Math 545, HW # 2

due 9-2-11

3. (a) Prove that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ form a G_δ-set in $\mathbb{R}$.

(b) Prove that the rational numbers $\mathbb{Q}$ do not form a G_δ-set in $\mathbb{R}$.

Hint: You may use without proof the following consequence of the Baire category theorem: The intersection of a collection of countably many dense open subsets of $\mathbb{R}$ is dense in $\mathbb{R}$.

4. Prove that measures are subadditive. More precisely, let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

5. Let μ be a measure on the Borel subsets of $\mathbb{R}$ such that $\mu(K) < \infty$ for every compact set $K \subseteq \mathbb{R}$. Define a function $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \mu((0, x])$ if $x > 0$, $F(0) = 0$, and $F(x) = -\mu((x, 0])$ if $x < 0$.

(a) Show that for all $a, b \in \mathbb{R}, a < b$ one has $\mu((a, b]) = F(b) - F(a)$.

(b) Show that F is nondecreasing and continuous from the right at each $b \in \mathbb{R}$.

(c) Let $x_0 \in \mathbb{R}$. Show that $\mu(\{x_0\}) \neq 0$ if and only if F is discontinuous at x_0. In fact, show that $\mu(\{x_0\})$ equals the size of the "jump" that F has at x_0. You need to define what is meant by "size of the jump".

If $\mu(\{x_0\}) \neq 0$, then we say that μ has a point mass at x_0.

6. Let X be an uncountable set. For $E \subseteq \mathcal{P}(\mathcal{X})$ set $\mu^*(E) = 0$ if E is finite or countably infinite and $\mu^*(E) = 1$ if E uncountable. Show that μ^* is an outer measure and determine the μ^*-measurable sets (with proof).