m-Isometric Commuting Tuples of Operators on a Hilbert Space

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Abstract

We consider a generalization of isometric Hilbert space operators to the multivariable setting. We study some of the basic properties of these tuples of commuting operators and we explore several examples. In particular, we show that the d-shift, which is important in the dilation theory of d-contractions (or row contractions), is a d-isometry. As an application of our techniques we prove a theorem about cyclic vectors in certain spaces of analytic functions that are properly contained in the Hardy space of the unit ball of $\mathbb{C}^d$.

1 Introduction

Much of the development of multivariable operator theory has arisen as a result of taking ideas and concepts that have been instrumental in the development of single variable operator theory and finding a "correct" generalization. Some examples include subnormal tuples, Taylor spectrum, and Hardy and Bergman spaces of regions in $\mathbb{C}^d$. Since the unilateral shift and other isometries played a pivotal role in the development of operator theory, in particular with the theory of contractions and polar decompositions, a large amount of research explores the multivariable analogues.

Originally much of this involved studying commuting isometries and the tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ on the Hardy space of the ball or polydisc. A more recent development along these lines was first introduced by Drury in [13] and was further developed and popularized by Arveson in a series of papers, [4], [5], and [6]. This line of thinking is that the "correct" generalization of the Hardy space H^2 of the unit disc $\mathbb{D}$ is the space H^2_d of analytic functions the unit ball $\mathbb{B}_d$ of $\mathbb{C}_d$. H^2_d is the space of analytic functions on $\mathbb{B}_d$ with reproducing kernel $k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle}$. The operator tuple M_z of multiplications by the coordinate functions on H^2_d is called the d-shift. The d-shift has played a role in the dilation

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theory of the d-contractions (also called row contractions), and in fact, Drury proved a von Neumann type inequality for d-contractions.

Since in one variable, the unilateral shift and isometries are intimately connected, one of the difficulties with the d-shift to this point is that it has not been related to any of the generalizations of isometries. The purpose of this article is to show that there is a strong relationship between the d-shift and the m-isometries that will be defined in the beginning of the next section. In our effort to develop this connection, we build off of the ideas of Jim Agler and Mark Stankus, see [1], [2], and [3], where they define an operator T to be a m-isometry if

$$\sum_{k=0}^{m}(-1)^k \left(\begin{array}{c} m \\ k \end{array} \right) (T^*)^{m-k}T^{m-k} = 0.$$

In the following we consider a multivariable generalization of these single variable m-isometries and explore some of their basic properties. We will find that through their spectral theory the m-isometric operator tuples are linked to the unit ball in $\mathbb{C}^d$, see Propositions 3.1 and 3.2. In fact for $m = 1$ the m-isometric operator tuples coincide with the so-called spherical isometries. Thus, for example M_z on the Hardy space of the ball is a 1-isometry. We then consider further examples and in particular we show that the d-shift is a d-isometry, see Theorems 4.1 and 4.2. It follows that any restriction of the d-shift to any of its invariant subspaces is a d-isometry. Thus, further study of d-isometric operator tuples may lead to new discoveries about the invariant subspace structure of the d-shift, much like the study of single variable two-isometries has lead to results about the invariant subspaces of the classical Dirichlet space, see e.g. [18], [19], and [20]. In fact, as an application of our techniques, we prove a Theorem about cyclic vectors in certain spaces of analytic functions that are properly contained in the Hardy space of the unit ball (Theorem 5.3). This result applies to the 2-shift.

In order to avoid confusion on the part of the reader we note that in [10] and [11], Curto and Vasilescu investigate certain operator tuples that are associated with a multiindex γ and that they call γ-isometries. Those operator tuples are associated with the polydisc and there is no relation with the operators considered here.

2 Basic Properties

In this Section we will prove some basic properties of m-isometric operator tuples. All of these results are fairly straightforward generalizations of the corresponding single variable results that were proved in [1] and [2].

Let $\mathcal{H}$ be a separable Hilbert space and $B(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Associated to each tuple of commuting operators, $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$, there is an associated function $Q_T : B(\mathcal{H}) \to B(\mathcal{H})$, defined by $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$, which has been found to be very useful in describing certain properties of T, see [4] or [15] for example.

Since $(I - Q_T)$ is an operator on $B(\mathcal{H})$, for each $m \geq 0$, denote $P_m(T)$ as $(I - Q_T)^m(I)$, i.e.

$$P_m(T) = \sum_{j=0}^{m} (-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) Q_T^j(I).$$

A commuting tuple, $T = (T_1, \ldots, T_d)$, is said to be an m-isometry if $P_m(T) = 0$. In order to work more easily with m-isometries the following lemma will be useful. We start by establishing our notation regarding multiindices. Let $\mathbb{Z}_+^d$ denote the set of all multiindices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \geq 0$, and for each of these multiindices we write $|\alpha| = \sum_{j=1}^d \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_d!$, and $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$.

Lemma 2.1. *If $T = (T_1, \ldots, T_d)$ is a commuting tuple of operators on a Hilbert space $\mathcal{H}$, then*

$$P_m(T) = \sum_{j=0}^{m} (-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} (T^\alpha)^* T^\alpha$$

and for all $f \in \mathcal{H}$

$$< P_m(T)f, f >= \sum_{j=0}^{m} (-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \left\| T^\alpha f \right\|^2.$$

Proof. The multinomial formula implies that $Q_T^j = \sum_{|\alpha|=j} \frac{j!}{\alpha!} (T^\alpha)^* T^\alpha$, thus the lemma follows immediately from the definition of $P_m(T)$. $\square$

Since $P_m(T) = (I - Q_T)^m(I)$, one sees that these operators can in fact be defined inductively using the equation

$$P_{m+1}(T) = (I - Q_T)(P_m(T)) = P_m(T) - Q_T(P_m(T)). \tag{2.1}$$

This type of inductive description of $P_m(T)$ is useful to see properties of m-isometries such as that if T is a m-isometry, then T is a $(m+n)$-isometry for all $n \geq 0$.

One may note at this point that there seems to be a strong relationship between the sets $\{P_m(T)\}_{m=0}^\infty$ and $\left\{Q_T^k(I)\right\}_{k=0}^\infty$. In fact, as the following lemma shows, these two sets of operators contain the same information from the tuple T since each set can be defined in terms of the other set.

Lemma 2.2. *For $k \geq 0$,*

$$Q_T^k(I) = \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!} P_j(T) \right) k^{(j)}$$

where $k^{(j)} = k \cdot (k-1) \cdots (k-j+1)$ for $j \geq 1$ and $k^{(0)} = 1$.

We note that for $j \geq k+1$ we have $k^{(j)} = 0$, so the sum only has finitely many nonzero summands.

Proof. For ease of notation in the proof, let $a_T(k) = \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!} P_j(T) \right) k^{(j)}$.

Note that

$$a_T(k+1) - a_T(k) = \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!} P_j(T) \right) (k+1)^{(j)} - \sum_{j=0}^{\infty} \left(\frac{(-1)^j}{j!} P_j(T) \right) k^{(j)}.$$

Since $k^{(j)} = 0$ for $j \geq k+1$,

$$\begin{aligned} & a_T(k+1) - a_T(k) = \\ &= \sum_{j=0}^{k+1} \left(\frac{(-1)^j}{j!} P_j(T) \right) (k+1)^{(j)} - \sum_{j=0}^{k} \left(\frac{(-1)^j}{j!} P_j(T) \right) k^{(j)} \\ &= \frac{(-1)^{k+1}}{(k+1)!} P_{k+1}(T)(k+1)! + \sum_{j=1}^{k} \left(\frac{(-1)^j}{j!} P_j(T) \right) \left((k+1)^{(j)} - k^{(j)} \right) \\ &= -\sum_{j=0}^{k} \frac{(-1)^j}{j!} P_{j+1}(T) k^{(j)}. \end{aligned}$$

We now use equation (2.1) to simplify the right side of the equation,

$$\begin{aligned} & a_T(k+1) - a_T(k) = \\ &= -\sum_{j=0}^{k} \frac{(-1)^j}{j!} \left(P_j(T) - Q_T(P_j(T)) \right) k^{(j)} \\ &= Q_T(a_T(k)) - a_T(k). \end{aligned}$$

So for $k \geq 1$, $a_T(k+1) = Q_T(a_T(k))$. Since $a_T(0) = I$, we have that $a_T(k) = Q_T^k(I)$. ☐

Because of the fact that if T is a m-isometry, then T is a $(m+n)$-isometry for all n, it will be useful to determine the smallest m for which T is a m-isometry. We define

$$\Delta_{T,m} := (-1)^{m-1} P_{m-1}(T).$$

If it is clear from the context what m is, we will sometimes write Δ_T instead of $\Delta_{T,m}$. The definition of $\Delta_{T,m}$ immediately implies that T is not a $(m-1)$-isometry if $\Delta_{T,m} \neq 0$. Another property involving $\Delta_{T,m}$ is a positivity condition.

Proposition 2.3. *If T is a m-isometry for some $m \geq 1$, then $\Delta_T \geq 0$.*

Proof. If T is a m-isometry, then $P_j(T) = 0$ for $j \geq m$. So

$$\lim_{k \to \infty} \frac{1}{k^{(m-1)}} Q_T^k(I) = \lim_{k \to \infty} \sum_{j=0}^{m-1} \left(\frac{(-1)^j}{j!} P_j(T) \right) \frac{k^{(j)}}{k^{(m-1)}}.$$

Since $\lim_{k\to\infty} \frac{k^{(j)}}{k^{(m-1)}} = \begin{cases} 0 & \text{if} \quad j < m-1 \\ 1 & \text{if} \quad j = m-1 \end{cases}$,

$$\lim_{k\to\infty} \frac{1}{k^{(m-1)}} Q_T^k(I) = \frac{(-1)^{m-1}}{(m-1)!} P_{m-1}(T) = \frac{1}{(m-1)!} \Delta_T. \tag{2.2}$$

Hence since for all T and k one has $Q_T^k(I) \geq 0$ the proposition follows.

□

One can also use the operator Δ_T to decompose T into pieces which are k-isometries, but not $k-1$ isometries. We start by looking at invariant subspaces of T, subspaces $\mathcal{M} \subset \mathcal{H}$ such that $T_i\mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, \ldots, d$. These next two propositions and their proofs are direct generalizations of Proposition 1.6 and 1.7 from [1].

Proposition 2.4. *If* $T = (T_1, \ldots, T_d)$ *is a* m*-isometry, then* $\ker(\Delta_T)$ *is invariant for each* T_i *and* $\tilde{T} = (T_1|\ker(\Delta_T), \ldots, T_d|\ker(\Delta_T))$ *is a* $(m-1)$*-isometry. Furthermore, if* $\mathcal{M} \subseteq \mathcal{H}$ *is invariant for* T *and* $T|\mathcal{M}$ *is a* $(m-1)$*-isometry, then* $\mathcal{M} \subseteq \ker(\Delta_T)$*.*

Proof. Since T is a m-isometry, from equation (2.1) we have that $P_{m-1}(T) - Q_T(P_{m-1}(T)) = P_m(T) = 0$. So

$$\Delta_T = Q_T(\Delta_T). \tag{2.3}$$

Thus for $x \in \mathcal{H}$,

$$\begin{aligned} \sum_{j=1}^{d} \langle \Delta_T T_j x, T_j x\rangle &= \sum_{j=1}^{d} \left\langle T_j^* \Delta T_j x, x\right\rangle \\ &= \langle Q_T(\Delta_T)x, x\rangle = \langle \Delta_T x, x\rangle . \end{aligned}$$

Since $\Delta_T \geq 0$, this implies that $\Delta_T x = 0$ if and only if $\Delta_T T_j x = 0$ for each $j = 1, \ldots, d$. Thus $\ker(\Delta_T)$ is invariant for each T_j.

For each j, let $\tilde{T_j} = T_j|\ker(\Delta_T)$ and set $\tilde{T} = (\tilde{T_1}, \ldots, \tilde{T_d})$. Then $\tilde{T}$ is a $(m-1)$-isometry, because for $x \in \ker(\Delta_T)$ the invariance of $\ker(\Delta_T)$ together with Lemma 2.1 implies that

$$\begin{aligned} < P_{m-1}(\tilde{T})x, x > &= < P_{m-1}(T)x, x > \\ &= (-1)^{m-1} < \Delta_T x, x >= 0. \end{aligned}$$

To prove the last statement of the proposition, let $\mathcal{M} \subseteq \mathcal{H}$, $T_j\mathcal{M} \subset \mathcal{M}$ for each j, and $T|\mathcal{M}$ be a $(m-1)$-isometry. If we fix $x \in \mathcal{M}$, then

$$\langle \Delta_T x, x\rangle = \left\langle (-1)^{m-1} P_{m-1}(T)x, x\right\rangle = \left\langle (-1)^{m-1} P_{m-1}(T|\mathcal{M})x, x\right\rangle = 0.$$

Since $\Delta_T \geq 0$ and since $x \in \mathcal{M}$ was arbitrary, $\mathcal{M} \subseteq \Delta_T$. □

Proposition 2.5. *If $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ is a m-isometry, then there is a unique subspace $\mathcal{M} \subset \mathcal{H}$ that is maximal with respect to the following properties:*

(i) $\mathcal{M}$ is reducing for T, and

(ii) $T|\mathcal{M}$ is a $(m-1)$-isometry.

Proof. The existence of subspaces $\mathcal{M}$ that are maximal with respect to (i) and (ii) follows from Zorn's Lemma. To prove uniqueness it suffices to establish that if $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy (i) and (ii), then $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2$ also satisfies (i) and (ii). $\mathcal{M}$ satisfies (i) by definition. To see that $\mathcal{M}$ satisfies (ii), we first notice that since $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy (ii), from Proposition 2.4, we have that $\mathcal{M}_1 \subseteq \ker(\Delta_T)$ and $\mathcal{M}_2 \subseteq \ker(\Delta_T)$. Thus $\mathcal{M} \subseteq \ker(\Delta_T)$ and so (ii) holds for $\mathcal{M}$. ☐

Following the methods of [1] we can use Proposition 2.5 to create a canonical decomposition of a m-isometry into a direct sum of l-isometries, $l \leq m$, where the l-isometries are pure in that they have no non-zero direct summand which is an $(l-1)$-isometry.

3 Spectral Properties

Associated to each commuting tuple $T = (T_1, \ldots, T_d)$ there are several different notions of a spectrum. These include the Taylor spectrum, $\sigma(T)$, the Harte spectrum, $\sigma_H(T)$, the left spectrum, $\sigma_l(T)$, the right spectrum, $\sigma_r(T)$, the Słodkowski spectra, $\sigma_{s,j,k}(T)$, and the joint approximate point spectrum, $\sigma_\pi(T)$. For a good description of each of these spectra and some of their properties the reader is referred to [12] and the references there. In this section, we will study how these different notions of spectra play out when the tuple is a m-isometry.

We will start this study by looking at different variations of the spectral radius. The first is the *geometric joint spectral radius* given by the formula

$$r_g(T) = \sup \{|z| : z \in \sigma(T)\} .$$

This definition appears to be dependent upon the choice of the Taylor spectrum. However, Chō and Żelazko proved in [9] that this definition is independent of the choice of the spectra that we have listed.

In addition to the geometric spectral radius, there is also an *algebraic joint spectral radius* associated with a tuple, T, which is given by

$$r_a(T) = \inf_k \left\{ \left\| \sum_{f \in F(k,d)} (T_f)^* T_f \right\|^{\frac{1}{2k}} \right\} \tag{3.1}$$

where $F(k,d)$ is the set of all functions from $\{1, \ldots, k\}$ to $\{1, \ldots, d\}$ and $T_f = T_{f(1)} \cdots T_{f(k)}$ for $f \in F(k,d)$. Note that since $\sum_{f \in F(k,d)} (T_f)^* T_f = Q_T^k(I)$, this algebraic spectral radius can be rewritten as

$$r_a(T) = \inf_k \left\| Q_T^k(I) \right\|^{\frac{1}{2k}} .$$

Another useful description of this spectral radius was given by Bunce, in [7], where he proves that

$$r_a(T) = \lim_{k\to\infty} \left\|Q_T^k(I)\right\|^{\frac{1}{2k}}.$$

It was also conjectured in [7] that the two spectral radii are equal as in the case when $d = 1$ with the spectral radius formula. This conjecture was proved true by Chō, Huruya, and Wrobel ([8]) in the case of a finite dimensional Hilbert space and by Müller and Sołtysiak ([16]) in the general Hilbert space context. Therefore, we will denote this spectral radius by $r(T)$ and use all of the different descriptions of the spectral radius to prove information regarding the spectral picture of T.

Proposition 3.1. *If T is a m-isometry, then $r(T) = 1$.*

Proof. In the case that T is a m-isometry, then in the proof of Proposition 2.3 we saw that

$$\lim_{k\to\infty} \frac{1}{k^{(m-1)}} Q_T^k(I) = \frac{1}{(m-1)!}\Delta_T. \tag{3.2}$$

We may assume that m is the smallest number such that T is a m-isometry. Thus we have that $\left\|\frac{1}{(m-1)!}\Delta_T\right\|$ is positive so that $\lim_{k\to\infty}\left\|\frac{1}{(m-1)!}\Delta_T\right\|^{\frac{1}{2k}} = 1$.

Since $(k-m+2)^{m-1} \le k^{(m-1)} \le k^{m-1}$ and

$$\lim_{k\to\infty} (k-m+2)^{\frac{m-1}{2k}} = \lim_{k\to\infty} k^{\frac{m-1}{2k}} = 1,$$

$\lim_{k\to\infty}\left(k^{(m-1)}\right)^{\frac{1}{2k}} = 1$. Hence

$$\lim_{k\to\infty} \|Q_T^k(I)\|^{\frac{1}{2k}} = \lim_{k\to\infty}\left(\frac{\|Q_T^k(I)\|}{k^{(m-1)}}\right)^{\frac{1}{2k}} = \lim_{k\to\infty}\left\|\frac{1}{(m-1)!}\Delta_T\right\|^{\frac{1}{2k}} = 1.$$

Therefore, $r(T) = 1$. ☐

Another method to find this spectral radius is to study the joint approximate point spectrum of T, $\sigma_\pi(T)$. Recall, from [12], that

$$\sigma_\pi(T) = \left\{\lambda \in \mathbb{C}^d : \begin{array}{c} \lim_{k\to\infty}\sum_{j=1}^d \|(T_j-\lambda_j)x_k\| = 0, \\ \text{for some sequence of unit vectors } \{x_k\} \end{array}\right\}.$$

This is equivalent to $\lim_{k\to\infty}(T_j-\lambda_j)x_k = 0$ for all $j = 1,\dots,d$. Since, for $\alpha_j > 1$, $\left(T_j^{\alpha_j} - \lambda_j^{\alpha_j}\right) = (T_j-\lambda_j)\left(\sum_{l=1}^{\alpha_j}\lambda_j^{l-1}T_j^{\alpha_j-l}\right)$, $\lambda \in \sigma_\pi(T)$ if and only if there is a sequence of unit vectors, $\{x_k\}$ such that $\lim_{k\to\infty}(T_j^{\alpha_j}-\lambda_j^{\alpha_j})x_k = 0$ for all $j = 1,\dots,d$, and $\alpha_j \ge 0$. Furthermore, by induction, for $\alpha \in \mathbb{Z}_+^d$ we have that

$$T^\alpha - \lambda^\alpha = \sum_{j=1}^d\left(\left(\prod_{i<j}\lambda_i^{\alpha_i}\right)\left(T_j^{\alpha_j}-\lambda_j^{\alpha_j}\right)\left(\prod_{i>j}T_i^{\alpha_i}\right)\right).$$

Therefore, $\lambda \in \sigma_\pi(T)$ if and only if there is a sequence of unit vectors, $\{x_k\}$ such that $\lim_{k\to\infty}(T^\alpha - \lambda^\alpha)x_k = 0$ for all $\alpha \in \mathbb{Z}_+^d$. Using this definition of the approximate point spectrum we have the following lemma.

Lemma 3.2. *If T is a m-isometry, then the joint approximate point spectrum of T is in the boundary of the unit ball.*

Proof. If λ is in the approximate point spectrum of T, then there is a sequence of unit vectors, $\{x_k\}$ such that $(T^\alpha - \lambda^\alpha)x_k \to 0$ for all $\alpha \in \mathbb{Z}_+^d$. Therefore,

$$0 = \langle P_m(T)x_k, x_k\rangle = \sum_{j=0}^{m}(-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|T^\alpha x_k\|^2$$

and letting $k \to \infty$ we have that

$$0 = \sum_{j=0}^{m}(-1)^j \left(\begin{array}{c} m \\ j \end{array} \right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha|^2 = (1 - |\lambda|)^m$$

and so $|\lambda| = 1$. $\square$

Since the approximate point spectrum of the tuple T is contained in the boundary of the unit ball, and since from [9] we know that the convex envelopes of all spectra coincide, thus again it follows that the spectral radius of T must be 1.

Example 3.3. If A is a single variable $m-$isometry, then $\sigma_\pi(A) \subseteq \partial\mathbb{D}$ and either $\sigma(A) = \overline{\mathbb{D}}$ or $\sigma(A) \subseteq \partial\mathbb{D}$. This was proved in [1] and it also follows from the Lemma above. One easily checks that $T = (A, 0, ..., 0)$ is an $m-$isometric operator tuple with $\sigma_\pi(T) \subseteq \partial\mathbb{D}\times 0\times ...\times 0$. This implies that $\sigma(T) = \overline{\mathbb{D}}\times 0\times ...\times 0$ or $\sigma(T) \subseteq \partial\mathbb{D} \times 0 \times ... \times 0$.

4 Examples

In this Section we will consider examples of m-isometric tuples that are built from single variable m-isometric operators in a more symmetric fashion than was done in the previous example. Throughout this section d will be a fixed positive integer, and we use $\mathbb{C}[z]$ to denote the algebra of polynomials in the variables $z_1, ..., z_d$. In our constructions we will use the slice functions $f_z : \mathbb{D} \to \mathbb{C}$ associated with a function $f : \overline{\mathbb{B}}_d \to \mathbb{C}$ and a point $z \in \partial\mathbb{B}_d$ by $f_z(w) = f(wz) = f(wz_1, ..., wz_d)$.

Theorem 4.1. *If*

(i) there is a $C > 0$ such that for each $z \in \partial\mathbb{B}_d$, there exists a Hilbert space, $\mathcal{H}_z$, of holomorphic functions in $\mathbb{D}$ such that multiplication by the independent variable, M_w, is a n-isometry with $\|M_w\|_{\mathcal{H}_z} \leq C$,

(ii) for all $i, j \geq 0$ and λ in the unit disc, the function $\phi_{i,j}(z) := \left\langle \lambda^i, \lambda^j \right\rangle_{\mathcal{H}_z}$ is Borel measurable on $\partial\mathbb{B}_d$,

(iii) and μ is a bounded Borel measure on the boundary of the unit ball,

then the space, $\mathcal{K}$, formed by completing $\mathbb{C}[z]$ with respect to the norm defined by the inner product $\langle p, q\rangle_{\mathcal{K}} := \int_{\partial\mathbb{B}_d} \langle p_z, q_z\rangle_{\mathcal{H}_z} \,\mathrm{d}\mu(z)$ is a Hilbert space on which the tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ is a n-isometry.

Proof. For each i, $1 \leq i \leq d$, and each polynomial p we have

$$\|z_i p\|^2_{\mathcal{K}} = \int_{\partial\mathbb{B}_d} \|(z_i p)_z\|^2_{\mathcal{H}_z} d\mu(z) = \int_{\partial\mathbb{B}_d} |z_i|^2 \|M_w p_z\|^2_{\mathcal{H}_z} d\mu(z) \leq C^2 \|p\|^2_{\mathcal{K}},$$

thus M_z extends to be a bounded operator tuple on $\mathcal{K}$.

Let $P_n(T) := \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) Q_T^j(I)$ as above, and define $\mathcal{K}$ as in the theorem. For $f \in \mathcal{K}$, Lemma 2.1 shows that

$$\langle P_n(M_z)f, f\rangle_{\mathcal{K}} = \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|M_z^\alpha f\|^2_{\mathcal{K}}\,.$$

Since $\|f\|^2_{\mathcal{K}} = \int_{\partial\mathbb{B}_d} \|f_z\|^2_{\mathcal{H}_z} \mathrm{d}\mu(z)$ this becomes

$$\langle P_n(M_z)f, f\rangle_{\mathcal{K}} = \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \int_{\partial\mathbb{B}_d} \|(M_z^\alpha f)_z\|^2_{\mathcal{H}_z} \,\mathrm{d}\mu(z).$$

Substituting $(M_z^\alpha f)_z = z^\alpha M_w^{|\alpha|} f_z$ into the above equation we have that

$$\langle P_n(M_z)f, f\rangle_{\mathcal{K}} = \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \int_{\partial\mathbb{B}_d} \left\| z^\alpha M_w^{|\alpha|} f_z \right\|^2_{\mathcal{H}_z} \mathrm{d}\mu(z).$$

With z^α being constant with respect to the norm $\|\cdot\|_{\mathcal{H}_z}$ the equation can be written as

$$\begin{aligned} \langle P_n(M_z)f, f\rangle_{\mathcal{K}} &= \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) \sum_{|\alpha|=j} \frac{j!}{\alpha!} \int_{\partial\mathbb{B}_d} |z^\alpha|^2 \left\| M_w^{|\alpha|} f_z \right\|^2_{\mathcal{H}_z} \mathrm{d}\mu(z) \\ &= \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) \int_{\partial\mathbb{B}_d} \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} |z^\alpha|^2 \right) \left\| M_w^j f_z \right\|^2_{\mathcal{H}_z} \mathrm{d}\mu(z). \end{aligned}$$

Finally, $\left(\sum_{|\alpha|=j}\frac{j!}{\alpha!}|z^\alpha|^2\right) = \|z\|^{2j} = 1$ for $z \in \partial\mathbb{B}_d$ and the equation becomes

$$\begin{aligned}
\langle P_n(M_z)f,f\rangle_{\mathcal{K}} &= \sum_{j=0}^{n}(-1)^j\begin{pmatrix} n \\ j \end{pmatrix}\int_{\partial\mathbb{B}_d}\left\|M_w^j f_z\right\|^2_{\mathcal{H}_z}\,\mathrm{d}\mu(z) \\
&= \int_{\partial\mathbb{B}_d}\sum_{j=0}^{n}(-1)^j\begin{pmatrix} n \\ j \end{pmatrix}\left\|M_w^j f_z\right\|^2_{\mathcal{H}_z}\,\mathrm{d}\mu(z) \\
&= \int_{\partial\mathbb{B}_d}\langle P_n(M_w)f_z,f_z\rangle_{\mathcal{H}_z}\,\mathrm{d}\mu(z).
\end{aligned}$$

Therefore, $P_n(M_w) = 0$ implies that $P_n(M_z) = 0$ and we have the desired result. □

In the special case of 2-isometries the previous theorem can be made more explicit, because a description of the spaces $\mathcal{H}_z$ as required in the hypothesis of the theorem is available, see [3], [17], or [18]. While this does create a large class of examples of multivariable 2-isometries, this class is not exhaustive, even if we are only interested in those cases where the resulting space $\mathcal{K}$ is a space of analytic functions in the unit ball of $\mathbb{C}^d$. For example, let $\mathcal{K}$ be the Hilbert space generated by taking the closure of the polynomials on the unit ball, $\mathbb{B}_2$, given by the norm

$$\|f\|^2_{\mathcal{K}} = \int_{\partial\mathbb{B}_2}|f|^2\mathrm{d}\sigma(z) + c_1\left|\frac{\partial f}{\partial z_2}(1,0)\right|^2 + c_2\int_{\partial\mathbb{D}}\left|\frac{f(\lambda,0)-f(1,0)}{\lambda-1}\right|^2\frac{|d\lambda|}{2\pi},$$

where σ denotes the normalized Lebesgue measure. To see that the tuple M_z is a 2-isometry one can compute that for every polynomial $f \in C[z_1,z_2]$

$$\|M_{z_1}f\|^2 + \|M_{z_2}f\|^2 = \|f\|^2 + (c_1+c_2)|f(1,0)|^2.$$

Then letting f equal $M_{z_1}f$ and $M_{z_2}f$ one has that

$$\begin{aligned}
&\|f\|^2 - 2\left(\|M_{z_1}f\|^2 + \|M_{z_2}f\|^2\right) + \\
\left(\left\|M_{z_1}^2 f\right\|^2 + \|M_{z_2}M_{z_1}f\|^2\right) + &\left(\|M_{z_1}M_{z_2}f\|^2 + \left\|M_{z_2}^2 f\right\|^2\right) \quad = 0,
\end{aligned}$$

so M_z is a 2-isometry. If $c_2 > 0$, then one easily shows that $M_z = (M_{z_1}, M_{z_2})$ is a tuple of bounded operators. Furthermore, if $c_1 > 0$, then the resulting space $\mathcal{K}$ cannot be obtained as in Theorem 4.1. We omit the verification of this last statement.

Another class of examples of m-isometries comes from the relationship between two spaces of analytic functions which we will call $\mathcal{H}_{a,d}$ and $\mathcal{K}_{a,d}$.

For integers $a, d > 0$ define $\mathcal{H}_{a,d}$ to be the space of analytic functions on the unit disk given by the norm $\|h\|^2_{\mathcal{H}_{a,d}} := \sum_{n=0}^\infty c_{a,d,n}\left|\hat{h}(n)\right|^2$ where $c_{a,d,0} = 1$ and $c_{a,d,n} = \frac{(d+n-1)(d+n-2)\cdots(d)}{(a+n-1)(a+n-2)\cdots(a)} = \frac{\Gamma(d+n)}{\Gamma(a+n)}\frac{\Gamma(a)}{\Gamma(d)}$ for $n \geq 1$ and $h(w) = \sum_{n=0}^\infty \hat{h}(n)w^n$

is the Taylor expansion of h. The properties of the Gamma function imply that $\lim_{n\to\infty}\frac{c_{a,d,n}}{n^{d-a}}=\frac{\Gamma(a)}{\Gamma(d)}$. Hence we observe that $\mathcal{H}_{a,d}=D_{d-a}$ with equivalence of norms, where $||h||^2_{D_\beta}=\sum_{n=0}^\infty (n+1)^\beta|\hat{h}(n)|^2$.

Also, for $a>0$, we let $\mathcal{K}_{a,d}$ be the space of analytic functions on the ball $\mathbb{B}_d$ with reproducing kernel $k_\lambda(z)=\frac{1}{(1-\langle z,\lambda\rangle)^a}$. The space $\mathcal{K}_{1,d}$ is the space H^2_d, $\mathcal{K}_{d,d}=H^2(\partial\mathbb{B}_d)$ is the Hardy space on the sphere, and $\mathcal{K}_{d+1,d}=L^2_a(\mathbb{B}_d)$ is the Bergman space on the ball. We further note that spectral information for the tuple M_z on $\mathcal{K}_{a,d}$ was established in [14].

Using Theorem 4.1, we will prove the following result that includes as special cases that M_z is a 1-isometry on the Hardy space of the unit ball, and that the d-shift is a d-isometry.

Theorem 4.2. *If d and a are positive integers with $d\geq a$, then*

$$\|f\|^2_{\mathcal{K}_{a,d}}=\int_{\partial\mathbb{B}_d}\|f_z\|^2_{\mathcal{H}_{a,d}}\mathrm{d}\sigma(z)$$

where $\mathrm{d}\sigma$ is the normalized Lebesgue measure on the unit sphere. Furthermore, the tuple $M_z=(M_{z_1},\ldots,M_{z_d})$ is a $(d-a+1)$-isometry on $\mathcal{K}_{a,d}$.

We note that implies that the square of the norm on $\mathcal{K}_{a,d}$ is equivalent to $\int_{\partial\mathbb{B}_d}\|f_z\|^2_{D_{d-a}}\mathrm{d}\sigma(z)$. This was certainly known, our main point here is that the exact expression for the norm gives that M_z is a $(d-a+1)$-isometry.

We will show that the hypothesis of Theorem 4.1 are satisfied in this case by proving two lemmas.

Lemma 4.3. *If d and a are positive integers with $d\geq a$, then M_w is a $(d-a+1)$-isometry on $\mathcal{H}_{a,d}$.*

Proof. If $d=a$, then the norm on $\mathcal{H}_{a,a}$ is given by $\|h\|^2_{\mathcal{H}_{a,a}}=\sum_{n=0}^\infty|\hat{h}(n)|^2$. So M_w is a 1-isometry on $\mathcal{H}_{a,a}$.

Assume for some $k\geq a$ that M_w is a $(k-a+1)$-isometry on $\mathcal{H}_{a,k}$. Then

$$\sum_{j=0}^{k-a+1}(-1)^j\begin{pmatrix}k-a+1\\ j\end{pmatrix}\|M_w^j(wf)\|^2_{\mathcal{H}_{a,k}}=0$$

for all f in $\mathcal{H}_{a,k}$.

Since

$$\begin{aligned}\|wh\|^2_{\mathcal{H}_{a,k+1}}-\|h\|^2_{\mathcal{H}_{a,k+1}} &= \sum_{n=1}^\infty c_{a,k+1,n}|\hat{h}(n-1)|^2-\sum_{n=0}^\infty c_{a,k+1,n}|\hat{h}(n)|^2\\ &= \sum_{n=0}^\infty\left(c_{a,k+1,n+1}-c_{a,k+1,n}\right)|\hat{h}(n)|^2\\ &= \frac{k-a+1}{k}\sum_{n=0}^\infty c_{a,k,n+1}|\hat{h}(n)|^2=\frac{k-a+1}{k}\|wh\|^2_{\mathcal{H}_{a,k}}\end{aligned}$$

for all h in $\mathcal{H}_{a,k+1}$, we have that

$$\begin{aligned}
\left\langle P_{(k+1)-a+1}(M_w)f, f\right\rangle &= \sum_{j=0}^{(k+1)-a+1} (-1)^j \begin{pmatrix} (k+1)-a+1 \\ j \end{pmatrix} \left\|M_w^j f\right\|^2_{\mathcal{H}_{a,k+1}} \\
= \quad \|f\|^2_{\mathcal{H}_{a,k+1}} &+ \sum_{j=1}^{k-a+1} (-1)^j \begin{pmatrix} k-a+2 \\ j \end{pmatrix} \left\|M_w^j f\right\|^2_{\mathcal{H}_{a,k+1}} \\
&+(-1)^{k-a+2} \left\|M_w^{k-a+2} f\right\|^2_{\mathcal{H}_{a,k+1}}
\end{aligned}$$

$$\begin{aligned}
= \quad & \|f\|^2_{\mathcal{H}_{a,k+1}} + (-1)^{k-a+2} \left\|M_w^{k-a+2} f\right\|^2_{\mathcal{H}_{a,k+1}} \\
& + \sum_{j=1}^{k-a+1} (-1)^j \begin{pmatrix} k-a+1 \\ j-1 \end{pmatrix} \left\|M_w^j f\right\|^2_{\mathcal{H}_{a,k+1}} \\
& + \sum_{j=1}^{k-a+1} (-1)^j \begin{pmatrix} k-a+1 \\ j \end{pmatrix} \left\|M_w^j f\right\|^2_{\mathcal{H}_{a,k+1}} \\
= \quad & \sum_{j=1}^{k-a+2} (-1)^j \begin{pmatrix} k-a+1 \\ j-1 \end{pmatrix} \left(\left\|M_w^j f\right\|^2_{\mathcal{H}_{a,k+1}} - \left\|M_w^{j-1} f\right\|^2_{\mathcal{H}_{a,k+1}}\right) \\
= \quad & \frac{k-a+1}{k} \sum_{j=1}^{k-a+2} (-1)^j \begin{pmatrix} k-a+1 \\ j-1 \end{pmatrix} \left\|M_w^{j-1} f\right\|^2_{\mathcal{H}_{a,k}} \\
= \quad & -\frac{k-a+1}{k} \sum_{j=0}^{k-a+1} (-1)^j \begin{pmatrix} k-a+1 \\ j \end{pmatrix} \left\|M_w^j (wf)\right\|^2_{\mathcal{H}_{a,k}} = 0.
\end{aligned}$$

Since the polynomials are dense in $\mathcal{H}_{a,k}$ we have by induction that for $d \geq a$, M_w is a $(d-a+1)$-isometry on $\mathcal{H}_{a,d}$. □

Lemma 4.4. *Let f be a function in $\mathcal{K}_{a,d}$. If d is an integer, then we have that*

$$\int_{\partial\mathbb{B}_d} \|f_z\|^2_{\mathcal{H}_{a,d}} \mathrm{d}\sigma(z) = \|f\|^2_{\mathcal{K}_{a,d}}$$

Proof. Recall that the multinomial formula implies that for $n > 0$ and $z, \lambda \in \mathbb{B}_d$ we have that

$$\langle z, \lambda\rangle^n = \sum_{\substack{\alpha \in \mathbb{Z}^d_+ \\ |\alpha| = n}} \frac{|\alpha|!}{\alpha!} z^\alpha \bar{\lambda}^\alpha.$$

Using an induction argument one can see that

$$k_\lambda(z) = \frac{1}{(1-\langle z, \lambda\rangle)^a} = \sum_{n=0}^{\infty} b_n \left(\langle z, \lambda\rangle\right)^n$$

where $b_0 = 1$ and $b_n = \frac{(a+n-1)!}{(a-1)!n!}$ for $n \geq 1$. By combining the previous two calculations we have that $k_\lambda(z) = \sum_{\alpha \in \mathbb{Z}_+^d} b_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha \bar{\lambda}^\alpha$. Since $k_\lambda(z) = \langle k_\lambda, k_z \rangle_{\mathcal{K}_a}$ it follows that the monomials in $\mathcal{K}_{a,d}$ are mutually orthogonal and

$$\|z^\alpha\|^2_{\mathcal{K}_{a,d}} = \frac{\alpha!}{b_{|\alpha|}|\alpha|!} = \frac{\alpha!}{a(a+1)\cdots(a+|\alpha|-1)}. \tag{4.1}$$

If d is an integer, then we have from the definition of the norm of $\mathcal{H}_{a,d}$ that

$$\int_{\partial\mathbb{B}_d} \|f_z\|^2_{\mathcal{H}_{a,d}} \mathrm{d}\sigma(z) = \int_{\partial\mathbb{B}_d} \sum_{n=0}^\infty c_{a,d,n} |\hat{f}_z(n)|^2 \mathrm{d}\sigma(z).$$

Then by switching the order of integration and summation, this becomes

$$\sum_{n=0}^\infty c_{a,d,n} \int_{\partial\mathbb{B}_d} |\hat{f}_z(n)|^2 \mathrm{d}\sigma(z).$$

Since $f_z(w) = \sum_{n=0}^\infty \hat{f}_z(n) w^n$ and $f_z(w) = f(zw) = \sum_\alpha \hat{f}(\alpha)(zw)^\alpha$, $\hat{f}_z(n) = \sum_{|\alpha|=n} \hat{f}(\alpha) z^\alpha$ and we now have that

$$\int_{\partial\mathbb{B}_d} \|f_z\|^2_{\mathcal{H}_{a,d}} \mathrm{d}\sigma(z) = \sum_{n=0}^\infty c_{a,d,n} \int_{\partial B_d} \left| \sum_{|\alpha|=n} \hat{f}(\alpha) z^\alpha \right|^2 \mathrm{d}\sigma(z).$$

Using that

$$\int_{\partial B_d} \left| \sum_{|\alpha|=n} \hat{f}(\alpha) z^\alpha \right|^2 \mathrm{d}\sigma(z) = \left\langle \sum_{|\alpha|=n} \hat{f}(\alpha) z^\alpha, \sum_{|\beta|=n} \hat{f}(\beta) z^\beta \right\rangle_{H^2(\partial\mathbb{B}_d)}$$

and that the monomials in $H^2(\partial\mathbb{B}_d) = \mathcal{K}_{d,d}$ are mutually orthogonal, the equation becomes

$$\int_{\partial\mathbb{B}_d} \|f_z\|^2_{\mathcal{H}_{a,d}} \mathrm{d}\sigma(z) = \sum_{n=0}^\infty c_{a,d,n} \sum_{|\alpha|=n} \left|\hat{f}(\alpha)\right|^2 \|z^\alpha\|^2_{H^2(\partial\mathbb{B}_d)}.$$

Then two applications of equation (4.1) and the definition of $c_{a,d,n}$ give us the desired result. ☐

It is now easy to check that the hypothesis of Theorem 4.1 have been met and we have proven Theorem 4.2.

5 An application: cyclic vectors in $\mathcal{K}_{d-1,d}$

We have already mentioned that reasonably good theorems are available which describe the structure of single variable two-isometric operators, see e.g. [1],

[2], [3], [17], [18], [19], [20], [21], [22]. We will now show that for $d > 1$ some of those results can be combined with Theorem 4.2 to prove a theorem about the two-isometric operator tuple M_z acting on $\mathcal{K}_{d-1,d}$. We will not present any further details, but we note that the same proof will show similar results for all two-isometric operator tuples M_z acting on spaces $\mathcal{K}$ as described by Theorem 4.1.

Throughout this Section we will fix an integer $d > 1$ and mostly consider the case $a = d - 1$. In this case one computes the coefficients $c_{d-1,d,n}$ from the definition of the single-variable space $\mathcal{H}_{d-1,d}$ as $c_{d-1,d,0} = 1$ and $c_{d-1,d,n} = 1 + \frac{n}{d-1}$ for $n > 0$. Thus, the space $\mathcal{H}_{d-1,d}$ equals the classical Dirichlet space D and the norm on $\mathcal{H}_{d-1,d}$ is equivalent to the norm

$$||h||_D^2 = ||h||_{H^2}^2 + \int_{\partial\mathbb{D}} D_\zeta(h)\frac{|d\zeta|}{2\pi},$$

where $D_\zeta(h) = \int_{\partial\mathbb{D}} \left|\frac{h(w)-h(\zeta)}{w-\zeta}\right|^2 \frac{|dw|}{2\pi}$ is the local Dirichlet integral of the H^2 function h at the point $\zeta \in \partial\mathbb{D}$. See [19] for more information on D_ζ. In particular, we note that $D_\zeta(h) = \infty$ at every point ζ, where the nontangential limit of h does not exist.

It now follows from Theorem 4.2 that an equivalent norm for $\mathcal{K}_{d-1,d}$ is given by

$$\int_{\partial\mathbb{B}_d} \|f_z\|_D^2 d\sigma(z), \;\; f \in \mathcal{K}_{d-1,d}.$$

An application of Fubini's Theorem and the rotation invariance of the measures shows that

$$\int_{\partial\mathbb{B}_d} \|f_z\|_D^2 d\sigma(z) = \int_{\partial\mathbb{B}_d} |f(z)|^2 d\sigma(z) + \int_{\partial\mathbb{B}_d} D_1(f_z) d\sigma(z).$$

Definition 5.1. A vector f is a cyclic vector for $\mathcal{K}_{a,d}$ if $\{pf : p \in \mathbb{C}[z]\}$ is dense in $\mathcal{K}_{a,d}$.

Remark 5.2. The constant functions are cyclic vectors since the polynomials are dense in $\mathcal{K}_{a,d}$.

Let $f \in \mathcal{K}_{a,d}$, then we will write $[f]$ for the smallest invariant subspace of the operator tuple of M_z acting on $\mathcal{K}_{a,d}$. Thus $[f]$ is the closure of the polynomial multiples of f in $\mathcal{K}_{a,d}$ and f is a cyclic vector for $\mathcal{K}_{a,d}$ if and only if $[f] = \mathcal{K}_{a,d}$.

If $a \geq d$, then it is easy to see and well-known that every bounded analytic function $\varphi \in H^\infty(\mathbb{B}_d)$ defines a bounded multiplication operator on $\mathcal{K}_{a,d}$, $M_\varphi f = \varphi f$ and $||M_\varphi|| = ||\varphi||_\infty$. Thus, whenever $a \geq d$ and $f, g \in \mathcal{K}_{a,d}$ with $|g(z)| \leq |f(z)|$ for all $z \in \mathbb{B}_d$, then one easily proves that $[g] \subseteq [f]$. Indeed, we set $\varphi = \frac{g}{f}$ and note that for $0 < r < 1$ we have $\varphi_r f \in [f]$, where $\varphi_r(z) = \varphi(rz)$, and $||\varphi_r f||_{\mathcal{K}_{a,d}} \leq ||f||_{\mathcal{K}_{a,d}}$. It follows that $\varphi_r f \to g$ weakly as $r \to 1$, thus $g \in [f]$ and the statement follows. For $a < d$ such a result may still be true, but the proof will have to be modified. By use of the remarks from above and the results from [19], Section 5, we will accomplish the case $a = d - 1$.

Theorem 5.3. *Let $f, g \in \mathcal{K}_{d-1,d}$ with $|g(z)| \leq |f(z)|$ for all $z \in \mathbb{B}_d$, then*

$$[g] \subseteq [f].$$

In particular, if g is cyclic in $\mathcal{K}_{d-1,d}$, then f is cyclic in $\mathcal{K}_{d-1,d}$.

Proof. We use the same approach as indicated above. Indeed we set $\varphi = \frac{g}{f}$ and note that for $0 < r < 1$ one easily shows using uniform convergence that $\varphi_r f \in [f]$. Thus, it remains to prove that $\|\varphi_r f\|_{\mathcal{K}_{d-1,d}}$ stays bounded as $r \to 1$. The inequality in the proof of Lemma 5.4 of [19] shows that there is a $c > 0$ such that for each $z \in \partial\mathbb{B}_d$ we have

$$D_1((\varphi_r f)_z) = D_1((\varphi_z)_r f_z) \leq c(D_1(f_z) + D_1(\varphi_z f_z)) = c(D_1(f_z) + D_1(g_z)).$$

Thus, it follows from the identity preceding Definition 5.1 that

$$\int_{\partial\mathbb{B}_d} \|(\varphi_r f)_z\|_D^2 d\sigma(z) \leq c \left(\int_{\partial\mathbb{B}_d} |g(z)|^2 d\sigma(z) + \int_{\partial\mathbb{B}_d} D_1(f_z) + D_1(g_z) d\sigma(z) \right).$$

The Theorem follows by the equivalence of norms as noted in the beginning of this section. ☐

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