3.6. Let $f(y) = \frac{1-e^{-y}}{y}$, then $f'(y) < 0$ for $y > 0$, so f is decreasing for $y>0$, and $\lim_{y\to 0^+} f(y) = 1$ (by l'Hospital's rule).

$g_n(x) = n\left(1-e^{-\frac{x}{n}}\right) = x\, f\left(\frac{x}{n}\right)$, hence $0 \le g_n(x) \le g_{n+1}(x) \to x$

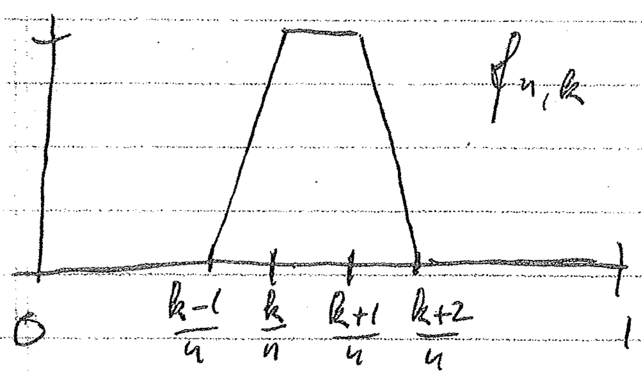
so by the monotone convergence theorem

$$\lim_{n\to\infty}\int_0^1 n\left(1-e^{-\frac{x}{n}}\right) dx = \int_0^1 \lim_{n\to\infty} n\left(1-e^{-\frac{x}{n}}\right) dx = \int_0^1 x\,dx = \frac{1}{2}$$

3.7. $f_n(x) = \chi_{[0,n]}(x)\,\frac{2-e^{-\frac{x}{n}}}{x^2}$. Then $0 \le f_n(x) \le \frac{2}{x^2}$ and $g(x) = \frac{2}{x^2}$, $g \in L^1((1,\infty))$, i.e. $\int_1^\infty \frac{2}{x^2}dx < \infty$. Hence by Lebesgue's dominated convergence theorem $\lim_{n\to\infty}\int_{[1,\infty)} f_n\, dm = \int_{[1,\infty)} \lim_{n\to\infty} f_n(x)\, dm$

$= \int_{[1,\infty)} \frac{1}{x^2}\, dx = 1$. (see class for why $\int_1^\infty \frac{2}{x}dx = \int_{[1,\infty)} \frac{2}{x}\, dm$).

3.8.



$$\begin{cases} n = 1, 2, 3, \dots \\ k = 0, 1, \dots n-1 \end{cases}$$

Then
$$\int_0^1 f_{n,k}\, dm = \begin{cases} \frac{2}{n} & k = 1, \dots n-2 \\ \frac{3}{2n} & k = 0 \text{ or } k = n-1 \end{cases}$$

so $\int_{[0,1]} f_{n,k}\, dm \to 0$ as $n \to \infty$

Let $\phi : \mathbb{N} \to \{(n,k) : n \in \mathbb{N},\ k = 0, \dots n-1\}$

be bijective, $g_j = f_{\phi(j)}$ satisfies the requirement.

4.1. Let μ be a Radon measure on $\mathbb{R}^n$

$$K = \{ x \in \mathbb{R}^n \ni \mu(B(x,r)) > 0 \ \forall r > 0 \}$$

Then K is closed.

Will show $V = \mathbb{R}^n \setminus K$ is open. If $x_0 \in V$, then $\exists r_0 > 0$ s.t. $\mu(B(x_0, r_0)) = 0$. If $x \in B(x_0, r_0)$ ~~and~~ then $\varepsilon = r_0 - \|x - x_0\| > 0$, and

$B(x, \varepsilon) \subseteq B(x_0, r_0)$ since if $y \in B(x, \varepsilon)$, then $\|x - y\| < \varepsilon$, hence $\|x_0 - y\| \le \|x_0 - x\| + \|x - y\|$

$$< \|x_0 - x\| + \varepsilon = r_0 .$$

Hence $\mu(B(x, \varepsilon)) \le \mu(B(x_0, r_0)) = 0$, so $x \in V$ and in fact $B(x_0, r_0) \subseteq V$. So V is open.

If $W \supseteq V$ ~~and~~ is open with $\mu(W) = 0$, then $\forall x \in W \ \exists r > 0 \ni B(x, r) \subseteq W$ hence $\mu(B(x,r)) \le \mu(W) = 0 \Rightarrow x \in V \Rightarrow W = V$.