonal normal operator with $\overline{\lambda_n}$ on the diagonal. Now use a result of Wermer from 1950 to finish it off.

Case 2: General \mathcal{H} .

Use case 1 and then apply the Bercovici-Chevreaux Theorem ($\mathbb{A} = \mathbb{A}_1$) to $M_z^*|_{\mathcal{M}^\perp}$ for appropriate \mathcal{M} .

For the construction of the interpolating sequence $\{\lambda_n\}$ and the rest of Theorem 1 one needs to know more about the relationship of when \mathcal{H} will admit nontangential limits on a set Δ and the asymptotic size of the reproducing kernel $\|k_\lambda\|$.

$$|f(\lambda)| = (1-|\lambda|^2) \left| \frac{f}{1-\bar{\lambda}z}(\lambda) \right| \leq (1-|\lambda|^2) \left\| \frac{f}{1-\bar{\lambda}z} \right\| \|k_\lambda\|$$

$$\left| \frac{f}{g}(\lambda) \right|^2 \leq \underbrace{\left((1-|\lambda|^2) \left\| \frac{f}{1-\bar{\lambda}z} \right\|^2 \right)}_{I_\lambda} \underbrace{\left((1-|\lambda|^2) \frac{\|k_\lambda\|^2}{|g(\lambda)|^2} \right)}_{II_\lambda}$$

(a) $M_z = PU|_{\mathcal{H}}$, U unitary dilation with spectral measure E :

$$\begin{aligned} I_\lambda &= (1-|\lambda|^2) \|P(1-\bar{\lambda}U)^{-1}f\|^2 \\ &\leq \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2} d \langle E(z)f, f \rangle \\ &= v_f(\lambda) = \text{harmonic function} \end{aligned}$$

(b) $E \subseteq \mathbb{T}$ closed, $\Omega_E = \bigcup_{z \in E} \Gamma_\alpha(z)$

If $II_\lambda \leq M$ in Ω_E , then

$$\left| \frac{f}{g}(\lambda) \right|^2 \leq M v_f(\lambda) \text{ in } \Omega_E, \text{ i.e. } f/g \in H^2(\Omega_E),$$

hence \mathcal{H} admits nontangential limits on E .

Def:

$$\Delta(\mathcal{H}) = \left\{ z \in \mathbb{T} : \text{nt-}\overline{\lim}_{\lambda \rightarrow z} (1 - |\lambda|^2) \frac{\|k_\lambda\|^2}{|g(\lambda)|^2} < \infty \right\}$$

Fact: Up to a.e. this definition is independent of $g \in \mathcal{H}$, $g \neq 0$.

Thm 3. (a) \mathcal{H} admits nt-limits on $\Delta(\mathcal{H})$.

(b) If \mathcal{H} admits nt-limits on E , then $E \subseteq \Delta(\mathcal{H})$ a.e.

If $\mathcal{H} \subseteq \text{Hol}(\mathbb{D})$ satisfies (1),(2),(3) and if $\mathcal{H} = P^2(\mu)$, $\text{supp}\mu \subseteq \overline{\mathbb{D}}$, then $d\mu|_{\mathbb{T}} = h \frac{|dz|}{2\pi}$, $h \in L^1$ and

$$I_\lambda = \int_{\mathbb{D}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}z|^2} |f|^2 d\mu + \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}z|^2} |f|^2 h \frac{|dz|}{2\pi}$$

↓

↓

0 a.e.

$w_f = |f|^2 h$ a.e.

$\implies I_\lambda \rightarrow w_f(z)$ a.e. as $\lambda \rightarrow z$ nontangentially.

Claim: If $f \neq 0$, then a.e.

$$\Delta(\mathcal{H}) \subseteq \{z : w_f(z) > 0\} \subseteq \{z : h(z) > 0\} =: \Sigma(\mathcal{H}).$$

Proof: For a.e z , if $I_\lambda \rightarrow w_f(z) = 0$ and $z \in \Delta(\mathcal{H})$, then

$$\text{nt-lim}_{\lambda \rightarrow z} \left(\frac{f}{g}\right)(\lambda) = 0.$$

Hence if $f \neq 0$ by Luzin-Privalov this happens on a set of measure 0.

Note: $\mathcal{H} = P^2(\mu)$

$$\|z^n f\| \rightarrow 0 \quad \forall f \iff h = 0 \iff |\Sigma(\mathcal{H})| = 0$$

Thm 4. *If $\mathcal{H} \subseteq \text{Hol}(\mathbb{D})$ satisfies (1), (2), (3) and if $\mathcal{H} = P^2(\mu)$, $\text{supp} \mu \subseteq \overline{\mathbb{D}}$, then*

$$\Delta(\mathcal{H}) = \Sigma(\mathcal{H}) \quad \text{a.e.}$$

and $\forall f \in \mathcal{H}$ and a.e. $z \in \mathbb{T}$

$$\begin{aligned} & \text{nt-} \lim_{\lambda \rightarrow z} \frac{|f(\lambda)|^2}{(1 - |\lambda|^2) \|k_\lambda\|^2} \\ &= w_f(z) = (|f|^2 h)(z) \\ &= \text{nt-} \lim_{\lambda \rightarrow z} (1 - |\lambda|^2) \left\| \frac{f}{1 - \bar{\lambda}z} \right\|^2 \end{aligned}$$

Works for $P^t(\mu)$, $1 \leq t < \infty$.

Proof: Rework the proof of Thomson's Theorem and use Tolsa's results on analytic capacity.

If $\mathcal{H} \neq P^2(\mu)$ use minimal co-isometric extension of M_z :

$$M_z = (S^* \oplus R) |_{\mathcal{H}} \quad \text{on } \mathcal{K}_1 \oplus \mathcal{K}_2 \supseteq \mathcal{H}$$

S^* = backward shift, R unitary, E = spectral measure for R

Facts: (a) $I_\lambda \rightarrow w_f(z) = 2\pi \frac{d\langle E(z)P_2f, P_2f \rangle}{|dz|}$ a.e. as $\lambda \rightarrow z$ nontangentially.

(b) $\exists w \in L^1(\mathbb{T})$ such that

$$\|E(F)\| = 0 \Leftrightarrow \int_F w(z)|dz| = 0.$$

(c) If $\mathcal{H} = P^2(\mu)$ as above, then $\{z \in \mathbb{T} : w(z) > 0\} = \{z \in \mathbb{T} : h(z) > 0\}$ a.e.

Def.: $\Sigma(\mathcal{H}) = \{z \in \mathbb{T} : w(z) > 0\}$

Facts:(a) $\|z^n f\| \rightarrow 0$ for all $f \in \mathcal{H}$, iff $|\Sigma(\mathcal{H})| = 0$.

(b) $\Delta(\mathcal{H}) \subseteq \Sigma(\mathcal{H})$ a.e.

(c) In general, $\Delta(\mathcal{H}) \neq \Sigma(\mathcal{H})$.

Thm 5. If \mathcal{H} satisfies (1),(2),(3) and if $\exists c > 0$ such that

$$\left\| \frac{z - \lambda}{1 - \bar{\lambda}z} f \right\| \geq c \|f\|$$

for all $f \in \mathcal{H}, \lambda \in \mathbb{D}$, then

$$\Delta(\mathcal{H}) = \Sigma(\mathcal{H}) \text{ a.e.}$$

and $\forall f \in \mathcal{H}$ and a.e. $z \in \mathbb{T}$

$$\begin{aligned} & \text{nt-} \lim_{\lambda \rightarrow z} \frac{|f(\lambda)|^2}{(1 - |\lambda|^2) \|k_\lambda\|^2} \\ & = w_f(z) \\ & = \text{nt-} \lim_{\lambda \rightarrow z} (1 - |\lambda|^2) \left\| \frac{f}{1 - \bar{\lambda}z} \right\|^2 \end{aligned}$$

Thm 6. Suppose $k_\lambda(z) = \frac{l_\lambda(z)}{1 - \bar{\lambda}z}$, where

$$l_\lambda(z) = \sum_{k \geq 0} \overline{\varphi_k(\lambda)} \varphi_k(z),$$

$\varphi_k \in N$, $N = \text{Nevanlinna class}$

If for a.e. $z \in \mathbb{T}$ we have

$$\text{nt-} \overline{\lim}_{\lambda \rightarrow z} l_\lambda(\lambda) = \infty \Rightarrow \sum_{k \geq 0} |\varphi_k(z)|^2 = \infty,$$

then **same conclusion.**