Math 545, HW # 1

due Friday 8-26-11

1. Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of a set X. The *superior limit of* $\{E_n\}_{n\in\mathbb{N}}$ is the set consisting of those points which belong to infinitely many E_n. The superior limit of $\{E_n\}_{n\in\mathbb{N}}$ is denoted by $\limsup_{n\to\infty} E_n$. Similarly, the *inferior limit of* $\{E_n\}_{n\in\mathbb{N}}$, $\liminf_{n\to\infty} E_n$, is the set consisting of all those points which belong to all but a finite number of the E_n. Clearly, $\liminf_{n\to\infty} E_n \subseteq \limsup_{n\to\infty} E_n$.

Prove:

(a) $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n\right)$ and $\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n\right)$.

(b) $(\liminf_{n\to\infty} E_n)^c = \limsup_{n\to\infty} E_n^c$ and $(\limsup_{n\to\infty} E_n)^c = \liminf_{n\to\infty} E_n^c$.

(c) If $\mathcal{M}$ is a σ-algebra such that $E_n \in \mathcal{M}$ for every $n \in \mathbb{N}$, then $\liminf_{n\to\infty} E_n$ and $\limsup_{n\to\infty} E_n \in \mathcal{M}$.

Hint: It would be good to start by formalizing the definitions. For example,

$$x \in \limsup_{n\to\infty} E_n \;\Leftrightarrow\; \exists\ n_1 < n_2 < n_3 < \ldots \forall j \in \mathbb{N} : x \in E_{n_j}.$$

2. The following will be Proposition 1.22. Prove it.

Suppose $F : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and continuous from the right at each point. Let $n \in \mathbb{N}$ or $n = \infty$. Then, if $a, b, a_i, b_i \in \mathbb{R}$ with $a < b, a_i < b_i$ for all $i \in \mathbb{N}$ and $(a, b] \subseteq \bigcup_{i=1}^{n} (a_i, b_i]$, then $F(b) - F(a) \leq \sum_{i=1}^{n} F(b_i) - F(a_i)$.

Hint: To warm up, first try to prove this for finitely many intervals ($n = 2$ or $n = 3$). For this it will be necessary to order the intervals somehow. When $n = \infty$, then in general the intervals cannot be ordered any more. For example, for all even i the interval $(a_i, b_i]$ may be contained in $(-\infty, 0]$ and for all odd i $(a_i, b_i]$ may be contained in $[0, \infty)$. The point is that the $a_i's$ and $b_i's$ may have many accumulation points. Thus if $n = \infty$, then modify the given intervals a little and use a compactness argument to reduce to the case of finitely many intervals.