Solutions to HW 9

Chapter 5 #4 Let $\sigma(E) = \int_E |f|\,d\mu$, then σ is a measure, let $E_n = \{x \in X : |f(x)| > n\}$, then

$\bigcap_n E_n = \{x \in X : |f(x)| = \infty\}$. Since $f \in L^1(\mu)$

we have $\mu(\bigcap_n E_n) = 0$, hence $\sigma(\bigcap_n E_n) = 0$

and since $E_n \supseteq E_{n+1}$ and $\sigma(X) = \|f\|_1 < \infty$
$\sigma(E_n) \le \sigma(X)$ $\forall n$, hence $\sigma(E_n) \to \sigma(\bigcap_n E_n) = 0$

Let $\varepsilon > 0$, choose $n \ni \sigma(E_n) < \varepsilon/2$. Set

$\delta = \frac{\varepsilon}{2n}$. If $A \in \mathcal{M}$ with $\mu(A) < \delta$, then

$$\int_A |f|\,d\mu = \int_{A\cap E_n} |f|\,d\mu + \int_{A\cap E_n^c} |f|\,d\mu \le \sigma(A\cap E_n) + n\mu(A\cap E_n^c)$$

$$\le \sigma(E_n) + n\mu(A) < \frac{\varepsilon}{2} + n\delta = \varepsilon.$$

#6 <u>Examples:</u>

(1) If $\mu(X) < \infty$, then $s < r \Rightarrow L^r(\mu) \subseteq L^s(\mu)$

$f \in L^r(\mu) \Rightarrow \int |f|^s d\mu \le \left(\int 1^p d\mu\right)^{\frac{1}{p}} \left(\int |f|^{sq} d\mu\right)^{\frac{1}{q}}$

so, if $\frac{r}{s} > 1$, $\frac{1}{p} = 1 - \frac{1}{q}$, then $\|f\|_s^s \le \mu(X)^{\frac{1}{p}} \|f\|_r^s$

Hence $f \in L^r(\mu) \Rightarrow f \in L^s(\mu)$.

(2) $\ell^p = \ell^p(\mathbb{N})$ with counting measure.

~~Thm~~ $s < r \Rightarrow \ell^s \subseteq \ell^r$

If $\sum_n |a_n|^s < \infty$, then $a_n \to 0$, so $\|a_n\|_\infty \leq M$ for some $M < \infty$. Hence

$$\sum |a_n|^r = \sum_n |a_n|^s |a_n|^{r-s} \leq M^{r-s} \sum |a_n|^s < \infty$$

Hence $\{a_n\} \in \ell^s$ implies $\{a_n\} \in \ell^r$

(3) $L^p[0,\infty) = L^q[0,\infty) \Leftrightarrow p = q$

~~Pf~~ $\Leftarrow$ ✓

$\Rightarrow$: Let $p \neq q$, without loss of generality suppose $p < q$. Note that

$$\int_1^\infty \frac{1}{x^\alpha} dx < \infty \Leftrightarrow \alpha > 1, \quad \int_1^\infty \frac{1}{x} dx = +\infty$$

Let $f(x) = \chi_{[1,\infty)}(x) \left(\frac{1}{x}\right)^{\frac{1}{p}}$, then $f \notin L^p[0,\infty)$, but $f \in L^q[0,\infty)$. Similarly $\int_0^1 \frac{1}{x} dx = +\infty$ and $\int_0^1 \frac{1}{x^\alpha} dx < \infty$, if $\alpha < 1$, so set

$g(x) = \chi_{[0,1]}(x) \left(\frac{1}{x}\right)^{\frac{1}{q}}$, then $g \notin L^q$, $g \in L^p$.