# A quantitative estimate for bounded point evaluations in $P^t(\mu)$-spaces.

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**Abstract.** In this note we explain how X. Tolsa's work on analytic capacity and an adaptation of Thomson's coloring scheme can be used to obtain a quantitative version of J. Thomson's theorem on bounded point evaluations for $P^t(\mu)$-spaces.

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## 1. Introduction

For $\lambda \in \mathbb{C}$ and $r > 0$ let $B(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| < r\}$, and let $M_c(\mathbb{C})$ denote the set of all compactly supported complex Borel measures in $\mathbb{C}$. Then for $\nu \in M_c(\mathbb{C})$, $r > 0$, and $\lambda \in \mathbb{C}$ we write

$$U_{|\nu|}(\lambda) = \int \frac{1}{|z-\lambda|} d|\nu|(z)$$

and

$$U_{|\nu|}(\lambda, r) = \int_{B(\lambda,r)} \frac{1}{|z-\lambda|} d|\nu|(z).$$

We will refer to $U_{|\nu|}$ as the potential of $\nu$. It is well-known that $U_{|\nu|}(\lambda) < \infty$ for [Area] a.e. $\lambda \in \mathbb{C}$. At every such $\lambda \in \mathbb{C}$ the Cauchy transform

$$C\nu(\lambda) = \int \frac{1}{z-\lambda} d\nu(z)$$

exists and $U_{|\nu|}(\lambda, r) \to 0$ as $r \to 0$. The purpose of this paper is to prove the following theorem.

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**Theorem 1.1.** *There exists an absolute constant $C > 0$ such that for every $\nu \in M_c(\mathbb{C})$ and for every $\lambda \in \mathbb{C}$ with $U_{|\nu|}(\lambda) = \int \frac{1}{|z-\lambda|} d|\nu|(z) < \infty$ there exists $r_0 > 0$ such that for all polynomials $p$ and for all $0 < r \leqslant r_0$ we have*

$$|p(\lambda)C\nu(\lambda)| \leqslant \frac{C}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)|dA(z).$$

*Here $r_0$ depends only on $|C\nu(\lambda)|$, $U_{|\nu|}(\lambda)$ and $U_{|\nu|}(\lambda, r)$ as $r \to 0$.*

The theorem is nontrivial only at points when $C\nu(\lambda) \neq 0$ and we will see that there is an absolute constant $K_0 > 0$ such that for all such points any $r_0$ satisfying

$$U_{|\nu|}(\lambda, r_0 + \sqrt{r_0}) + \sqrt{r_0}U_{|\nu|}(\lambda) \leqslant K_0 \ |C\nu(\lambda)|$$

will work.

The insight that such a theorem can be used to establish bounded point evaluations for $P^t(\mu)$-spaces that are proper subspaces of $L^t(\mu)$ is a part of what J. Thomson calls "Brennan's trick", see Theorem 1.1 of [8] and also see Section 2 below. Although as far as we know Theorem 1.1 has never been stated before in full generality, versions of it have been implicitly derived for annihilating measures in [1] and [2]. In fact, we shall see that it follows fairly easily from our paper [1], and it can also be deduced from Brennan's paper [2]. Thus we think of the current paper mostly as an expository note, and we plan to take this opportunity to once more carefully explain how X. Tolsa's theorem on analytic capacity, [9] and an adaptation of Thomson's coloring scheme, [8] come together to prove the current result. In Section 5 we explain how the current approach can also be used to establish that every bounded point evaluation must either arise because of an atom of $\mu$ or it must be an analytic bounded point evaluation.

## 2. Thomson's theorem

Let $\mu$ be a positive finite compactly supported measure in the complex plane $\mathbb{C}$, let $1 \leqslant t < \infty$ and let $P^t(\mu)$ denote the closure of the polynomials in $L^t(\mu)$. In 1991 James Thomson proved the following theorem, [8].

**Theorem 2.1.** *(J. Thomson) If $P^t(\mu) \neq L^t(\mu)$, then there are a $\lambda_0 \in \mathbb{C}$ and a constant $c > 0$ such that*

$$|p(\lambda_0)| \leqslant c \left( \int |p|^t d\mu \right)^{1/t}$$

*for every polynomial $p$.*

The point $\lambda_0$ is called a bounded point evaluation for $P^t(\mu)$. In fact, Thomson proved that every bounded point evaluation for $P^t(\mu)$ is either a point mass for $\mu$ or it is an analytic bounded point evaluation, i.e. the constant $c$ can be chosen so that there is $\varepsilon_0 > 0$ such that $|p(\lambda)| \leqslant c \left(\int |p|^t d\mu\right)^{1/t}$ for every polynomial $p$ and every $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon_0$.

Thomson's proof contains a basic construction, but at its core it is a proof by contradiction and it originally was not clear which points $\lambda_0$ occur and how the $c$ depends on $\mu$ and $\lambda_0$. After the papers [2] and [1] were written we received a note from J. Thomson which showed that a careful analysis of his original proof does show that point evaluations occur at every point where some annihilating measure has finite potential and nonzero Cauchy transform.

The following observation and the realization of its usefulness goes back to Brennan, [4, 2, 5]. It shows that Theorem 1.1 gives some information on how certain changes of the measure would affect the $\lambda_0$ and $c$.

**Lemma 2.2.** *(J. Brennan) Let $\mu$ be a compactly supported positive measure, let $1 \leqslant t < \infty$, and let $1 < t' \leqslant \infty$ satisfy $1/t + 1/t' = 1$. If $G \in L^{t'}(\mu)$ such that with $d\nu = Gd\mu$ we have $\int p d\nu = 0$ for all polynomials $p$, and if $r, C_0 > 0$ such that*

$$|p(\lambda)| \leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)| dA(z),$$

*then*

$$|p(\lambda)| \leqslant \frac{2\pi C_0}{r} \|G\|_{t'} \|p\|_t.$$

*Proof.* In this paper we shall repeatedly use the inequality

$$\int_{z\in\Delta} \frac{1}{|w-z|} \frac{dA(z)}{\pi} \leqslant 2\sqrt{\frac{A(\Delta)}{\pi}} \tag{2.1}$$

for $w \in \mathbb{C}$, $\Delta \subseteq \mathbb{C}$ (see [7], pages 2-3). Thus in particular,

$$\int_{B(\lambda,r)} \frac{1}{|w-z|} dA(z) \leqslant 2\pi r,$$

for all $\lambda, w \in \mathbb{C}$. If $\int p d\nu = 0$ for every polynomial $p$, then $\int \frac{p(w)-p(z)}{w-z} d\nu(w) = 0$ for all $z \in \mathbb{C}$ and hence $p(z)C\nu(z) = C(p\nu)(z)$ for a.e. $z \in \mathbb{C}$. Thus,

$$\begin{aligned} |p(\lambda)| &\leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)| dA(z) \\ &= \frac{C_0}{r^2} \int_{B(\lambda,r)} |C(p\nu)(z)| dA(z) \\ &\leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} \int \frac{|p(w)G(w)|}{|w-z|} d\mu(w) dA(z) \\ &= \frac{C_0}{r^2} \int \int_{B(\lambda,r)} \frac{1}{|w-z|} dA(z) |p(w)G(w)| d\mu(w) \\ &\leqslant \frac{2\pi C_0}{r} \int |pG| d\mu \leqslant \frac{2\pi C_0}{r} \|G\|_{t'} \|p\|_t \end{aligned}$$

□

Note that in the above setting the largest choice of $r$ as given by Theorem 1.1 will give the best bound for the point evaluation. If one is interested in rational approximation, then there may be an advantage to applying the theorem with smaller values of $r$. Let $R^t(\mu)$ denote the closure in $L^t(\mu)$ of the rational functions with no poles in the support of $\mu$. It is well-known that for $1 \leqslant t \leqslant 2$ there are measures $\mu$ such that $R^t(\mu) \neq L^t(\mu)$, but $R^t(\mu)$ does not have any bounded point evaluations, see [3, 6]. Nevertheless the above setup can be used to obtain bounded point evaluations for $R^t(\mu)$ in case the support of $\mu$ satisfies an extra condition.

Suppose that $R^t(\mu) \neq L^t(\mu)$ and let $G \in L^{t'}(\mu)$ be such that $d\nu = Gd\mu$ annihilates the rational functions with poles outside the support of $\mu$. Let $\lambda$, $r_0 > 0$ be as in Theorem 1.1, let $0 < r < r_0$ and let $q$ be a rational function with no poles in $\overline{B(\lambda, r)} = \{z : |z-\lambda| \leqslant r\}$. By Runge's theorem $q$ can be uniformly approximated on $\overline{B(\lambda, r)}$ by polynomials, hence the conclusion of Theorem 1.1 remains valid with $q$ in place of $p$. If $q$ also has no poles in the support of $\mu$, then the proof of Lemma 2.2 shows that

$$|q(\lambda)| \leqslant \frac{2\pi C}{r} \frac{1}{|C\nu(\lambda)|} \|G\|_{t'} \|q\|_t.$$

Another application of Runge's Theorem now implies that this last inequality remains valid for each rational function $q$ which has no poles in the support of $\mu$, if each component of the complement of the support of $\mu$ has a point in $\mathbb{C} \setminus \overline{B(\lambda, r)}$. This implies that if $R^t(\mu) \neq L^t(\mu)$ and if there is $\varepsilon > 0$ such that all components of the complement of the support of $\mu$ have diameter $\geqslant \varepsilon$, then $R^t(\mu)$ has bounded point evaluations. This result is due to Brennan, see Theorem 1 of [5].

## 3. Some auxiliary lemmas

Our argument will make essential use of Xavier Tolsa's work on analytic capacity. For a compact $K \subseteq \mathbb{C}$ we define the *analytic capacity* of $K$ by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}_\infty \setminus K), |f(z)| \leqslant 1 \ \forall z \in \mathbb{C}^\infty \setminus K\}$$

where

$$f'(\infty) = \lim_{z \to \infty} z[f(z) - f(\infty)].$$

A good source for basic information about analytic capacity is [7].

A related capacity, $\gamma_+$, is defined by

$$\gamma_+(K) = \sup\{\sigma(K) : \sigma \geqslant 0, \text{spt } \sigma \subseteq K, C\sigma \in L^\infty(\mathbb{C}), |C\sigma(z)| \leqslant 1 \text{ for A-a.e. } z \in \mathbb{C}\}.$$

Since $C\sigma$ is analytic in $\mathbb{C}_\infty \setminus \operatorname{spt} \mu$ and $(C\mu)'(\infty) = -\mu(K)$ we have

$$\gamma_+(K) \leqslant \gamma(K)$$

for all compact $K \subseteq \mathbb{C}$. In 2001, Tolsa proved the astounding result that $\gamma_+$ and $\gamma$ are actually equivalent [9]:

**Theorem 3.1 (Tolsa).** *There is an absolute constant $A_T$ such that*

$$\gamma(K) \leqslant A_T \gamma_+(K)$$

*for all compact sets $K \subseteq \mathbb{C}$.*

**Lemma 3.2.** *Suppose $\omega$ is a compactly supported bounded function times area measure. We then have the following weak-type inequality for analytic capacity*

$$\gamma([\operatorname{Re} C\omega \geqslant a]) \leqslant \frac{A_T}{a}\|\omega\| \qquad \text{for all } a > 0,$$

*where $A_T$ is Tolsa's constant.*

For a general compactly supported measure $\omega$, $C\omega$ is only defined $A$-almost everywhere, so $\gamma([\operatorname{Re} C\omega \geqslant a])$ might not even make sense. The restriction we have put on $\omega$ avoids this problem since it implies that $C\omega$ is continuous and the set $[\operatorname{Re} C\omega \geqslant a]$ is compact. A proof of this Lemma can be found in [1], but we note that it is a standard argument that follows easily from the definitions that $\gamma_+$ satisfies the weak-type inequality

$$\gamma_+([\operatorname{Re} C\omega \geqslant a]) \leqslant \frac{1}{a}\|\omega\| \qquad \text{for all } a > 0.$$

Thus Lemma 3.2 follows immediately from Tolsa's Theorem.

**Lemma 3.3.** *There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. Let $E \subset \operatorname{clos}\mathbb{D}$ be compact with $\gamma(E) < \epsilon_1$. Then*

$$|p(0)| \leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \qquad \text{for all } p \in \mathcal{P}.$$

This is Lemma B of [1] and it is proved directly by an adaptation of Thomson's coloring scheme. In fact, using Thomson's terminology for sets $E$ with sufficiently small analytic capacity it turns out that the measure $\chi_{\mathbb{D}\setminus E} dA$ gives rise to a sequence of heavy barriers around 0.

One can use the previous two lemmas to prove the following fact:

**Theorem 3.4.** *There are constants $\epsilon_0 > 0$ and $C_0 < \infty$ such that the following is true. If $\nu$ is a compactly supported measure in $\mathbb{C}$, and $\nu = \nu_1 + \nu_2$ where $\nu_1$ and $\nu_2$ are compactly supported measures in $\mathbb{C}$ with*

$$\operatorname{Re} C\nu_1 \geqslant 1 \qquad \text{a.e. } [A] \text{ in } \operatorname{clos}\mathbb{D}$$

*and*

$$\|\nu_2\| < \epsilon_0,$$

*then*

$$|p(0)| \leqslant C_0 \int_{|w|<1} |p(w) C\nu(w)| dA(w) \qquad \text{for all } p \in \mathcal{P}.$$

*Proof.* Let $\nu$, $\nu_1$, $\nu_2$ satisfy the hypotheses of Theorem 3.4 with $\epsilon_0 = \epsilon_1/2A_T$. By convolving with $\frac{n^2}{\pi}\chi_{B(0,\frac{1}{n})}$ and taking limits as $n \to \infty$, we see that we may assume that the measures $\nu$, $\nu_1$, $\nu_2$ are all compactly supported bounded functions times area measures, so that $C\nu$, $C\nu_1$, $C\nu_2$ are continuous, and the set $E = [-\operatorname{Re} C\nu_2 \geqslant \frac{1}{2}]$ is compact. We apply Lemma 3.2 with $a = \frac{1}{2}$ to $-\nu_2$ to get

$$\gamma(E) \leqslant 2A_T\|\nu_2\| < \epsilon_1. \tag{3.1}$$

For $w \in (\operatorname{clos}\mathbb{D}) \setminus E$ we have

$$|C\nu(w)| \geqslant \operatorname{Re} C\nu(w) > 1 - \frac{1}{2} = \frac{1}{2}. \tag{3.2}$$

By (3.1) $E$ satisfies the hypotheses of Lemma 3.3, hence for $p \in \mathcal{P}$ we can apply that lemma together with (3.2) to obtain

$$\begin{aligned} |p(0)| &\leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \\ &\leqslant 2C_1 \int_{w\in(\operatorname{clos}\mathbb{D})\setminus E} |p(w)C\nu(w)| \frac{dA(w)}{\pi}. \end{aligned}$$

This proves Theorem 3.4 with $C_0 = 2C_1$. □

## 4. The proof of Theorem 1.1

**Lemma 4.1.** *Let $\nu \in M_c(\mathbb{C})$ with $U = \int \frac{1}{|z|} d|\nu|(z) < \infty$, and write $U(r) = \int_{|z|<r} \frac{1}{|z|} d|\nu|(z)$.*

*Then for any $r > 0$ we have*

$$\frac{1}{r}|\nu|(B(0,r)) \leqslant U(r)$$

*and*

$$\frac{1}{\pi r^2}\int_{|w|<r} |C\nu(w) - C\nu(0)| dA(w) \leqslant 2U(r+\sqrt{r}) + \frac{2\sqrt{r}}{3}U.$$

*Proof.* Let $r > 0$. The first inequality is trivial. We will establish the second one. We have

$$\int_{|w|<r} |C\nu(w) - C\nu(0)| dA(w) \leqslant \int_{z\in\mathbb{C}} \left(\int_{|w|<r} \frac{|w|}{|w-z|} dA(w)\right) \frac{1}{|z|} d|\nu|(z).$$

The estimate (2.1) implies that $\int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \leqslant 2\pi r^2$ for all $z \in \mathbb{C}$. Thus

$$\int_{|z|<r+\sqrt{r}} \int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \frac{1}{|z|} d|\nu|(z) \leqslant 2\pi r^2 U(r+\sqrt{r}).$$

If $|z| \geqslant r + \sqrt{r}$, then we use

$$\int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \leqslant \int_{|w|<r} \frac{|w|}{|z|-|w|} dA(w)$$
$$\leqslant \frac{1}{\sqrt{r}} \int_{|w|<r} |w| dA(w) = \frac{2\pi r^{5/2}}{3}.$$

Hence

$$\int_{|z|\geqslant r+\sqrt{r}} \int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \frac{1}{|z|} d|\nu|(z) \leqslant \frac{2\pi r^{5/2}}{3} U.$$

The lemma follows. $\square$

**Lemma 4.2.** *Let $\varepsilon_0, C_0 > 0$ be as given by Theorem 3.4. Let $\nu \in M_c(\mathbb{C})$ with $\int \frac{1}{|z|} d|\nu|(z) < \infty$ and $C\nu(0) \neq 0$.*

*Suppose that $r > 0$ satisfies*

$$\int_{|w|<r} |C\nu(w) - C\nu(0)| \frac{dA(w)}{\pi} + 2r|\nu|(B(0,r)) < \frac{9}{32} r^2 |C\nu(0)| \tag{4.1}$$

*and*

$$\frac{1}{r}|\nu|(B(0,r)) < \frac{1}{4}\varepsilon_0 |C\nu(0)|. \tag{4.2}$$

*Then*

$$|p(0)C\nu(0)| \leqslant \frac{8C_0}{r^2} \int_{|w|<r} |p(w)C\nu(w)| dA(w)$$

*for every polynomial $p$.*

Lemma 4.1 implies that if the potential $U_{|\nu|}(0)$ is finite and if $C\nu(0) \neq 0$, then the hypothesis of this lemma is satisfied for all sufficiently small $r > 0$. Thus it is clear that Lemma 4.2 implies Theorem 1.1.

*Proof.* Set $C\nu(0) = a \neq 0$. For $r > 0$ satisfying (4.1) and (4.2) set $\nu_1 = \nu|\mathbb{C} \setminus B(0,r)$, $\nu_2 = \nu|B(0,r)$. We have

$$\int_{B(0,r)} |C\nu_2| \frac{dA}{\pi} = \int_{|z|<r} \left| \int_{|w|<r} \frac{d\nu(w)}{w-z} \right| \frac{dA(z)}{\pi}$$
$$\leqslant \int_{|w|<r} \int_{|z|<r} \frac{dA(z)}{\pi|w-z|} d|\nu|(w)$$
$$\leqslant 2r|\nu|(B(0,r)),$$

where we have used (2.1). Hence by (4.1)

$$\int_{B(0,r)} |C\nu_1 - a| \frac{dA}{\pi} \leqslant \int_{B(0,r)} |C\nu - a| \frac{dA}{\pi} + \int_{B(0,r)} |C\nu_2| \frac{dA}{\pi} \leqslant \frac{9}{32} r^2 |a|. \tag{4.3}$$

The Bergman space estimate ([10])

$$|f(z)| \leqslant \frac{1}{(1-|z|^2)^2} \int_{\mathbb{D}} |f| \frac{dA}{\pi}$$

valid for $f$ analytic in $\mathbb{D}$ and $z \in \mathbb{D}$, rescales to

$$|f(z)| \leqslant \frac{r^2}{(r^2 - |z|^2)^2} \int_{B(0,r)} |f| \frac{dA}{\pi}$$

for $f$ analytic in $B(0,r)$ and $z \in B(0,r)$. We apply this with $f = C\nu_1 - a$ to get

$$|C\nu_1(z) - a| \leqslant \frac{16}{9r^2} \int_{B(0,r)} |C\nu_1 - a| \frac{dA}{\pi}$$

for $|z| \leqslant \frac{1}{2}r$, and combining this with (4.3) we obtain that

$$|C\nu_1(z) - a| \leqslant \frac{1}{2}|a| \tag{4.4}$$

uniformly in $|z| \leqslant \frac{1}{2}r$.

We now define measures $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ by the formulas

$$\begin{aligned} \hat{\nu}(E) &= \frac{4}{ar}\nu\left(\frac{r}{2}E\right) \\ \hat{\nu}_1(E) &= \frac{4}{ar}\nu_1\left(\frac{r}{2}E\right) \\ \hat{\nu}_2(E) &= \frac{4}{ar}\nu_2\left(\frac{r}{2}E\right). \end{aligned}$$

A calculation shows that

$$C\hat{\nu}_1(z) = \frac{2}{a}C\nu_1\left(\frac{r}{2}z\right).$$

From (4.2) and (4.4) we now see that

$$\|\hat{\nu}_2\| < \epsilon_0 \tag{4.5}$$

and

$$|C\hat{\nu}_1(z) - 2| \leqslant 1 \qquad \text{for } |z| \leqslant 1. \tag{4.6}$$

Clearly $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$. By (4.5) and (4.6) we thus see that $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ satisfy the hypotheses of Theorem 3.4, so

$$|p(0)| \leqslant C_0 \int_{|w|<1} |p(w)C\hat{\nu}(w)| dA(w)$$

for every polynomial $p$. It follows that each polynomial $p$ satisfies

$$|p(0)| \leqslant \frac{8C_0}{|a|r^2} \int_{|w|<r} |p(w)C\nu(w)| dA(w)$$

and the Lemma follows. □

## 5. Analytic bounded point evaluations

Thomson shows that all bounded point evaluations for $P^t(\mu)$ either come from atoms of the measure $\mu$ or they are analytic bounded point evaluations, i.e. if $\lambda_0$ is a bounded point evaluation for $P^t(\mu)$ and if $\mu(\{\lambda_0\}) = 0$, then there are $C, \varepsilon > 0$ such that for all $\lambda \in B(\lambda_0, \varepsilon)$ and for all polynomials $p$ we have

$$|p(\lambda)|^t \le C \int |p|^t d\mu.$$

This fact also follows from our approach.

In fact, by a simple translation and rescaling argument Lemma 3.3 implies

**Lemma 5.1.** *There are absolute constants $\epsilon_2 > 0$ and $C_2 < \infty$ with the following property. Let $E \subset \operatorname{clos}\mathbb{D}$ with $\gamma(E) < \epsilon_2$. Then*

$$|p(\lambda)| \leqslant C_2 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \qquad \text{for all } p \in \mathcal{P} \text{ and all } |\lambda| < 1/2.$$

It is clear that Lemma 5.1 implies that the constants of Theorem 3.4 can be adjusted in such a way that the conclusion will be

$$|p(\lambda)| \leqslant C_0 \int_{|w|<1} |p(w)C\nu(w)| dA(w) \qquad \text{for all } p \in \mathcal{P} \text{ and all } |\lambda| < 1/2.$$

Thus the proof of Lemma 4.2 implies

**Theorem 5.2.** *There exists an absolute constant $C > 0$ such that for every $\nu \in M_c(\mathbb{C})$ and for every $\lambda_0 \in \mathbb{C}$ with $U_{|\nu|}(\lambda_0) = \int \frac{1}{|z-\lambda_0|} d|\nu|(z) < \infty$ there exist $r_0 > 0$ such that for all polynomials $p$, for all $0 < r \leqslant r_0$, and all $|\lambda - \lambda_0| < r/2$ we have*

$$|p(\lambda)C\nu(\lambda_0)| \leqslant \frac{C}{r^2} \int_{B(\lambda_0,r)} |p(z)C\nu(z)| dA(z).$$

*Here $r_0$ depends only on $|C\nu(\lambda_0)|$, $U_{|\nu|}(\lambda_0)$ and $U_{|\nu|}(\lambda_0, r)$ as $r \to 0$.*

Theorem 5.2 implies the statement made about analytic bounded point evaluations for $P^t(\mu)$ made at the beginning of this section. In fact, if $\mu$ is any compactly supported measure in $\mathbb{C}$, if $1 \leqslant t < \infty$, and if $\lambda_0$ is a bounded point evaluations for $P^t(\mu)$ with $\mu(\{\lambda_0\}) = 0$, then there is $h \in L^{t'}(\mu)$ such that

$$p(\lambda_0) = \int phd\mu$$

for all polynomials $p$. It then follows that the measure $d\nu(z) = (z - \lambda_0)hd\mu(z)$ satisfies the hypothesis of Theorem 5.2 and $C\nu(\lambda_0) \neq 0$. Thus Theorem 5.2 and Lemma 2.2 prove the desired result. We note that this reasoning together with the explanations near the end of Section 2 also shows that every bounded point evaluation for $R^t(\mu)$ that lies in the interior of the support of $\mu$ must either come from an atom of $\mu$ or be an analytic bounded point evaluation for $R^t(\mu)$.

## References

[1] Alexandru Aleman, Stefan Richter, and Carl Sundberg. Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces. *Ann. of Math.*, 169(2), 2009.

[2] Dzh. È. Brennan. Thomson's theorem on mean-square polynomial approximation. *Algebra i Analiz*, 17(2):1–32, 2005.

[3] James E. Brennan. Invariant subspaces and rational approximation. *J. Functional Analysis*, 7:285–310, 1971.

[4] James E. Brennan. Point evaluations, invariant subspaces and approximation in the mean by polynomials. *J. Funct. Anal.*, 34(3):407–420, 1979.

[5] James E. Brennan. The structure of certain spaces of analytic functions. *Comput. Methods Funct. Theory*, 8(1-2):625–640, 2008.

[6] Claes Fernstrøm. Bounded point evaluations and approximation in $L^p$ by analytic functions. In *Spaces of analytic functions (Sem. Functional Anal. and Function Theory, Kristiansand, 1975)*, pages 65–68. Lecture Notes in Math., Vol. 512. Springer, Berlin, 1976.

[7] John Garnett. *Analytic capacity and measure*. Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin, 1972.

[8] James E. Thomson. Approximation in the mean by polynomials. *Ann. of Math. (2)*, 133(3):477–507, 1991.

[9] Xavier Tolsa. Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003.

[10] Dragan Vukotić. A sharp estimate for $A^p_\alpha$ functions in $\mathbf{C}^n$. *Proc. Amer. Math. Soc.*, 117(3):753–756, 1993.

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