Math 545 HW3 Solutions

#7. Let $A_n = (-n, n]$, then $\mathbb{R} = \bigcup_n (-n, n]$

and $m_F(A_n) = F(n) - F(-n) < \infty$ for each n. so $(\mathbb{R}, \mathcal{L}_F, m_F)$ is σ-finite.

It is finite $\Leftrightarrow$ $m_F(\mathbb{R}) < \infty$

$m_F(\mathbb{R}) = m_F(\bigcup_n (-n, n]) = \lim_{n\to\infty} m_F((-n, n]) = \lim_{n\to\infty} F(n) - F(-n)$

Hence it is a finite measure space

$\Leftrightarrow$ $\lim_{n\to\infty} F(n) = A < \infty$ and

$\lim_{n\to-\infty} F(-n) = B > -\infty$.

(8) at the end

(9) $\Rightarrow$. Assume that $(X, \mathcal{M}, \mu)$ is locally finite.

<u>To show</u>: μ has no ∞ atoms

Let $C \in \mathcal{M}$. If $\mu(C) < \infty$, then C is not an ∞ atom. If $\mu(C) = \infty$, then since $(X, \mathcal{M}, \mu)$ is locally finite: $\sup\{\mu(B) : B \subseteq C, \mu(B) < \infty\} = \infty$ hence $\exists B \subseteq C$ with $\mu(B) < \infty$ and $\mu(B) > 0$, so C is not an ∞ atom.

$\Leftarrow$: Suppose (X, m, μ) has no infinite atoms.
Let $C \in m$. To show
$\mu(C) = \sup\{\mu(B) : B \subseteq C, \mu(B) < \infty\}$.
If $\mu(C) < \infty$, then clearly we have equality (this follows from the monotonicity of the measure).
If $\mu(C) = \infty$, then C is not an infinite atom.
Suppose $\mu(C) \neq \sup\{\mu(B) : B \subseteq C, \mu(B) < \infty\}$,
$B \in m$
then $\sup\{\mu(B) : B \subseteq C, \mu(B) < \infty\} = M < \infty$.
$B_n \in m$
Let $B_n \subseteq C$, $\mu(B_n) < \infty$ and $\mu(B_n) \to M$
(the B_n's exist by def. of sup). Then $B = \bigcup_{n=1}^{\infty} B_n \subseteq C$
$B \in m$. Also set $A_k = \bigcup_{n=1}^{k} B_n$, so $A_k \subseteq C$ and $A_k \in m$

$$\mu(A_k) = \mu\left(\bigcup_{n=1}^{k} B_n\right) \leq \sum_{n=1}^{k} \mu(B_n) \leq kM < \infty,$$

so $\mu(A_k) < \infty$ and hence $\mu(A_k) \leq M$ and

$B = \bigcup_{k=1}^{\infty} A_k$ so by a theorem from class

$\lim_{k\to\infty} \mu(A_k) = \mu(B)$. Also $B_k \subseteq A_k \subseteq C$, so

$\mu(B_k) \leq \mu(A_k) \leq M$, so $\mu(B) = M < \infty$.

$\mu(C) = \infty$, so $\mu(C \setminus B) = \infty$. $C \setminus B$ is not an infinite

atom, so $\exists\, A \subseteq C \setminus B$ $\mu(A) > 0$, $\mu(A) < \infty$, and
$A \cup B \subseteq C$, $\mu(A \cup B) = \mu(A) + \mu(B) > M$ contradiction

8. (a) Let $E \subseteq \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, then

$$m^*(\alpha E + \beta) = |\alpha|\, m^*(E)$$

Pf: $\alpha = 0$ (check out separately)

$\alpha \neq 0$. Set $f(x) = \alpha x + \beta$. f is 1-1 and onto $\mathbb{R} \to \mathbb{R}$

If $E \subseteq \bigcup_n (a_n, b_n]$, $a_n < b_n$, then

$$f(E) \subseteq f\Big(\bigcup_n (a_n, b_n]\Big) = \bigcup_n f((a_n, b_n]).$$

We know that $f((a_n, b_n]) = \begin{cases} (\alpha a_n + \beta, \alpha b_n + \beta] & \alpha > 0 \\ [\alpha b_n + \beta, \alpha a_n + \beta) & \alpha < 0 \end{cases}$

so $\underbrace{f((a_n, b_n])}_{\text{Borel set}} \in \mathcal{L}_F$ and $m^*(f((a_n, b_n]))$

$= m(f((a_n, b_n])) = |\alpha|(b_n - a_n)$ (Done explicitly for $\alpha > 0$, and it is easy to see for $\alpha < 0$ also).

m^* is an outer measure, hence

$$m^*(f(E)) \le \sum_n m^*(f((a_n, b_n])) = |\alpha| \sum_n (b_n - a_n)$$

Now take the inf over those $(a_n, b_n]$ to get

$$m^*(f(E)) \le |\alpha|\, m^*(E)$$

This is valid for all $E \subseteq \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, $|\alpha| \neq 0$.

Next set $F = \alpha E + \beta$, $\alpha' = \frac{1}{\alpha}$, $\beta' = -\frac{\beta}{\alpha}$

So $g(x) = \alpha' x + \beta' = \frac{x-\beta}{\alpha}$, so $E = g(F)$

Then $m^*(E) = m^*(g(E))$

$\leq \frac{1}{|\alpha|} m^*(F) = \frac{1}{|\alpha|} m^*(\alpha E + \beta)$

So $|\alpha| m^*(E) = m^*(\alpha E + \beta)$ $\forall E \subseteq \mathbb{R}, \alpha, \beta \in \mathbb{R}$.

(b) If $E \in \mathcal{L}$, then $\alpha E + \beta \in \mathcal{L}$.

<u>Note</u>: Since f is 1-1 and onto we have

$$f(A' \cap E) = f(A') \cap f(E), \quad f(E^c) = f(E)^c$$

So, if $A \subseteq \mathbb{R}$ and $A' = g(A)$, then

$$A \cap f(E) = f(A') \cap f(E) = f(A' \cap E)$$
$$A \cap f(E)^c = f(A') \cap f(E^c) = f(A' \cap E^c)$$

and

$$m^*(A) = m^*(f(A')) = |\alpha| m^*(A')$$

$$= |\alpha| \left(m^*(A' \cap E) + m^*(A' \cap E^c) \right)$$

$$= m^*(f(A' \cap E)) + m^*(f(A' \cap E^c))$$

$$= m^*(A \cap f(E)) + m^*(A \cap f(E)^c)$$

so $f(E) \in \mathcal{L}$