# Solutions to HW5

12. <u>Claim</u>: $|\rho_A(x) - \rho_A(y)| \le d(x,y)$ $\forall x,y \in X$, Assume $A \neq \emptyset$

Let $\varepsilon > 0$. Then $\exists\, a \in A \ni d(y,a) \le \rho_A(y) + \varepsilon$. Then
$\rho_A(x) \le d(x,a) \le d(x,y) + d(y,a) \le d(x,y) + \rho_A(y) + \varepsilon$
$\Rightarrow \rho_A(x) - \rho_A(y) \le d(x,y) + \varepsilon$. True $\forall\, \varepsilon > 0$, hence
$\rho_A(x) - \rho_A(y) \le d(x,y)$. By symmetry
$\rho_A(y) - \rho_A(x) \le d(y,x) = d(x,y)$ so
$|\rho_A(x) - \rho_A(y)| \le d(x,y)$. Hence $\rho_A$ is uniformly continuous. Take $\delta = \varepsilon$.

(b) If $x \in F$, then $\rho_F(x) \leq d(x,x) = 0$ so this is true whether or not $F$ is closed.

<u>Ex</u>: $F = (0,1) \subseteq \mathbb{R}$, $x = 0 \notin F$, yet $\rho_F(x) = 0$.

So suppose $F$ is closed in $X$. We must show that if $x \in X$ with $\rho_F(x) = 0$, then $x \in F$. We will prove the contrapositive: If $x \notin F$, then $\rho_F(x) > 0$. Let $x \notin F$, i.e. $x \in F^c$ which is open. Hence $\exists\, \varepsilon > 0$ such that $B(x, \varepsilon) \subseteq F^c$. If $a \in F$, then $d(x,a) \geq \varepsilon$ because if $d(x,a) < \varepsilon$, then $a \in B(x,\varepsilon) \subseteq F^c$. Hence $\rho_F(x) = \inf\{d(x,a) : a \in F\} \geq \varepsilon$. In particular, $\rho_F(x) \neq 0$.

(c) The function $f(x) = \dfrac{\rho_{V^c}(x)}{\rho_{V^c}(x) + \rho_F(x)}$ was given. Note that the denominator could only be $0$, if $\rho_{V^c}(x)$ and $\rho_F(x)$ are both $0$. But $F$ and $V^c$ are closed, so by (b) this can only happen if $x \in V^c \cap F = \emptyset$. Thus $\rho_{V^c}(x) + \rho_F(x) \neq 0$, and $f$ is continuous. It is easy to check that $f = 1$ on $F$ and $f = 0$ on $V^c$.

Chapter 2

2. (a) If $a_n \in [-\infty, \infty]$, then $\inf_n a_n \in [-\infty, \infty]$, $\sup_n a_n \in [-\infty, \infty]$ and $-\inf_n a_n = \sup_n (-a_n)$ and $-\sup_n a_n = \inf_n (-a_n)$

so $\overline{\lim_{n\to\infty}} (-a_n) = \inf_{n\geq 1} \sup_{k\geq n} (-a_k) = \inf_{n\geq 1} \{-(\inf_{k\geq n} a_k)\}$

$= -\sup_{n\geq 1} \inf_{k\geq n} a_k = -\underline{\lim}_{n\to\infty} a_n$

(b) Let $a_n, b_n \in [-\infty, \infty]$ for each $n$.

If either $\overline{\lim}\, a_n = \infty$ or $\overline{\lim}\, b_n = \infty$, then since we are assuming the other $\overline{\lim} > -\infty$ there is nothing to prove. So we may assume that $\overline{\lim}\, a_n < +\infty$ and $\overline{\lim}\, b_n < +\infty$. Then $\exists N\ \forall n \geq N$ $a_n < \infty$, $b_n < \infty$ and in fact, $a_n \leq (\overline{\lim_{k\to\infty}} a_k) + 1$, $b_n \leq \overline{\lim_{k\to\infty}} b_k + 1$.

If $\overline{\lim_{n\to\infty}} a_n = -\infty$, then $\underline{\lim}\, a_n = -\infty$, i.e. $a_n \to -\infty$,

so $a_n + b_n \leq a_n + \overline{\lim_{k\to\infty}} b_k + 1$ $\forall n \geq N$ and

$\overline{\lim_{n\to\infty}} a_n + b_n = -\infty$. So we may also assume

$\overline{\lim}\, a_n > -\infty$, $\overline{\lim}\, b_n > -\infty$.

Since $\{\sup_{k\geq n} a_k\}_{n=1}^{\infty}$ is monotonically nonincreasing

we have $\inf_{n\geq 1} \sup_{k\geq n} a_k = \lim_{n\to\infty} \sup_{k\geq n} a_k$, and

$\inf_{n\geq 1} \sup_{k\geq n} b_k = \lim_{n\to\infty} \sup_{k\geq n} b_k$ and these limits are $\in \mathbb{R}$,

Thus $\lim_{n\to\infty} \left(\sup_{k\geq n} a_k + \sup_{k\geq n} b_k\right) = \lim_{n\to\infty} \sup_{k\geq n} a_k + \lim_{n\to\infty} \sup_{k\geq n} b_k$

For $n \geq N$ we have

$$\sup_{k\geq n}(a_k+b_k) \leq \cancel{\sup_{k\geq n}(a_k+b_k)} \leq \left(\sup_{k\geq n} a_k\right)+\left(\sup_{k\geq n} b_k\right)$$

so $\inf_{m\geq 1} \sup_{k\geq m}(a_k+b_k) \leq \sup_{k\geq n}(a_k+b_k) \quad \forall n \geq N$

$$\overline{\lim_{n\to\infty}}(a_n+b_n) \leq \lim_{n\to\infty} \sup_{k\geq n}(a_k+b_k) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$$

ex: $a_n = (-1)^n$, $b_n = -(-1)^n$, $a_n+b_n = 0 \ \forall n$

$\overline{\lim}\, a_n = 1$, $\overline{\lim}\, b_n = 1$

(c) If $a_n \leq b_n \ \forall n$, then $\inf_{k\geq n} a_k \leq \inf_{k\geq n} b_k$

$$\leq \sup_{n\geq 1} \inf_{k\geq n}(b_k)$$

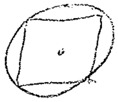
$$= \underline{\lim}_{n\to\infty} b_n$$

True for all $n \geq 1$, so $\sup_{n\geq 1} \inf a_k \leq \underline{\lim}_{n\to\infty} b_n$,

i.e. $\underline{\lim}\, a_n \leq \underline{\lim}\, b_n$

3 (a.) Set $f(x) = g_{V^c}(x)$, $x \in \mathbb{R}^n$, so $f$ is continuous and $K_j = f^{-1}([0,j]) \cap f^{-1}([\frac{1}{j},\infty))$, so $K_j$ is closed (since inverse images of closed sets are closed). Since $K_j$ is bounded, it follows that $K_j$ is compact. It is easy that $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots \subseteq V$, so $\bigcup_{j=1}^{\infty} K_j \subseteq V$. If $x \in V$, then $g_{V^c}(x) = \varepsilon > 0$, since $V^c$ is closed, so $x \in K_j$ for all $j$ with $j \geq \|x\|$ and $\frac{1}{j} \leq \varepsilon$.

(b). Choose $K_j$'s as above. For each $x \in K_j$ $\exists$ $\varepsilon_{x,j} \ni B(x, \varepsilon_{x,j}) \subseteq V$ (since $V$ is open). Let $R_j(x)$ be the largest inscribed rectangle in $B(x, \varepsilon_{x,j})$ with center at $x$, so $R_j(x) = \{y \in \mathbb{R}^n : |x_i - y_i| < \frac{1}{\sqrt{n}} \varepsilon_{x,j},\ i = 1, \dots, n\}$, then $R_j(x) \subseteq V$ is an open rectangle containing $x$. Since $K_j \subseteq \bigcup_{x \in K_j} R_j(x)$



finitely many of those $R_j(x)$ will cover $K_j$. Say $x_{j1}, \dots, x_{jn_j} \in K \ni K_j \subseteq \bigcup_{i=1}^{n_j} R_j(x_{ji})$.

Then $\{R_j(x_{ji}) : j = 1, 2, \dots,\ 1 \leq i \leq n_j\}$ is the collection of open rectangles that covers $V$.

4. ~~[illegible]~~) Let $s, t$ be measurable simple functions range $s = \{\alpha_1, \ldots \alpha_n\}$, range $t = \{\beta_1, \ldots \beta_m\} \subseteq [0, \infty)$
$E_i = s^{-1}(\{\alpha_i\})$, $F_j = t^{-1}(\{\beta_j\})$, then $E_i \cap E_j = \emptyset \ \forall i \neq j$ and $\bigcup_{i=1}^{n} E_i = X$. Similarly $\bigcup_{j=1}^{m} F_j = X$, $F_i \cap F_j = \emptyset \ \forall i \neq j$.

Let $1 \le i \le n$, $1 \le j \le m$, $A_{ij} = E_i \cap F_j$.

~~Then~~ If $\mu(A_{ij}) \neq 0$, then $A_{ij} \neq \emptyset$, so $\exists x \in A_{ij}$ and $\alpha_i = s(x) \le t(x) \le \beta_j$. Hence if $E \in \mathfrak{M}$

$$\sum_{i=1}^{n} \alpha_i \mu(E_i \cap E) = \sum_{i=1}^{n} \alpha_i \mu\left((E_i \cap E) \cap \left(\bigcup_{j=1}^{m} F_j\right)\right)$$

$$= \sum_{i=1}^{n} \alpha_i \mu\left(\bigcup_{j=1}^{m} (E_i \cap E \cap F_j)\right) \underset{\uparrow \ F_j\text{'s are disjoint}}{=} \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \mu(E_i \cap E \cap F_j)$$

$$= \sum_{i,j} \alpha_i \mu(A_{ij} \cap E) = \sum_{\mu(A_{ij})=0} \alpha_i \mu(A_{ij} \cap E) + \sum_{\substack{\mu(A_{ij}) \\ \neq 0}} \alpha_i \mu(A_{ij} \cap E)$$

$$\le 0 + \sum_{\substack{\mu(A_{ij}) \\ \neq 0}} \beta_j \mu(A_{ij} \cap E) = \sum_{\mu(A_{ij})=0} \beta_j \mu(A_{ij} \cap E) + \sum_{\mu(A_{ij}) \neq 0} \beta_j \mu(A_{ij} \cap E)$$

$$= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n} \mu(E_i \cap E \cap F_j) \underset{\uparrow \ E_i\text{'s are disjoint}}{=} \sum_{j=1}^{m} \beta_j \mu\Big(\underbrace{\bigcup_{i=1}^{n} E_i}_{X} \cap E \cap F_j\Big)$$

$$= \sum_{j=1}^{m} \beta_j \mu(F_j \cap E)$$

This should have been proven in class before $\int_E f \, d\mu$ was defined for more general functions $f$.