Introduction to Gluing Methods and Applications

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Lecture V: Gluing Method:

an inner-outer gluing scheme

applied to vortex dynamics of Euler flows

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Lecture IV: Parabolic Gluing Method:

an inner-outer gluing scheme

applied to energy-critical Fujita equation

$$(P) \qquad u_t = \Delta u + u^3 \quad \text{in } \mathbb{R}^4 \times (0, T)$$

Type II blow-up in general domains

Theorem 1 **del Pino-Musso-Wei-Zhou (2019)** For each $T > 0$ sufficiently small there exists an initial condition u_0 such that the solution of Problem (P) blows-up at time T exactly at q. It looks at main order like

$$u(x,t) \sim u_0 = \frac{2\sqrt{2}\lambda(t)}{\lambda^2 + |x - \xi(t)|^2}$$

where

$$\|u(\cdot,t)\|_{L^\infty} \sim \frac{1}{\lambda(t)} \sim \frac{|\log(T-t)|^2}{T-t}, \xi(t) \to q, \text{ as } t \to T.$$

This blow-up is co-dimensional 1 stable.

Inner-Outer Gluing

We look for the solution of the form

$$u_0 + \eta\left(\frac{x-\xi}{R\lambda}\right)\phi + \psi$$

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ϕ solves the inner problem, which is solved only in

$$|x-\xi| < R(t)\lambda(t)$$

and ψ solves the outer problem, and η is a suitable cut-off. Both equations form a nonlinear parabolic system.

Inner-Outer Gluing

Inner Problem:

$$\partial_t \phi = \Delta\phi + 3u_0^2\phi + \eta E_0$$
$$+ 3u_0^2\psi + \text{quadratic terms}$$
$$|x| < 2R\lambda(t)$$

Outer Problem:

$$\psi_t = \Delta\psi + \text{small terms in } \psi$$
$$+ (1-\eta)E_0 + 2\nabla\eta\nabla\phi + \phi\Delta\eta$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^4, 0 < t < T$$

Inner variables and linearized inner equation

Inner variables:

$$y := \frac{x-\xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

Inner variables and linearized inner equation

Inner variables:

$$y := \frac{x-\xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

In the inner variables the approximation u_0 is such that

$$\lambda u_0 \to W(y) = \frac{2\sqrt{2}}{1+|y|^2} \quad \text{as } \tau \to \infty.$$

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The inner equation is

$$\begin{aligned} \phi_\tau &= \Delta\phi + 3W^2\phi \\ &\quad + \text{error} + 3W^2\psi + \text{small linear terms} \end{aligned}$$

Construction of Linear Theory for Mode Zero

$$\begin{cases} \phi_\tau = \Delta_y \phi + 3W^2(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = \phi(r) & \\ \int hZ_5 = 0 & \tau \in (\tau_0, +\infty) (\text{orthogonality}) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. (\text{instability}) \end{cases}$$

Result: Let $\nu, \sigma \in (0,1)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma}(\log R)}{1+|y|^4}$$

Step 1: Solving the elliptic part

By finite-dimensional reduction method (Lecture One)

$$L_0[H] := \Delta H + 3W^2(y)H = -h \text{ in } \mathbb{R}^4,$$

$$H(y) \to 0 \quad \text{as} \quad |y| \to \infty$$

$$\int hZ_5 = 0$$

admits a solution H satisfies the estimate

$$|H(y,\tau)| \lesssim \frac{\tau^{-\nu}}{(1+|y|)^{\sigma}}$$

Step 2: Linear Theory for Mode Zero without orthogonality condition

.

We claim that for all sufficiently large $R > 0$ there exists $\Phi = \Phi(y,\tau)$ and $c = c(\tau)$ which solve Problem

$$\begin{cases} \Phi_\tau = \Delta_y \Phi + 3W^2(y)\Phi + H - c(\tau)Z_0, & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \Phi = \Phi(r) & \\ \Phi(R,\tau) = 0 & \tau \in (\tau_0, +\infty) \\ \int_{B_R} \Phi Z_0 = 0 & \tau \in (\tau_0, +\infty) \\ \Phi(y,\tau_0) = 0 & |y| < 2R \end{cases}$$

and satisfy the estimates

$$|\Phi(y,\tau)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma} \log R}{1 + |y|^2}$$

Step 3

Finally we take

$$\phi = L_0[\Phi]$$

Recall

$$\Phi_\tau = L_0[\Phi] + H - cZ_0$$

$$L_0[\Phi_\tau] = (L_0[\Phi])_\tau = L_0[L_0[\Phi]] + L_0[H] - cL_0[Z_0]$$

Then ϕ satisfies

$$\phi_\tau = \Delta\phi + 3W^2\phi + h - c\mu_0 Z_0$$

and

$$|\phi| \lesssim \frac{R^{4-\sigma}\log R}{1+|y|^4}$$

This is exactly the solution we have constructed for mode zero.

Proof of Step 2: Linear Theory Without Orthogonality Condition

This is a linear theory without any orthogonality condition. The estimates are very bad in the interior!!!

$$\begin{cases} \Phi_\tau = \Delta_y \Phi + 3W^2(y)\Phi + H - c(\tau)Z_0, & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \Phi = \Phi(r) & \\ \Phi(R,\tau) = 0 & \tau \in (\tau_0, +\infty) \\ \int_{B_R} \Phi Z_0 = 0 & \tau \in (\tau_0, +\infty) \\ \Phi(y, \tau_0) = 0 & |y| < 2R \end{cases}$$

$$|H| \lesssim \tau^{-\nu} \frac{1}{(1+|y|)^\sigma}$$

$$|\Phi(y,\tau)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma} \log R}{1+|y|^2}$$

Step 2.1: Concentrating the error

Let $\eta(s)$ be the smooth cut-off function, and consider $\eta_\ell(y) = \eta(|y| - \ell)$, for a large but fixed number ℓ independently of R. By standard parabolic theory, there exists a unique solution Φ_1 to

$$\Phi_{1,\tau} = \Delta\Phi_1 + 3W^2(1-\eta_\ell)\Phi_1 + H(y,\tau) - c(\tau)Z_0 = 0 \text{in} B_{2R} \times (\tau_0, \infty)$$

$$\Phi_1 = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \text{ in } B_{2R},$$

Key observation: $\Delta + 3W^2(1-\eta_\ell)$ satisfies Maximum Principle. By barrier method, the function Φ_1 satisfies the bound

$$|\Phi_1| \lesssim \tau^{-\nu} R^{2-\sigma}$$

Setting $\Phi = \Phi_1 + \Phi_2$. Then Φ_2 satisfies

$$\Phi_{2,\tau} = \Delta\Phi_2 + 3W^2\Phi_2 + 3W^2\Phi_1(1-\eta) - c(\tau)Z_0 \text{in } B_{2R} \times (\tau_0, \infty)$$
$$\Phi_2 = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0, \infty), \quad \Phi_2(\cdot, \tau_0) = 0 \text{ in } B_{2R}.$$

We can adjust c such that

$$\int_{B_{2R}} \Phi_2(\cdot, \tau)\, Z_0 \;=\; 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty).$$

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Existence follows from linear parabolic theory. A Priori Estimates???

Testing the equation against Φ_2 and integrating in space, we obtain the relation

$$\partial_\tau \int_{B_R} \Phi_2^2 + \int_{B_R} \left[|\nabla \Phi_2|^2 - 3W^2 |\Phi_2|^2 \right] = \int_{B_R} (3W^2(1-\eta)\Phi_1)\Phi_2.$$

Since dimension is 4 and $\Phi_2 = 0$ on ∂B_R, there exists $C > 0$ such that, for any $\phi \in H_0^1(B_R)$ with $\int \phi Z_0 = 0$, the following inequality holds

$$\int_{B_R} \left[|\nabla \phi|^2 - 3W^2 |\phi|^2 \right] \geq \frac{C}{R^2 \log R} \int_{B_R} \phi^2.$$

Using that $\Phi_2(\cdot, \tau_0) = 0$ and Gronwall's inequality, we readily get the L^2-estimate

$$\begin{aligned} \|\Phi_2(\cdot,\tau)\|_{L^2(B_{2R})} &\lesssim \tau^{-\nu} R^2 \log R \|3W^2(1-\eta)\Phi_1)\Phi_2\|_{L^2(B_{2R})} \\ &\lesssim \tau^{-\nu} R^{4-\sigma} \end{aligned}$$

Now, using standard parabolic estimates in the equation satisfied by Φ_2 we obtain then that on any large fixed radius $M > 0$,

$$\|\Phi_2(\cdot,\tau)\|_{L^\infty(B_M)} \lesssim \tau^{-\nu} R^{4-\sigma} \log R \quad \text{for all} \quad \tau > \tau_0.$$

Since the right hand side has a fast decay at infinity and taking into account that we are in dimension 4, outside B_ℓ we can dominate the solution by a barrier of the order $\tau^{-\nu}|y|^{-2}$. As a conclusion, also using local parabolic estimates for the gradient, we find that

$$|\Phi_2(y,\tau)| \ \lesssim \tau^{-\nu}\frac{R^{4-\sigma}\log R}{1+|y|^2}$$

Combining the estimates of Φ_1 and Φ_2:

$$|\Phi| \leq |\Phi_1| + |\Phi_2| \lesssim \frac{R^{4-\sigma}\log R}{1+|y|^2}$$

2. Construction at modes 1 to 4.

As we can see in mode 0, the estimates are somewhat deteriorated inside the inner regime, and this will result in difficulties when solving the inner problem. One can observe that, for modes 1 to 4, the kernel function for the corresponding linearized operator decays like $\frac{1}{|y|^3}$ which is in L^2. We shall carry out the construction for modes 1 to 4 by means of the blow-up argument.

$$\begin{cases} \phi_\tau = \Delta_y \phi + 3W^2(y)\phi + h(y,t), & y \in \mathbb{R}^4, \tau \in (\tau_0, +\infty) \\ \phi = \sum_{j=1}^4 \phi_j(r)\Theta_j \\ \int hZ_j = 0, j = 1, ..., 4, & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = 0. \end{cases}$$

Main Result: Let $\nu \in (0,1), \sigma \in (1,2)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then there exists a solution ϕ such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{(1+|y|)^{\sigma}}$$

We first claim that

$$\int_{\mathbb{R}^4} \phi Z_i = 0 \text{ for all } \tau \in [\tau_0, +\infty), \ \ i = 1, \cdots, 4.$$

In fact

$$\int \phi_\tau Z_i = \int (\Delta_y \phi + 3W^2(y)\phi + h(y,t))Z_i = 0$$

$$(\int \phi Z_i)_\tau = 0, \int \phi(y,0) Z_i dy = 0$$

$$\int \phi Z_i = 0$$

Make it rigorous by $Z_i \eta_{R'}$, where $\eta_{R'} := \eta(\frac{|y|}{R'})$ and η is the standard cut-off function.

We prove the a priori estimates by contradiction. Suppose that there exist sequences $\tau^k \to +\infty$ and $y_k \in \mathbb{R}^4$ such that

$$\max \tau^\nu (1+|y|)^\sigma |\phi(y)| = \tau_k^\nu (1+|y_k|)^\sigma |\phi(y_k)| = 1$$

$$|h^k| \lesssim o(1)\tau^{-\nu}(1+|y|)^{-2-\sigma}$$

$$\begin{cases} \phi_\tau^k = \Delta_y \phi^k + 3W^2(y)\phi^k + h^k(y,t), & \tau \in (\tau_0, +\infty) \\ \phi^k = \sum_{j=1}^4 \phi_j^k(r)\Theta_j & \\ \int h^k Z_j = 0, j = 1, ..., 4, & \tau \in (\tau_0, +\infty) \\ \phi^k(y, \tau_0) = 0. & \end{cases}$$

Case 1. $|y_k| \le M$

In this case, up to a subsequence, $\phi^k \to \phi_\infty$ uniformly on compact subsets with $\phi_\infty \neq 0$ and ϕ_∞ is an ancient solution to

$$\begin{cases} \partial_\tau \phi_\infty = \Delta \phi_\infty + 3W^2(y)\phi_\infty \ \text{ in } \ \mathbb{R}^4 \times (-\infty, 0], \\ \int_{\mathbb{R}^4} \phi_\infty(y,\tau) \cdot Z_j(y)dy = 0 \ \text{ for all } \ \tau \in (-\infty, 0], \ j = 1, \cdots, 4, \\ \phi_\infty = \sum_{j=1}^{4} \phi_j(r)\Theta_j \\ |\phi_\infty(y,\tau)| \le \dfrac{1}{1+|y|^\sigma} \ \text{ in } \ \mathbb{R}^4 \times (-\infty, 0] \end{cases}$$

Note that the orthogonality conditions above are well-defined if $\sigma > 1$.

We claim that $\phi_\infty = 0$.

Case 2: $|y_k| \to +\infty$

Suppose there exists y_k with $|y_k| \to +\infty$ such that

$$(\tau_k)^{\nu}(1+|y_k|^{\sigma})|\phi_k(y_k,\tau_k)| \geq \frac{1}{2}.$$

Let

$$\tilde{\phi}_k(z,\tau) := (\tau_k)^{\nu}|y_k|^{\sigma}\phi_k(|y_k|z, |y_k|^2\tau+\tau_k).$$

Then $\tilde{\phi}_k \to \phi_\infty \neq 0$ uniformly on compact subsets of $\mathbb{R}^4 \setminus \{0\} \times (-\infty, 0]$ with ϕ_∞ satisfying ancient solution of heat equation

$$\begin{cases} \phi_{\infty,\tau} = \Delta\phi_\infty, & \text{in } \mathbb{R}^4 \setminus \{0\} \times (-\infty, 0], \\ |\phi_\infty(z,\tau)| \leq |z|^{-\sigma}, & \text{in } \mathbb{R}^4 \setminus \{0\} \times (-\infty, 0]. \end{cases}$$

Removable singularities for ancient solutions of heat equation: $\phi_\infty = 0$.

Final step: the reduced equations for $\lambda(t)$ and $\xi(t)$

By the linear theory, approximately, orthogonality conditions

$$\int_{\mathbb{R}^4} hZ_j(y)dy \approx 0 \quad \text{for all } j = 1, \cdots, 5, \; t \in (0, T)$$

should be satisfied.

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should be satisfied.

What is h?

$$h \sim E_1 \sim 3W^2(\phi^0[\lambda] + \psi)$$

The orthogonality conditions above imply that

$$\begin{cases} \dot{\xi}(t) \sim 0 \\ \int_0^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds = \psi(q, T) + o(1) \end{cases}$$

Equation for λ

$$\begin{cases} \int_0^{t-\lambda(t)^2} \frac{\dot{\lambda}(s)}{t-s} ds = -c_0 = \psi(q, T) \\ \lambda(T) = 0 \end{cases}$$

How do we solve this nonlocal ODE?

We expect the solution to be at main order given by

$$\lambda_0(t) = \kappa \frac{T-t}{\log^2(T-t)}$$

for some $\kappa > 0$.

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To solve it, we need another 20 pages!

Applications and New Developments of Parabolic Inner-Outer Gluing Scheme

1. Finite-time Blow-ups for Hramonic Map Flow in dimension two

Theorem I.1. [Davila, del Pino, Wei (Invent Math 2019)] Given k distinct points $q_1, \cdots, q_k \in \Omega$ and $T > 0$ sufficiently small, then there exists initial condition u_0 and boundary condition $u_{\partial\Omega}$ such that the solution u to the harmonic map flow (HMF)

$$(HMF) \quad \begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u & \text{in } \ \Omega \times (0, T) \\ u(x, 0) = u_0 & \text{in } \ \Omega \\ u = u_{\partial\Omega} & \text{in } \ \partial\Omega \times (0, T) \\ u : \Omega \times (0, T) \to \mathbb{S}^2 & \end{cases}$$

blows up precisely at $q_1, ..., q_k$ as $t \to T$ with blow-up rate

$$\|\nabla u\|_{L^\infty} \sim \frac{\log^2(T - t)}{T - t}, t < T$$

Continuation of Bubbling

Theorem I.2 [Davila, del Pino, Wei (2019)]
Let $u_q(x,t)$ be the solution in Theorem I.1. Then u_q can be continued as an H^1-weak solution in $\Omega \times (0, T+\delta)$, which is continuous except at the points (q_i, T), with the property that,

$$\|\nabla u\|_{L^\infty} \sim \begin{cases} \frac{|\log(T-t)|^2}{T-t}, & \text{for } t < T \\ \frac{|\log(t-T)|^2}{t-T}, & \text{for } t > T \end{cases}$$

New Perspective: Continuation of Bubbling

Spontaneous Bubbling

Theorem I.3 [Davila, del Pino, Wei (2019)]
Given points $q_1, \ldots, q_k$ in Ω, there exists an H^1-weak solution $u(x,t)$ of the harmonic map flow problem in $\Omega \times (0, T+\delta)$ which is continuous except at the points (q_i, T), it is smooth in $\Omega \times (0, T]$ and has spontaneous reverse bubbling at the points q_i in the form

$$\|\nabla u\|_{L^\infty} \sim \begin{cases} O(1), & \text{for } t < T \\ \frac{|\log(t-T)|}{t-T}, & \text{for } t > T \end{cases}$$

New Perspective: Spontaneous Bubbling

2. Infinite-time blow-up in entire space

.

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^n. \end{cases} \qquad (P0)$$

infinite-time blow-up: $\lim_{t \to +\infty} \|u(\cdot, t)\|_{L^\infty} = +\infty$

Fila-King Conjecture (2010): Infinite time blow-up **should only happen** in low dimensions 3 and 4.

Theorem (del Pino, Musso, Wei (2018))

Given $\gamma > 1$ *there exists a positive global solution* $u(x,t)$ *to problem*

$$\begin{cases} u_t = \Delta u + u^5 & \text{in} \quad \mathbb{R}^3 \quad \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

for a u_0 *such that*

$$\lim_{|x|\to\infty} |x|^\gamma u_0(x) \;=:\; A > 0, \qquad (G)$$

and as $t \to +\infty$ *it has the blow up near the origin like*

$$u(x,t) \sim \left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{\frac{1}{2}},$$

$$\lambda(t) \;\sim\; \begin{cases} t^{1-\gamma} & \text{if} \quad 1 < \gamma < 2, \\ t^{-1}\ln^2 t & \text{if} \quad \gamma = 2, \\ t^{-1} & \text{if} \quad \gamma > 2. \end{cases}$$

3. Bubble-Tower

Infinite time sign-changing bubbling tower also exist:

Theorem (del Pino, Musso, Wei (2019))

Let $n \geq 7, p = \frac{n+2}{n-2}$. There is a global solution $u_q(x,t)$ of problem (P_∞) of the form

$$u_q(x,t) \sim \lambda^{-\frac{n-2}{2}} U\left(\frac{x-q}{\lambda}\right) - \mu^{-\frac{n-2}{2}} U\left(\frac{x-q}{\mu}\right)$$

with

$$\lambda \sim t^{-\frac{1}{n-4}}, \ \mu(t) \sim t^{-\frac{3n-10}{(n-6)(n-4)}}$$

New Perspective: bubble-tower.

4. Infinite time blow-up for 1/2-harmonic map flows

Theorem (Sire, Wei, Zheng, (2019))

Let $q_1, \cdots, q_k$ be distinct points in $\mathbb{R}$, there exist an initial datum $u_0 : \mathbb{R} \times [0, \infty) \to \mathbb{S}^1$ and smooth functions $\xi_j(t) \to q_j$, $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \cdots, k$, such that the solution u_q of half harmonic map flow

$$\begin{cases} u_t = -(-\Delta)^{\frac{1}{2}} u + \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|u(x)-u(s)|^2}{|x-s|^2} ds \right) u & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}, \end{cases}$$

has form

$$u_q \sim \sum_{j=1}^{k} U \left(\frac{x - \xi_j(t)}{\mu_j(t)} \right),$$

where U is the canonical least energy half-harmonic map, $\mu_j(t) \sim e^{-\kappa_j t}$.

New Perspective: fractional bubbling.

5. Finite/Infinite Time Blow-up in Keller-Segel System

$$\textit{Keller} - \textit{Segel} \quad \begin{cases} u_t = \Delta u - \nabla(u\nabla v), \ x \in \mathbb{R}^2, t > 0, \\ -\Delta v = u \end{cases}$$

- Ghoul-Masmoudi CPAM 2018: infinite time blow-up in the radial case
- Davila-del Pino-Musso-Wei (2020): infinite time blow-up in general case; stability of blow-up

$$\|u(\cdot, t)\|_{L^\infty} \sim \log t \quad \text{as } t \to +\infty$$

- Davila-del Pino-Musso-Wei (in progress): Finite time blow-ups; clusters of blow-up

$$\|u(\cdot, t)\|_{L^\infty} \sim (T-t)^{-1} e^{\sqrt{2}\sqrt{|\log(T-t)|}}$$

New Perspective: reaction-diffusion system

6. Finite time blow-up in Nematic Liquid Crystal Flow

Theorem ([Lai-Lin-Wang-Wei-Zhou, CPAM]): Given k distinct points $q_1, \cdots, q_k \in \Omega \subset R^2$, if $T, \epsilon_0 > 0$ are sufficiently small, then there exists initial condition (v_0, d_0) such that the solution (v, d) to the nematic liquid crystal flow (NLCF)

Nematic Liquid Crystal Flow (NLCF):

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = \Delta v - \epsilon_0 \nabla \cdot (\nabla d \odot \nabla d) & \text{in } \Omega \times (0, T) \\ \nabla \cdot v = 0 & \text{in } \Omega \times (0, T) \\ \partial_t d + v \cdot \nabla d = \Delta d + |\nabla d|^2 d & \text{in } \Omega \times (0, T) \\ |d| = 1 & \text{in } \Omega \times (0, T) \end{cases}$$

blows up precisely at those points as $t \to T$.

New Perspective: non-variational

Lecture V: Gluing Method:

an inner-outer gluing scheme

applied to vortex dynamics of Euler flows

The Euler equation for the velocity of an incompressible fluid confined to $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla p & \text{in } \Omega \times (0, T) \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \times (0, T) \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 & \text{in } \Omega \end{cases} \qquad \text{(E)}$$

$\mathbf{u}(x, t) : \bar{\Omega} \times [0, T) \to \mathbb{R}^2$, $p(x, t) : \bar{\Omega} \to \mathbb{R}$.
Ω smooth, bounded, simply connected domain in $\mathbb{R}^2$ or entire space.

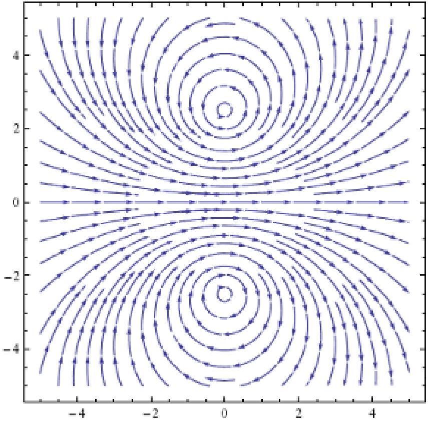
Vorticity-stream formulation

Vorticity :$\omega := \nabla \times \mathbf{u}$

$$\begin{cases} \omega_t + \nabla^\perp \Psi \cdot \nabla \omega = 0 & \text{in } \Omega \times (0, T) \\ -\Delta \Psi = \omega & \text{in } \Omega \times (0, T) \\ \Psi = 0 & \text{on } \partial\Omega \times (0, T) \\ \mathbf{u} = \nabla^\perp \Psi & \end{cases} \quad \text{(V)}$$

Well-posedness:
Wolibner (1933), Yudovich (1963): there is global existence and uniqueness if $\omega_0 \in L^\infty$. If ω_0 is regular, then $\omega(t)$ is regular for all times .

We are interested in describing the evolution of solutions to system (VS) whose vorticities $\omega(x, t)$ are **very concentrated** around a finite number of points.



Singular Vortex Solutions

Singular (formal) vortex solutions: Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921), C.C. Lin (1941).

$$\omega(x,t) = \sum_{j=1}^{k} 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi(x,t) = \sum_{j=1}^{k} 8\pi\kappa_j G(x, \xi_j(t))$$

so that $-\Delta\Psi = \omega$. Green's function

$$G(x,\xi) := \Gamma(|x-\xi|) - H(x,\xi)$$

$$-\Delta G(x,\xi) = 8\pi\delta(x-\xi) \quad \text{in } \Omega, \quad G(x,\xi) = 0 \quad \text{on } \partial\Omega$$

$$\Gamma(|x|) = 4\log\frac{1}{|x|}$$

Formally we compute

$$\omega_t + \nabla\Psi^\perp \nabla\omega =$$

$$\sum_{j=1}^{k} \left(-\dot{\xi}_j + \nabla_x^\perp (-\kappa_j^2 H(x,\xi_j) + \sum_{i\neq j} \kappa_i\kappa_j G(x,\xi_i)) \right) \nabla\delta(x-\xi_j).$$

Hence

$$\omega(x,t) = \sum_{j=1}^{k} 8\pi\kappa_j\delta(x-\xi_j), \quad \Psi(x,t) = \sum_{j=1}^{k} \kappa_j G(x,\xi_j)$$

is a "solution" of Euler if $\xi = (\xi_1,\dots,\xi_k)$ solves the ODE system

$$\dot{\xi}_j(t) = \nabla_x^\perp (-\kappa_j^2 H(x,\xi_j(t)) + \sum_{i\neq j} \kappa_i\kappa_j G(\xi_j(t),\xi_i(t)))$$

Equivalently, if and only if $\xi = (\xi_1, \ldots, \xi_j)$ solves the Hamiltonian system

$$\dot{\xi}_j = \nabla^{\perp}_{\xi_j} K(\xi), \quad j = 1, \ldots, k, \tag{K}$$

where

$$K(\xi) := \sum_{i<j} \kappa_i \kappa_j G(\xi_i, \xi_j) - \frac{1}{2} \sum_{i=1}^{k} \kappa_i^2 H(\xi_i, \xi_i).$$

This is the **Kirchhoff-Routh function**.

Dynamical Correspondence: Desingularization of vortices

Given a solution $\xi(t)$ of the system

$$\dot{\xi}_j = \nabla^{\perp}_{\xi_j} K(\xi), \quad j = 1, \ldots, k,$$

we want to find a regular solution $(\omega_\varepsilon, \Psi_\varepsilon)$ to Euler such that as $\varepsilon \to 0$ and with $\mathbf{u}_\varepsilon = \nabla^{\perp}\Psi_\varepsilon$

$$\omega_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^{k} 8\pi\kappa_j \delta(x - \xi_i(t)), \tag{1}$$

$$\psi_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^{k} 8\pi\kappa_j G(x, \xi_i(t)), \tag{2}$$

Previous results

- Marchioro and Pulvirenti, 1993:

Given a colissionless solution $\xi(t)$ of System (K) there is a solution $(\omega_\varepsilon, \Psi_\varepsilon)$ of (V) with

$$\omega_\varepsilon(x,t) \rightharpoonup \omega^s(x,t) = 8\pi \sum_{j=1}^{N} \kappa_j \delta(x - \xi_i(t)),$$

$$\Psi_\varepsilon(x,t) \rightharpoonup \Psi^s(x,t) = 8\pi \sum_{j=1}^{N} \kappa_j G(x, \xi_j(t)),$$

in the distributional sense.

• **Our purpose:** To solve the N-vortex desingularization problem via gluing methods, providing precise asymptotic behavior of the solution near the vortices.

We find a solution in the form

$$\Psi_\varepsilon = \Psi_{0\varepsilon} + \psi_\varepsilon, \quad \omega_\varepsilon = \omega_{0\varepsilon} + \phi_\varepsilon, \quad -\Delta\Psi_{0\varepsilon} = \omega_{0\varepsilon}$$

where $\omega_{0\varepsilon}$ and $\Psi_{0\varepsilon}$ are explicit ε-regularizations of

$$\omega^s(x,t) = \sum_{j=1}^{N} \kappa_j \delta(x - \xi(t))$$

$$\Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j \Big[\frac{1}{4\pi} \log \frac{1}{|x - \xi_j(t)|^2} - H(x, \xi_j(t))\Big]$$

and we have control on the ε-smallness of ψ_ε and ϕ_ε in stronger norms.

We choose the regularization

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{4\pi} \log \frac{1}{|x-\xi_j(t)|^2+\varepsilon^2} - \kappa_j H(x,\xi_j(t)),$$

$$\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{1}{\pi(1+|y|^2)^2}.$$

We have $\omega_{0\varepsilon} = -\Delta\Psi_{0\varepsilon}$ and $\int_{\mathbb{R}^2} U_0 = 1$. We get

$$\omega_{0\varepsilon} \rightharpoonup \sum_{j=1}^{N} \kappa_j \delta(x-\xi_i(t)).$$

Theorem (Dávila, del Pino, Musso, Wei, (2020))

Let $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ *be a collisionless solution of System* (K). *There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* (V) *of the form*

$$\boxed{\begin{aligned} \omega_\varepsilon(x,t) &= \omega_{0\varepsilon}(x,t) + \phi_\varepsilon(x,t) \\ \Psi_\varepsilon(x,t) &= \Psi_{0\varepsilon}(x,t) + \psi_\varepsilon(x,t) \end{aligned}}$$

where for some $0 < \sigma < 1$ *and all* $(x,t) \in \Omega \times (0,T)$ *we have*

$$\boxed{\begin{aligned} |\phi_\varepsilon(x,t)| \ &\le \ \varepsilon^\sigma \sum_{j=1}^{k} \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \\ |\psi_\varepsilon(x,t)| + \varepsilon |D_x \psi_\varepsilon(x,t)| \ &\le \ \varepsilon^2. \end{aligned}}$$

Ingredients in the construction

- Improvement of the approximation in powers of ε using elliptic and transport equations. (We will need improvements up to order $O(\varepsilon^8)$.)
- Inner-Outer Gluing Scheme: Setting up the problem as a coupled system of inner problem near the singularities and and an outer problem more regular (the inner-outer gluing scheme)
- A priori estimates to solve by a continuation (degree) argument. (It will be impossible by contraction mapping.)

New ingredients of gluing for Euler:

- The inner problem is highly degenerate;
- The outer problem is transport equation–lack of regularity.

$$\omega_t + \nabla^{\perp}(-\Delta)^{-1}\omega\nabla\omega = 0, \omega(x,0) = \omega_0 \in L^\infty$$

$$\|\omega\|_{L^\infty} \lesssim e^{C_1 e^{C_2 t}}$$

Kiselev-Sverak (Annals of Math 2014)

Step 1: Improvement of Error

Let us show how to improve the approximation near the core of the singularity ξ_j. Let

$$E(\omega) := \omega_t + \nabla_x^{\perp}\Psi \cdot \nabla_x \omega, \quad -\Delta_x \Psi = \omega.$$

We look for a (local) solution of $E(\omega) = 0$ of the form

$$\Psi = \Psi_{0\varepsilon}(x,t) + \kappa_j \psi(y,t), \quad \omega = \frac{\kappa_j}{\varepsilon^2} U_0(y) + \frac{\kappa_j}{\varepsilon^2}\phi(y,t), \quad y = \frac{x - \xi_j(t)}{\varepsilon}$$

where

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \kappa_j \log \frac{1}{(\varepsilon^2 + |x - \xi_j(t)|^2)^2} - \kappa_j H(x, \xi_j(t)).$$

$$\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x - \xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}.$$

Then in terms of the y-variable we get

$$\varepsilon^2 E(\omega) = \varepsilon^2 \phi_t + (-\varepsilon\dot{\xi} + \nabla_y^{\perp}\Psi_{0\varepsilon} + \kappa_j \nabla_y^{\perp}\psi) \cdot \nabla_y (U_0 + \phi), \quad -\Delta_y \psi = \phi$$

We decompose

$$\Psi_{0\varepsilon}(x,t) = -\kappa_j \log 8\varepsilon^2 + \kappa_j \Gamma_0(y) \\ -\kappa_j H(x,\xi_j) + \sum_{i\neq j} \kappa_i \log \frac{1}{(\varepsilon^2 + |x-\xi_i|^2)^2} - \kappa_i H(x,\xi_i).$$

Hence

$$\Psi_{0\varepsilon}(x,t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + constant$$

$$\varphi(x) = -\kappa_j H(x,\xi_j) + \sum_{i\neq j} \kappa_i G(x,\xi_i).$$

By assumption $\dot{\xi} = \nabla_x^\perp \varphi(\xi)$. Then we find

$$-\dot{\xi} + \nabla_y^\perp \Psi_{0\varepsilon}(\xi_j + \varepsilon y) = \kappa_j \nabla^\perp(\Gamma_0 + \mathcal{R})$$

with $\mathcal{R} = O(\varepsilon^2 |y|^2)$.

Thus

$$\varepsilon^2 E(\omega) = \varepsilon^2 \phi_t + \kappa_j \nabla_y^\perp (\Gamma_0(y) + \mathcal{R} + \psi) \cdot \nabla_y (U_0 + \phi),$$

Let $f(u) = e^u$. Using that $U_0 = f(\Gamma_0)$ a direct computation gives

$$\begin{aligned} \varepsilon^2 E(\omega) =& \varepsilon^2 \phi_t - \kappa_j \nabla_y^\perp \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) \\ & + \kappa_j \nabla^\perp \mathcal{R} \cdot \nabla U_0 + \kappa_j \nabla^\perp \mathcal{R} \nabla \phi + \nabla^\perp \psi \nabla \phi. \end{aligned}$$

The error term in the equation $\varepsilon^2 E(\omega) = 0$ is $\nabla^\perp \mathcal{R} \cdot \nabla U_0 = O(\varepsilon^2 \rho^{-3})$. We obtain a reduction in the error by approximately solving

$$\nabla_y^\perp \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) - \nabla^\perp \mathcal{R} \cdot \nabla U_0 = 0$$

In polar coordinates $y = \rho e^{i\theta}$ we see that

$$\nabla_y^\perp \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) = \frac{1}{1+\rho^2}\frac{\partial}{\partial\theta}[\Delta_y\psi + f'(\Gamma_0)\psi]$$

$$\mathcal{L}[\phi] := \frac{1}{1+|y|^2}\frac{\partial}{\partial\theta}[\Delta\psi + f'(\Gamma_0)\psi] = 0$$

This operator is highly degenerate:

- All radial functions in its kernel;
- there is also the kernel of

$$\Delta\psi + f^{'}(\Gamma_0)\psi = 0, \ \psi = c_0\frac{1-|y|^2}{1+|y|^2} + c_1\frac{\partial\Gamma_0}{\partial y_1} + c_2\frac{\partial\Gamma_0}{\partial y_2}$$

But our equation is equivalent to

$$\frac{\partial}{\partial\theta}[\Delta_y\psi + f'(\Gamma_0)\psi + f'(\Gamma_0)\mathcal{R}] = 0.$$

and we check that the larger terms of $\mathcal{R}$ only contain Fourier modes 2 and 3.

Indeed

$$\Delta_y\psi + f'(\Gamma_0)\psi = -f'(\Gamma_0)\bar{\mathcal{R}}$$

where $\bar{\mathcal{R}}$ does not have mode zero terms. The operator is nicely invertible (it behaves like Laplacian) in Fourier modes 2 and higher.

The largest term in $\bar{\mathcal{R}}$ is at mode 2, $\varepsilon^2 D_x^2\varphi(\xi)[y]^2$ One then gets a bounded solution ψ of size $O(\varepsilon^2)$ so that the new error is smaller in ε.

We have $\phi = -\Delta\psi = O(\varepsilon^2\rho^{-2})$.

The new error at main order is $O(\varepsilon^4\rho^{-2})$. In fact, for $\omega = U_0 + \phi$,

$$\varepsilon^2 E(\omega) = \varepsilon^2\phi_t + \kappa_j\nabla^\perp\mathcal{R}\nabla\phi + \kappa_j\nabla^\perp\psi\nabla\phi.$$

All these terms come with ε^4. In fact $\phi = O(\varepsilon^2\rho^{-2})$ and we obtain

$$\varepsilon^2 E(\omega) = O(\varepsilon^4\rho^{-2}).$$

This is not enough. The equation for a new ψ and ϕ adjusting the previous improvement will come as

$$\varepsilon^2\phi_t - \kappa_j\nabla_y^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + Q(\phi) + O(\varepsilon^4\rho^{-2})$$

with Q quadratic and one loses ε^2 in solving the full operator because of the $\varepsilon^2\phi_t$ term.

Iterating the "elliptic" argument one can actually arrive to an error $O(\varepsilon^8\rho^2)$ To improve (this is too big far away) one solves the pure transport equation

$$\varepsilon^2\phi_t + \nabla^\perp(\Gamma_0 + \mathcal{R})\cdot\nabla\phi = O(\varepsilon^8\rho^2), \quad \phi(y,0) = 0.$$

(characteristics give a nice way to represent the solution). The solution ϕ is size $O(\varepsilon^6\rho^2)$. Substituting in the full operator one creates and error $O(\varepsilon^6\rho^{-2})$. This is a full improvement everywhere sufficient for the method to work.

Step 2: Inner-Outer Gluing Scheme

We now set up an inner-outer gluing scheme to find a true solution to Euler flow.

Let us assume $\kappa_j = 1$. We consider smooth cut-off functions

$$\eta_j(x,t) = \eta_0\left(\frac{|x-\xi_j(t)|}{\delta}\right)$$

where $\eta_0(s) = 1$ if $s < 1$, $\eta_0(s) = 0$ for $s > 2$. We look for a solution the form

$$\omega(x,t) = \omega_\varepsilon^*(x,t) + \varepsilon^{-2}\sum_{j=1}^{k}\eta_j\,\phi_j\left(\frac{x-\xi_j(t)}{\varepsilon}, t\right) + \phi^{out}(x,t)$$

$$\Psi(x,t) = \Psi_\varepsilon^*(x,t) + \sum_{j=1}^{k}\eta_j\,\psi_j\left(\frac{x-\xi_j(t)}{\varepsilon}, t\right) + \psi^{out}(x,t)$$

We want that (ω, Ψ) given by

$$\omega = \omega_\varepsilon^* + \varepsilon^{-2} \sum_{j=1}^{k} \eta_j \, \phi_j + \phi^{out}, \quad \Psi = \Psi_\varepsilon^* + \sum_{j=1}^{k} \eta_j \, \psi_j + \psi^{out}$$

satisfies

$$\begin{cases} \omega_t + \nabla_x \Psi^\perp \cdot \nabla_x \omega = 0 & \text{in } \Omega \times [0, T] \\ -\Delta_x \Psi = \omega & \text{in } \Omega \times [0, T] \\ \Psi = 0 & \text{on } \partial\Omega \times [0, T] \end{cases}$$

for which it suffices that the functions

$$(\phi_j(y, t), \psi_j(y, t)), \quad (\phi^{out}(x, t), \psi^{out}(x, t))$$

satisfy the **inner-outer gluing system**

Inner-Outer Gluing Scheme

$$(Inner) \quad \begin{cases} \varepsilon^2 \partial_t \phi_j + \nabla_y^{\perp} \Gamma_0 \cdot \nabla(\Delta \psi_j + f'(\Gamma_0)\psi_j) \\ + \nabla_y^{\perp} \psi^{out} \cdot \nabla U_0 \\ + Q_j + E_j = 0 \quad \text{in } \mathbb{R}^2 \times [0, T] \\ - \Delta_y \psi_j = \phi_j, \end{cases}$$

for $j = 1, \ldots, k$, coupled with $\phi^{out}(\cdot, 0) = 0$ and

$$(Outer) \quad \begin{cases} \partial_t \phi^{out} + \nabla_x^{\perp}[\Psi_* + \eta_j \psi_j + \psi^{out})] \cdot \nabla_x \phi^{out} \\ + \varepsilon^{-2} \phi_j \partial_t \eta_j + \nabla_x^{\perp}(\eta_j \psi_j + \psi^{out}) \cdot \nabla_x \eta_j \\ + Q_{out} + E_{out} = 0 \quad \text{in } \Omega \times [0, T] \\ \\ \Delta_x \psi^{out} + \phi^{out} + \psi_j \Delta_x \eta_j + 2\nabla_x \eta_j \cdot \nabla \psi_j = 0, \\ \psi^{out} = 0, \quad \text{on } \partial\Omega \times [0, T]. \end{cases}$$

We solve System (Inner)-(Outer) by a continuation (degree) argument.
We drop the index j, and the inner problem (I) becomes approximately

$$\boxed{\begin{aligned} \varepsilon^2\phi_t - \nabla^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{in } \mathbb{R}^2\times[0,T] \end{aligned}}$$

with $E = O(\varepsilon^5\rho^{-3})$.

Gluing-Outer Problem: Estimates for Transport Equation

The outer problem is a transport equation:

$$\begin{cases} \phi_\tau + \nabla_y^\perp(\Gamma_0 + R(y, \varepsilon^2\tau))\nabla_y\phi = E(y, \varepsilon^2\tau), & \text{in } \mathbb{R}^2 \\ \phi(y, 0) = 0, & \text{in } \mathbb{R}^2 \end{cases}$$

where

$$E(y,t) = \varepsilon^8 O(|y|^2)$$

$$\partial_t R,\ R = O(\varepsilon^2(1+|y|^2)),$$

$$\tau = \varepsilon^{-2} t$$

Results on Outer Problem

- (Gradient estimate)
  $$|\phi(y,\tau)| \le C\varepsilon^6(1+\rho^2), \quad |D_y\phi(y,\tau)| \le C\varepsilon^6(1+\rho)$$
  for $\rho = |y| \le \frac{\delta}{\varepsilon}$, $0 \le \tau \le \frac{T}{\varepsilon^2}$.
- (Propagation of support) There exist numbers $R_0 > 0, \beta > 0$ such that for any sufficiently small ε, all $R > R_0$ and any locally bounded function E such that
  $$E(y,\varepsilon^2\tau) = 0 \forall (y,\tau) \in B_R(0) \times [0, \frac{T}{\varepsilon^2}]$$
  we have that the solution satisfies
  $$\phi(y,\tau) = 0 \forall (y,\tau) \in B_\beta R(0)[0, \frac{T}{\varepsilon^2}]$$

Proof: precise estimates of the characteristics.

Gluing: Inner Problem

For the inner problem we need solve in $\mathbb{R}^2$

$$\varepsilon^2\phi_t + \nabla^\perp\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

$$\varepsilon^2\phi_t + \frac{1}{1+|y|^2}\frac{\partial}{\partial\theta}[\Delta\psi + f'(\Gamma_0)\psi] + E(y,t) = 0, \quad \phi(y,0) = 0$$

$$\varepsilon^2\phi_t + \mathcal{L}[\phi] + E(y,t) = 0, \quad \phi(y,0) = 0$$

where

$$\mathcal{L}[\phi] := \frac{1}{1+|y|^2}\frac{\partial}{\partial\theta}[\Delta\psi + f'(\Gamma_0)\psi]$$

$$\mathcal{L}[\phi] := \frac{1}{1+|y|^2}\frac{\partial}{\partial\theta}[\Delta\psi + f'(\Gamma_0)\psi] = 0$$

As before, this operator is highly degenerate:

- All radial functions in its kernel;
- there is also the kernel of

$$\Delta\psi + f^{'}(\Gamma_0)\psi = 0,\ \ \psi = c_0\frac{1-|y|^2}{1+|y|^2} + c_1\frac{\partial\Gamma_0}{\partial y_1} + c_2\frac{\partial\Gamma_0}{\partial y_2}$$

We use energy method.

The natural way to measure the size of the error and the solution ϕ is through the norm

$$\|\phi U_0^{-\frac{1}{2}}\|_{L^2}^2 = \int_{\mathbb{R}^2} E(y,t)^2 U_0^{-1}(y)\, dy$$

We observe that for $E = O(\varepsilon^6 \rho^{-2})$

$$\|E U_0^{-\frac{1}{2}}\|_{L^2} = O(\varepsilon^5)$$

A central ingredient is an L^2-a priori estimate that involves both the elliptic and the time derivative at the same time. A simplified version of the key estimate is as follows

A priori estimates:

If ϕ solves in $\mathbb{R}^2$

$$\varepsilon^2\phi_t + \nabla^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

and satisfies the orthogonality conditions

$$\int_{B(0,\delta\varepsilon^{-1})} y_i \phi(y,t)\,dy = 0, \quad i = 1,2,$$

$$\int_{\mathbb{R}^2} \phi(y,t)\,dy = \int_{\mathbb{R}^2} \phi(y,t)\frac{1-2|y|^2}{1+|y|^2}U_0(y)\,dy = 0,$$

then the following estimate holds

$$\|\phi(\cdot,t)U_0^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} \le C\varepsilon^{-2}|\log\varepsilon| \sup_{\tau\in[0,T]} \|E(\cdot,\tau)U_0^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}$$

The proof involves estimating a quadratic form in the sphere via stereographic projection.

$$\varepsilon^2\phi_t + \nabla^\perp\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$

We use the test function

$$g = \frac{\phi}{U_0} - (-\Delta)_{\mathbb{R}^2}\phi$$

so that

$$U_0 g = -(\Delta\psi + f'(\Gamma_0)\psi).$$

$$\varepsilon^2\partial_t \int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} U_0^{-1}\nabla^\perp\Gamma_0\nabla(U_0^2 g^2) + 2\int_{\mathbb{R}^2} Eg$$

The second integral is zero for

$$\int_{\mathbb{R}^2} U_0^{-1}\nabla^\perp\Gamma_0\nabla(U_0^2 g^2) = -\int_{\mathbb{R}^2} \nabla\cdot(U_0^{-1}\nabla^\perp\Gamma_0)\nabla(U_0^2 g^2)$$

and since Γ_0 and U_0 are radial,

$$\nabla\cdot(U_0^{-1}\nabla^\perp\Gamma_0) = 0.$$

Thus

$$\varepsilon^2 \partial_t \int_{\mathbb{R}^2} \phi g = 2 \int_{\mathbb{R}^2} Eg \le C\varepsilon^{-2} \| E U_0^{-\frac{1}{2}} \|_{L_2} \| \| g U_0^{\frac{1}{2}} \|_{L^2}$$

and integrating,

$$\varepsilon^2 \int_{\mathbb{R}^2} \phi g(\cdot, t) \le \max_{\tau \in (0,T)} C \| E(\cdot, \tau) U_0^{-\frac{1}{2}} \|_{L_2} \| \| g(\cdot, \tau) U_0^{\frac{1}{2}} \|_{L^2}.$$

Under the orthogonality conditions assumed on ϕ we can prove the following Poincare inequality:

$$\frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \phi^2 U_0^{-1} \le \int_{\mathbb{R}^2} \phi g$$

while we always have

$$\int_{\mathbb{R}^2} g^2 U_0 \le C \int_{\mathbb{R}^2} \phi^2 U_0^{-1}.$$

From these inequalities the desired estimate follows.

To prove the Poincare inequality

$$\frac{\gamma}{|\log \varepsilon|}\int_{\mathbb{R}^2}\phi^2 U_0^{-1} \leq \int_{\mathbb{R}^2}\phi g$$

we set $\tilde{\phi} = U_0^{-1}\phi$. Using stereographic projection we see that

$$\int_{S^2}\tilde{\phi}^2 = \int_{\mathbb{R}^2}\phi^2 U_0^{-1}$$

while

$$\int_{S^2}\tilde{\phi} = \int_{\mathbb{R}^2}\phi = 0.$$

Besides

$$\int_{\mathbb{R}^2}\phi g = \int_{S^2}\tilde{\phi}(\tilde{\phi} - 2(-\Delta_{S^2})^{-1}\tilde{\phi}).$$

Expanding $\tilde{\phi}$ in the orthonormal basis in $L^2(S^2)$ of spherical harmonics we get

$$\tilde{\phi} = \sum_{j=0}^{\infty} \tilde{\phi}_j e_j(z) = \sum_{j=0}^{3} \phi_j e_j + \tilde{\phi}^{\perp},$$

where $-\Delta_{S^2} e_j = \langle_j e_j$.

Here $\lambda_0 = 0$ and e_0 is constant, while $\lambda_1 = \lambda_2 = \lambda_3 = 2$, with $e_j(z) = z_j$. Thus $\tilde{\phi}_0 = 0$ and also $\tilde{\phi}_3 = 0$ because of our orthogonality condition:

$$\int_{\mathbb{R}^2} \phi(y,t)dy = \int_{\mathbb{R}^2} \phi(y,t)\frac{1-2|y|^2}{1+|y|^2} U_0(y)dy = 0,$$

$$\int_{\mathbb{R}^2} \phi g = \sum_{j=4}^{\infty} \left(1 - \frac{2}{\lambda_j}\right) \tilde{\phi}_j^2 \sim \|\tilde{\phi}^{\perp}\|_{L^2(S^2)}^2$$

and this is zero only if $\tilde{\phi} = cz_j$. We also have

$$0 = \int_{B_R} \phi y_j = c\tilde{\phi}_j + O(\|\tilde{\phi}^{\perp}\|_{L^2(S^2)}) |\log R|^{\frac{1}{2}}$$

with $R = \delta\varepsilon^{-1}$ which gives

$$\tilde{\phi}_j = O(\|\tilde{\phi}^{\perp}\|_{L^2(S^2)}) |\log \varepsilon|^{\frac{1}{2}}.$$

From here it follows that

$$\int_{\mathbb{R}^2} \phi g \geq \gamma |\log \varepsilon|^{-1} \int_{S^2} \tilde{\phi}^2$$

as we wanted.

The above energy method gives the L^2 bound for the vorticity, which is not enough for the transport equation. ($\phi \in L^2, \psi \in H^2$.) To obtain L^p-estimates, with $p > 2$, we multiply this by $|\phi|^{p-2}\phi$ and use the transport equation: Unfortunately we don't have cancellations and we obtain lots of boundary terms:

$$\|\phi\|_{L^p} \leq C\varepsilon^{-2} \sup_{0\leq t\leq T} (\|E\|_{L^p} + \|U_0\psi\|_{L^p} + \|\rho^{-5}\nabla\psi\|_{L^p}).$$

We need the following decaying estimate:

$$\|U_0\psi\|_{L^p} + \|\rho^{-5}\nabla\psi\|_{L^p} \leq C\|\phi\, U_0^{-1/2}\|_{L^2}.$$

Interpolation estimates.

Summary of Inner-Outer Gluing Method

- Inner-outer Gluing method is a very flexible and efficient method to construct and analyze solutions with singularities/concentrations or interfaces. In each problem one has to identify the corresponding Inner Problem (less regular) and Outer Problem (more regular).

Summary of Inner-Outer Gluing Method

- Inner-outer Gluing method is a very flexible and efficient method to construct and analyze solutions with singularities/concentrations or interfaces. In each problem one has to identify the corresponding Inner Problem (less regular) and Outer Problem (more regular).
- It can deal with point spikes, concentrations on higher dimensional sets, finite/infinite time blow-ups, single/mutiple/towered blow-ups, forward/reverse blow-ups, slow-decaying errors, nonlocal dynamics, fractional Laplacian, general nonradial domains, systems or non-variational problems, transport equations.

Thanks for your attention