

A model for two-isometric operator tuples with finite defect.

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joint work with Carl Sundberg

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September, 2011

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Example

$(M_z, H^2(\partial B_d))$ is a spherical isometry.

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$$\Delta_{n+1} = \left(\sum_{i=1}^d T_i^* \Delta_n T_i \right) - \Delta_n$$

Defn

T is an n-isometric operator tuple, if $\Delta_n = 0$.

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Facts:

If T is an n-iso \Rightarrow

- ▶ $\Delta_k = 0$ for all $k \geq n$
- ▶ $\Delta_{n-1} \geq 0$,
- ▶ but for $n \geq 3$ it is possible that $\Delta_k \not\geq 0$ for some $1 \leq k \leq n-2$

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Drury-Arveson space
- ▶ $\mathcal{H}(k) \subseteq \text{Hol}(\mathbb{B}_d)$ with reproducing kernel,

$$k_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^j},$$

$j = 1, \dots, d$, then $(M_z, \mathcal{H}(k))$ is a $(d - j + 1)$ -isometry.

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- ▶ If T is an n -isometric tuple, $\mathcal{M} \in \text{Lat} T_i \quad \forall i$,
then $T|_{\mathcal{M}}$ is an n -isometric tuple.

$$\langle \Delta_n x, x \rangle = \sum_{j=1}^n (-1)^j \binom{n}{j} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} \|T^\alpha x\|^2 = 0 \quad \forall x$$

More Examples of two-isometric tuples

- ▶ If $d = 1$, then two-isometric operators on finite dimensional spaces must be unitary.
- ▶ $d = 2$

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T_2^2 = 0$$

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$$\Delta_2 = T_1^* \Delta_1 T_1 + T_2^* \Delta_1 T_2 - \Delta_1 = T_2^{*2} T_2^2 = 0$$

More generally

If $\alpha = (\alpha_1, \dots, \alpha_d) \in \partial \mathbb{B}_d$, $V_i : \mathbb{C}^m \rightarrow \mathbb{C}^n$, $\sum_{i=1}^d \overline{\alpha_i} V_i = 0$, then

$$S = (S_1, \dots, S_d),$$

$$S_i = \begin{pmatrix} \alpha_i I_n & V_i \\ 0 & \alpha_i I_m \end{pmatrix}$$

defines a two-isometric d-tuple.

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If $\alpha = (\alpha_1, \dots, \alpha_d) \in \partial \mathbb{B}_d$, $V_i : \mathbb{C}^m \rightarrow \mathbb{C}^n$, $\sum_{i=1}^d \overline{\alpha_i} V_i = 0$, then

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Thm

If T is a two-isometric d -tuple on a finite dimensional space, then

$$T = U \oplus S,$$

where U is spherical unitary and S is a direct sum of operator tuples as above.

From now on let $T = (T_1, \dots, T_d)$ be a **two-isometric tuple** without nontrivial finite dimensional reducing subspace.

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$$\text{Thus } \sum_{i=1}^d T_i^* \Delta_1 T_i - \Delta_1 = 0 \Leftrightarrow$$

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" T induces a spherical isometry w.r.t. $\|x\|_* = \|Dx\|$ "

Spherical isometries are jointly subnormal (Athavale), hence for all $x_0 \in \mathcal{H}$ there is $\mu \in M_+(\partial B_d)$ with

$$\|Dp(T)x_0\|^2 = \int_{\partial B_d} |p|^2 d\mu \quad \forall p \in \mathbb{C}[z]$$

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For $d = 1$ one can often recover $\|p(T)x_0\|$ by use of this μ .
For $d > 1$ this is not the case.

$d = 1$ Model theorem, Agler-Stankus version

If $T \in \mathcal{B}(\mathcal{H})$ is a 2-isometry with $D \neq 0$,

if $x_0 \in \ker(\|D\| - D)$, $\|x_0\| = 1$,

if

$$\|Dp(T)x_0\|^2 = \int |p|^2 d\mu$$

then

$$\|p(T)x_0\|^2 = \|p\|_{AS}^2 = \int_{\partial\mathbb{D}} |p|^2 \frac{d\mu}{\|\mu\|} + \int_{\partial\mathbb{D}} D_\alpha(p) d\mu(\alpha)$$

$$\|\mu\| = \|D\|^2$$

$$D_\alpha(f) = \int_{\partial\mathbb{D}} \frac{|f(z) - f(\alpha)|^2}{|z - \alpha|^2} \frac{|dz|}{2\pi} \quad (\text{local Dirichlet integral})$$

$$d \geq 1$$

Defn

Let $\alpha \in \partial \mathbb{B}_d$, $\sigma \in M_+(\partial \mathbb{B}_d)$, then

$$\sigma \in \mathcal{M}_\alpha$$

$$\Leftrightarrow$$

- ▶ $\sigma(\partial \mathbb{B}_d) = 1$,
- ▶ $\sigma(\{\alpha\}) = 0$, and
- ▶

$$\int_{\partial \mathbb{B}_d} p(z) \frac{(1 - \langle z, \alpha \rangle)^2}{\|z - \alpha\|_{\mathbb{C}^d}^2} d\sigma(z) = 0 \quad \forall p \in \mathbb{C}[z].$$

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Defn

Let $\alpha \in \partial\mathbb{B}_d$, $\sigma \in \mathcal{M}_\alpha$, then define the local Dirichlet integral

$$D_\alpha(p) = D_\alpha(p, \sigma) = \int_{\partial\mathbb{B}_d} \frac{|p(z) - p(\alpha)|^2}{\|z - \alpha\|_{\mathbb{C}^d}^2} d\sigma(z)$$

For $\sigma \in \mathcal{M}_\alpha$ we have

$$\sum_{i=1}^d D_\alpha(z_i p, \sigma) = D_\alpha(p, \sigma) + |p(\alpha)|^2 \quad \forall p \in \mathbb{C}[z].$$

Thus:

Let $\mu \in M_+(\partial\mathbb{B}_d)$, $\mu \neq 0$, let $\sigma_\alpha \in \mathcal{M}_\alpha$, continuously depending on $\alpha \in \partial\mathbb{B}_d$, then define

$$\|p\|_{AS}^2 = \int_{\partial\mathbb{B}_d} |p|^2 \frac{d\mu}{\|\mu\|} + \int_{\partial\mathbb{B}_d} D_\alpha(p, \sigma_\alpha) d\mu(\alpha).$$

Let $\mathcal{H}_{AS} = \mathcal{H}_{AS}(\mu, \sigma_\alpha)$ be the completion of the polys w.r.t. $\|\cdot\|_{AS}$.

Then (M_z, \mathcal{H}_{AS}) is a 2-isometric d-tuple with

- ▶ $\|Dp\|_{AS}^2 = \int_{\partial\mathbb{B}_d} |p|^2 d\mu \quad \forall p \in \mathbb{C}[z]$
- ▶ $1 \in \ker(\|D\| - D), \|1\|_{AS} = 1.$

Theorem

Let T be a two-isometric d -tuple with

- ▶ $\text{rank } D < \infty$,
- ▶ T has no nontrivial finite dimensional reducing subspace.

Let $x_0 \in \ker(\|D\| - D)$, $\|x_0\| = 1$. Then

$$\|p(T)x_0\|^2 = \|p\|_{AS}^2 + \langle Ep, p \rangle_{AS} \quad \forall p \in \mathbb{C}[z].$$

Here

- ▶ $\|Dp(T)x_0\|^2 = \int |p|^2 d\mu = \sum_{j=1}^N \mu_j |p(\alpha_j)|^2$
- ▶ $\exists \sigma_j \in \mathcal{M}_{\alpha_j} \quad j = 1, \dots, N$
- ▶ $\mathcal{H}_{AS} = \mathcal{H}_{AS}(\mu, \{\sigma_j\})$
- ▶ $E \in \mathcal{B}(\mathcal{H}_{AS})$ with
 - ▶ E is s.a. with $I + E \geq 0$
 - ▶ $E1 = 0$
 - ▶ $\sum_{i=1}^d \langle Ez_i p, z_i p \rangle = \langle Ep, p \rangle \quad \forall p \in \mathbb{C}[z]$