Math 545, Solutions to HW #2

(3a) $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ-set in $\mathbb{R}$.

pf: $\forall q \in \mathbb{Q}$ the set $V_q = \mathbb{R} \setminus \{q\}$ is open and clearly $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} V_q$. Since $\mathbb{Q}$ is countable $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ-set.

(3b) $\mathbb{Q}$ is <u>not</u> a G_δ-set in $\mathbb{R}$

Suppose $\mathbb{Q}$ is a G_δ-set, i.e. spse $\exists W_n$, open $n = 1, 2, \ldots$ such that $\mathbb{Q} = \bigcap_{n=1}^{\infty} W_n$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \subseteq W_n$ for each n, we have that each W_n is dense in $\mathbb{R}$.

Since $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ-set $\mathbb{R} \setminus \mathbb{Q} = \bigcap_n V_n$, V_n open and dense in $\mathbb{R}$. Then

$$\emptyset = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \left(\bigcap_n W_n\right) \cap \left(\bigcap_n V_n\right)$$

is a countable intersection of dense open sets, but this contradicts the Baire Category Theorem.

#4. to show: measures are subadditive

Let $A_n \in \mathcal{M}$ to show: $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$

Pf: As in the proof of Theorem 1.4 for the given $A_n \in \mathcal{M}$ we construct $E_n \in \mathcal{M}$ that are mutually disjoint and satisfy $\bigcup_n A_n = \bigcup_n E_n$ and $E_n \subseteq A_n$ for each n (see page 3 of the notes). Then

$$\mu(\bigcup_n A_n) = \mu(\bigcup E_n) \overset{\mu \text{ is a measure}}{=} \sum_n \mu(E_n)$$

$$\leq \sum_n \mu(A_n) \text{ by the monotonicity of } \mu.$$

#5. Let μ be a measure on the Borel subsets of $\mathbb{R}$ with $\mu(K) < \infty$ $\forall K$ compact. Set

$$F(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$$

(a) $\forall a < b$ $\mu((a,b]) = F(b) - F(a)$

First note that $\mu((c,d]) \leq \mu([c,d]) < \infty$ $\forall c < d$, $c, d \in \mathbb{R}$. Now consider 3 cases for $a < b$:

① $0 \leq a < b$: If $a = 0$, then $F(b) - F(a) = F(b) = \mu(0,b] = \mu(a,b]$.

If $a > 0$, then $F(b) - F(a) = \mu(0,b] - \mu((0,a]) = \mu((a,b])$ since $(a,b] = (0,b] \setminus (0,a]$.

(2) $a < 0 < b$ and (3) $a < b \leq 0$ are similar.

Thus $F(b) - F(a) = \mu((a,b])$ for all $a, b \in \mathbb{R}$, $a < b$

#(5b) Hence if $x < y$, then $F(y) - F(x) = \mu((x,y]) \geq 0$, so F is nondecreasing.

Let $b \in \mathbb{R}$. To show: F is cont. from the right at b, i.e. $\lim_{x \to b+} F(x) = F(b)$. Step 1: Set $A_n = (b, b+\frac{1}{n}]$, then

$A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) = \mu(b, b+1] \leq \mu[b, b+1] < \infty$, so $\mu(A_n) \to \mu(\bigcap_{n=1}^{\infty} A_n)$ by a Theorem from class.

$\bigcap A_n = \bigcap_{n=1}^{\infty} (b, b+\frac{1}{n}) = \emptyset$. Hence

$F(b+\frac{1}{n}) - F(b) = \mu((b, b+\frac{1}{n}]) = \mu(A_n) \xrightarrow{n \to \infty} \mu(\bigcap_{j=1}^{\infty} A_j) = \mu(\emptyset) = 0$

So $\lim_{n \to \infty} F(b+\frac{1}{n}) = F(b)$. Step 2: If $F(x)$ is

nondecreasing and if $x_k \in \mathbb{R}$, $x_k \geq 0$ $\forall k$ $x_k \to 0$,

and if $\lim_{n \to \infty} F(b+\frac{1}{n}) = F(b)$, then $\lim_{k \to \infty} F(b+x_k) = F(b)$

(advanced calculus fact). Let $\varepsilon > 0$, then $\exists n_0 \in \mathbb{N}$ $\ni 0 \leq F(b+\frac{1}{n_0}) - F(b) < \varepsilon$. $x_k \to 0$ so $\exists K_0 \in \mathbb{N}$ such

that $\forall k \geq K_0$ $0 \leq x_k < \frac{1}{n_0}$. Thus $F(b+x_k) \leq F(b+\frac{1}{n_0})$

and $0 \leq F(b+x_k) - F(b) \leq F(b+\frac{1}{n_0}) - F(b) < \varepsilon$ $\forall k \geq K_0$

5c The proof of this is similar to the continuity proof of (5b).

Fact: Since $F(x)$ is nondecreasing for all $b \in \mathbb{R}$

$F(b^-) = \lim_{x \to b^-} F(x) = \sup\{F(x) : x < b\}$ exists.

The size of the jump at b is $F(b) - F(b^-)$.

We have $\lim_{n\to\infty} F(b - \frac{1}{n}) = F(b^-)$ (since the limit ① exists (by the fact, which I did not prove here))

Set $B_n = (b - \frac{1}{n}, b]$, then $\bigcap_{n=1}^{\infty} B_n = \{b\}$.

$\mu(B_1) = \mu(b-1, b] \leq \mu([b-1, b]) < \infty$, thus by a theorem from class $\mu(B_n) \to \mu(\bigcap_{n=1}^{\infty} B_n) = \mu(\{b\})$

But $\mu(B_n) = F(b) - F(b - \frac{1}{n}) \Rightarrow F(b) - F(b^-)$ as $n \to \infty$

hence $\mu(\{b\}) = F(b) - F(b^-)$ the size of the jump

and $\mu(\{b\}) = 0 \Leftrightarrow F(b) = F(b^-) \Leftrightarrow F$ is cont. at b

(since we already know the continuity on the right)

6 Let X be uncountable, let $E \subseteq X$

$$\mu^*(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is uncountable} \end{cases}$$

<u>claim</u>: μ^* is an outer measure

(a) $\mu^*(\emptyset) = 0$ since $\emptyset$ is countable

(b) If $A \subseteq B$, then we ~~have 2 case~~ consider 2 cases: B countable, B uncountable. If B is countable, then A is countable, so $\mu^*(A) = \mu^*(B) = 0$. If B is not countable, then $\mu^*(B) = 1$ and no matter what A is we must have $\mu^*(A) \leq 1 = \mu^*(B)$.

(c) Let $A_n \subseteq X$. <u>Case 1</u>: ~~cond. of the~~ all A_n's are countable, $\mu^*(A_n) = 0 \ \forall n$. Then $\bigcup_{n=1}^{\infty} A_n$ is countable and $\mu^*(\bigcup_n A_n) = 0 \leq \sum_n \mu^*(A_n)$.

<u>Case 2</u>: If one A_n (or more of them) is uncountable, then $\sum_n \mu^*(A_n) \geq 1$. Hence $\mu^*(\bigcup_n A_n) \leq 1 \leq \sum \mu^*(A_n)$

<u>claim</u>: $E \subseteq X$ is μ^*-measurable $\Leftrightarrow$ either E or E^c is countable

<u>pf</u>: Spse either E or E^c is countable and $A \subseteq X$.

<u>Case 1</u>: A countable $\Rightarrow \mu^*(A \cap E) = \mu^*(A \cap E^c) = \underbrace{\mu^*(A)}_{=0}$

so $\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$

<u>Case 2</u>: A ~~count~~ uncountable, so $\mu^*(A) = 1$

If E is countable, then $A \cap E$ is countable and since $A = (A \cap E) \cup (A \cap E^c)$ we must have that $A \cap E^c$ is uncountable

so $\mu^*(A \cap E) + \mu^*(A \cap E^c) = 0 + 1 = 1 = \mu^*(A)$.

The case where E^c is countable is very similar to this.

This shows that if E or E^c is countable, then E is measurable.

Now assume that neither E nor E^c are countable, so both are uncountable. Take $A = X$. Then

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(E) + \mu^*(E^c) = 2$$

$> 1 = \mu^*(X)$, so E is not μ^*-measurable