Math 545, Fall 2011
Solutions to HW #4

#) μ^* is an outer measure

(11a) (i) Since $\emptyset \in \mathcal{M}$ we have $\mu^*(\emptyset) \leq \mu(\emptyset) = 0$, so $\mu^*(\emptyset) = 0$

(ii) If $E \subseteq F$, then whenever $B \in \mathcal{M}$ with $F \subseteq B$ we have $E \subseteq B$, so $\mu^*(E) \leq \mu(B)$. Now take the inf over such B to get $\mu^*(E) \leq \mu^*(F)$.

(iii) Let $A_1, A_2, \ldots \subseteq X$. <u>To show</u>: $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$

We may suppose $\mu^*(A_n) < \infty \ \forall n$, otherwise this is clear.

Let $\varepsilon > 0$. For each n choose $B_n \in \mathcal{M}$ s.t. $A_n \subseteq B_n$, $\mu^*(A_n) + \frac{\varepsilon}{2^n} > \mu(B_n)$. This is possible by the definition of $\mu^*(A_n)$. Then $\bigcup_n A_n \subseteq \bigcup_n B_n \in \mathcal{M}$

so $\mu^*(\bigcup_n A_n) \leq \mu(\bigcup_n B_n) \underset{\substack{\uparrow \\ \text{subadditivity of } \mu}}{\leq} \sum_n \mu(B_n) \leq \sum_n \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right)$

so $\mu^*(\bigcup_n A_n) \leq \left(\sum_n \mu^*(A_n)\right) + \varepsilon$. $\varepsilon > 0$ arbitrary, nothing else depends on $\varepsilon \Rightarrow \mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$.

(b) Let $E \in \mathcal{M}$, $A \subseteq X$. <u>To show</u>: $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Spse $\mu^*(A) < \infty$, let $A \subseteq B$, $B \in \mathcal{M}$, then $A \cap E \subseteq B \cap E \in \mathcal{M}$ (since $B, E \in \mathcal{M}$) and $A \cap E^c \subseteq B \cap E^c \in \mathcal{M}$, hence by definition of μ^* $\mu^*(A \cap E) \leq \mu(B \cap E)$, $\mu^*(A \cap E^c) \leq \mu(B \cap E^c)$

so $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu(B \cap E) + \mu(B \cap E^c)$

$\underset{\substack{\uparrow \\ \text{by additivity of } \mu}}{=} \mu(B)$

Now take inf over such B, get

$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$.

(c) If $A \in \mathcal{M}$, then $A \subseteq A \in \mathcal{M}$, so $\mu^*(A) \leq \mu(A)$, by definition. If $A \subseteq B$, $B \in \mathcal{M}$, then by monotonicity of μ: $\mu(A) \leq \mu(B)$.

Take inf over such B, get $\mu(A) \leq \mu^*(A)$.

#13 ① claim: If $A \in \mathcal{L}_F$, $\varepsilon > 0$, then $\exists V$ open $A \subseteq V$ and $m_F(V \setminus A) \leq \varepsilon$.

case 1: $m_F(A) < \infty$.

By definition of m_F $\exists\ a_n < b_n \ni A \subseteq \bigcup_{n\geq 1} (a_n, b_n]$ and $m_F(A) + \frac{\varepsilon}{2} \geq \sum_{n\geq 1} F(b_n) - F(a_n)$. For each n choose $\delta_n > 0$ s.t. $F(b_n+\delta_n) - F(b_n) < \frac{\varepsilon}{2^{n+1}}$ (by right continuity of F).

Set $V = \bigcup_{n\geq 1} (a_n, b_n+\delta_n)$. Then $A \subseteq V$, V open and

$$m_F(V) \leq m_F\left(\bigcup_{n\geq 1} (a_n, b_n+\delta_n]\right) \leq \sum_{n\geq 1} m_F((a_n, b_n+\delta_n])$$

$$= \sum_{n\geq 1} F(b_n+\delta_n) - F(a_n) = \sum_{n\geq 1} F(b_n+\delta_n) - F(b_n) + \sum_{n\geq 1} F(b_n) - F(a_n)$$

$$\leq \sum_{n\geq 1} \frac{\varepsilon}{2^{n+1}} + m_F(A) + \frac{\varepsilon}{2} \leq \varepsilon + m_F(A) < \infty$$

Hence $m_F(V \setminus A) \leq \varepsilon$ (here we used $m_F(V) < \infty$, $m_F(A) < \infty$)

case 2: Any $A \in \mathcal{L}_F$.

Set $A_n = A \cap (-n, n]$, so $m_F(A_n) \leq m_F((-n,n])$ $= F(n) - F(-n) < \infty$ for each $n \in \mathbb{N}$. Also $A_n \subseteq A_{n+1} \subseteq A_{n+2} \ldots$ and $\bigcup_{n\geq 1} A_n = A$, By case 1 for each n choose V_n open $\ni A_n \subseteq V_n$ and $m_F(V_n \setminus A_n) < \frac{\varepsilon}{2^n}$

Since $A^c \subseteq A_n^c$ we have $V_n \setminus A \subseteq V_n \setminus A_n$ hence $m_F(V_n \setminus A) < \frac{\varepsilon}{2^n}$. Set $V = \bigcup_n V_n$, then V is open and since $\forall n$ $V_n \supseteq A_n$ we have $V \supseteq \bigcup_n A_n = A$.

Furthermore, $m_F(V \setminus A) = m_F((\bigcup_n V_n) \setminus A = m_F(\bigcup_n (V_n \setminus A))$

$$\leq \sum_n m_F(V_n \setminus A) \leq \sum \frac{\varepsilon}{2^n} = \varepsilon.$$

(2) Take $\varepsilon = \frac{1}{n}$, so by (1) $\exists\, V_n$ open $\ni m_F(V_n \setminus A) \to 0$
$A \subseteq V_n$, clearly then $m_F(V_n) \to m_F(A)$. By setting
$W_n = \bigcap_{k=1}^{n} V_k$ we may assume $W_n \supseteq W_{n+1} \supseteq W_{n+2} \supseteq \cdots$, $m_F(W_n \setminus A) \to 0$.

(3) Let $A \in \mathcal{L}_F$, then $\exists\, K_n$ compact, $K_n \subseteq A \ni m_F(K_n) \to m_F(A)$
~~As above choose~~ Since $A^c \in \mathcal{L}_F$ we may choose
$W_n \supseteq W_{n+1} \supseteq \cdots \ni m_F(W_n \setminus A^c) \to 0$. Since $W_n \setminus A^c = A \setminus W_n^c$
we have $m_F(A \setminus E_n) \to 0$ where $E_n = W_n^c$ is closed.
Set $K_n = E_n \cap [-n, n]$, so K_n is compact,
$K_n \subseteq K_{n+1} \subseteq \cdots$ since $E_n \subseteq E_{n+1} \subseteq \cdots$ $\forall n$. Thus,
$m_F(K_n) \to m_F(\bigcup_n K_n) = m_F(\bigcup_n E_n)$. We have $\forall n$
$K_n \subseteq E_n \subseteq A$ and $m_F(A) = m_F(A \setminus E_n) + m_F(E_n)$
So $m_F(E_n) \to m_F(A) \geq m_F(\bigcup_{k\geq 1} E_k) \geq m_F(E_n)$
Thus, $m_F(\bigcup_{k\geq 1} E_k) = m_F(A)$ and $m_F(K_n) \to m_F(A)$.

Also note that $E = \bigcup_{k\geq 1} E_k$ is a F_σ-set, contained in A with
$m_F(E) = m_F(A)$.
Of course, $G = \bigcap_n W_n$ is a G_δ-set with $A \subseteq G$ and
$m_F(G) = m_F(A)$.

Note that this does not mean that $E = A = G$!

#(d)) ...

#14 a. $\mu(\emptyset) = \mu^*(\emptyset) = 0$

<u>To show</u>: If $\{E_n\}_n \subseteq \mathfrak{M}$ are mutually disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

μ^* is an outer measure, hence $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ hence $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$ and we have to show the other inequality. Since μ is finitely additive we have for each $N \in \mathbb{N}$: $\sum_{n=1}^{N} \mu(E_n) = \mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu^*\left(\bigcup_{n=1}^{N} E_n\right) \leq \mu^*\left(\bigcup_{n=1}^{\infty} E_n\right)$

so $\forall N$ $\sum_{n=1}^{N} \mu(E_n) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$. Now let $N \to \infty$

$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$. //

#15 a. $A = [0,1] \cap \mathbb{Q}$, then $\bar{A} = [0,1] = B = \bar{B}$

$B = [0,1]$ $\quad m(A) = 0$, $m(B) = 1$

(b) Suppose that $\bar{A} \neq \bar{B}$. Since $A \subseteq B \subseteq \bar{B}$ we have $\bar{A} \subseteq \bar{B}$. Hence $\exists\, x \in \bar{B} \setminus \bar{A}$. Since $\bar{A}^c$ is open $\exists\, \varepsilon > 0 \ni (x-\varepsilon, x+\varepsilon) \cap \bar{A} = \emptyset$. Since $x \in \bar{B}$ we must have $(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}) \cap B \neq \emptyset$. Let $x_0 \in B \cap (x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$. Then $(x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}) \cap \bar{A} = \emptyset$, so $(x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}) \cap \bar{B} \subseteq \bar{B} \setminus \bar{A}$, hence $m((x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}) \cap \bar{B}) \leq m(\bar{B} \setminus \bar{A}) = 0$ contradicting the hypothesis.
$(= m(\bar{B}) - m(\bar{A}) = 0$ since $m(\bar{B}) < \infty)$