Introduction to Gluing Methods and Applications

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What is gluing method?

In this lecture series, I will introduce the Inner-Outer Gluing Scheme for singularity formations in nonlinear PDE problems.

The key idea is to "Zoom In" the singularity region and decouple the whole nonlinear PDE problem into two problems: the inner problem, which is only solved near the singularity, captures the essential geometric information of the singularities; the outer problem, which is solved in the whole space, sums all global and external effects.

This scheme grows out of the finite/infinite dimensional reduction method.

Outline of Lectures

Lecture I: Finite-dimensional reduction method

(Model Example: $\varepsilon^2 \Delta u - V(x)u + u^p = 0$)

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(Model Example: $u_t = \Delta u + u^{\frac{n+2}{n-2}}$)

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Lecture IV: Gluing method for Euler equation

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Lecture V: Reverse gluing method

(Model Example: $\Delta u + u - u^3 = 0$)

Lecture I: Finite-dimensional reduction method

April 1, 2021

What is finite dimensional Liapunov-Schmidt reduction method?

Let X, Y be Banach spaces and $S(u)$ be a C^1 nonlinear map from X to Y. To find a solution to the nonlinear equation

$$S(u) = 0, \tag{0.1}$$

a natural way is to find approximations first and then to look for genuine solutions as (small) perturbations of approximations.

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Example: $S(u) = \varepsilon^2 \Delta u - V(x)u + u^3$

Assume that U_λ are the approximations, where $\lambda \in \Lambda$ is the parameter (we think of Λ as the configuration space). Writing $u = U_\lambda + \phi$, then solving $S(u) = 0$ amounts to solving

$$L[\phi] + E + N(\phi) = 0, \tag{0.2}$$

where

$$L[\phi] = S'(U_\lambda)[\phi], \text{ Linearized Operator}$$

$$E = S(U_\lambda), \text{ First Error}$$

$$N(\phi) = S(U_\lambda + \phi) - S(U_\lambda) - S'(U_\lambda)[\phi]. \text{Nonlinear Terms}$$

In order to solve (0.2), we try to invert the linear operator L so that we can rephrase the problem as a fixed point problem. That is, when L has a uniformly bounded inverse in a suitable space, one can rewrite the equation (0.2) as

$$\phi = -L^{-1}[E + N(\phi)] = \mathcal{A}(\phi).$$

What is left is to use fixed point theorems such as contraction mapping theorem.

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Infinite-dimensional reduction or parabolic gluing method: eigenfunction space may be infinite dimensional, and the operator L may not be Fredholm !!!

Assuming that $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\}$ is a basis of the eigenfunction space associated to small eigenvalues of L, we can divide the procedure of solving (0.2) into two steps:

Step I: For any $\lambda \in \Lambda$ we solve the projected nonlinear problem

$$\begin{cases} S(U_\lambda + \phi) = L[\phi] + E + N(\phi) = \sum_{j=1}^{n} c_j \mathcal{Z}_j, \\ \langle \phi, \mathcal{Z}_j \rangle = 0, \ \forall\, j = 1, \ldots, n, \end{cases}$$

where c_j may be constant depending on the form of $\langle \phi, \mathcal{Z}_j \rangle$.

Step II: solving the reduced problem

$$c_j(\lambda) = 0, \ \forall\, j = 1, \ldots, n,$$

by adjusting λ in the configuration space Λ.

Some History

- Floer-Weinstein (1986) first introduced this method for nonlinear Schrodinger equation

$$\epsilon^2 \Delta u - V(x)u + u^p = 0, u > 0, \; u \in H^1(\mathbb{R}^N). \tag{0.3}$$

  Oh (1989) generalized to higher dimensional case

- Bahri-Coron (1988) developed the reduction method for critical exponent problems.

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \tag{0.4}$$

- New developments: Ambrosetti-Malchiodi (2006), Bahri-Li-Rey (1995), Del Pino-Felmer-Musso (2003), Del Pino-Kowalczyk-Musso (2005), Gui-Wei (2000), Lin-Ni-Wei (2007), Wei-Yan (2010), Li-Wei-Xu (2017), etc

Model Problem: Nonlinear Schrodinger equation in dimension N

We start with the following model problem to illustrate the idea of finite dimensional reduction method:

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + u^p = 0 & \text{in } \mathbb{R}^N \\ u > 0, & u(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{0.5}$$

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Assumptions on p: $1 < p < \infty$ if $N \leq 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$.

Assumptions on V:

$$V(x) \in C^2_{loc}, \quad 0 < \alpha \leq V(x) \leq \beta < +\infty. \tag{0.6}$$

Building Block

Building Block: Fix a point $P \in \mathbb{R}^N$ and let

$$x = P + \varepsilon y$$

Then (0.5) becomes

$$\Delta_y u - V(P + \varepsilon y)u + u^p = 0$$

Formally let $\varepsilon = 0$ we get

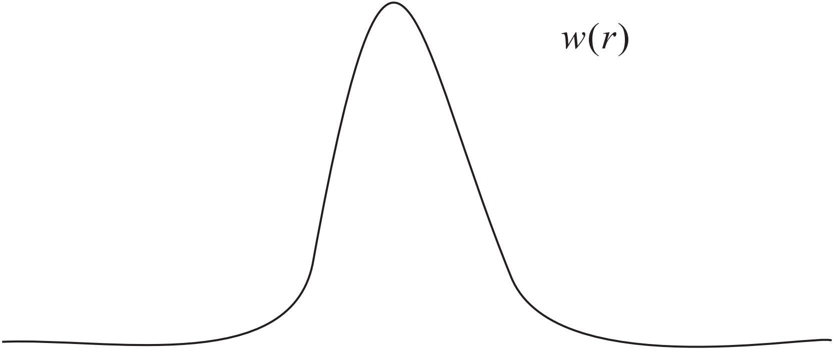
$$\Delta_y u - V(P)u + u^p = 0$$

whose solution is given by $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda} x)$ where $\lambda = V(P)$ and w is a solution to

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\ w > 0, & w(x) \to 0, \quad \text{as } |x| \to \infty \end{cases} \qquad (0.7)$$

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

$$\begin{cases} w'' + \frac{N-1}{r} w' - w + w^p = 0 & r \in (0, \infty) \\ w'(0) = 0,\ 0 < w \text{ in } (0, \infty) & w(|x|) \to 0, \quad \text{as } |x| \to \infty \end{cases} \tag{0.8}$$



We collect the following basic properties of w:

- There exist a unique solution $w(r)$ to (0.8); (Kwong 1989)
- $w(r)$ satisfies the decay estimate $w(r) = A_0 r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r}))$;
- $w(r)$ is nondegenerate, i.e., the only bounded solution to

$$L(\phi) = \Delta\phi + pw^{p-1}\phi - \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N) \qquad (0.9)$$

  is a linear combination of the functions $\frac{\partial w}{\partial y_j}(x)$, $j = 1, \ldots, N$.

Approximate Solutions

We want to solve the problem

$$\begin{cases} S(u) := \varepsilon^2 \Delta u - V(x)u + u^p = 0 & \text{in} \quad \mathbb{R}^N \\ u > 0 & u(x) \to 0, \quad \text{as} \quad |x| \to \infty \end{cases} \tag{0.10}$$

Fix a point $P \in \mathbb{R}^N$ and let

$$w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda} x)$$

Then

$$S(w_\lambda(\frac{x-P}{\varepsilon})) = (V(P) - V(P + \varepsilon y)) w_\lambda(y) \approx 0$$

Formulation of the problem

Similarly given k points $P_1, ..., P_k$, as long as

$$|P_i - P_j| >> \varepsilon$$

then

$$S(\sum_{j=1}^{m} w_{V(P_j)}(\frac{x - P_j}{\varepsilon})) \approx 0$$

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Conversely we ask: can we find a true solution to

$$S(u) := \varepsilon^2 \Delta u - V(x)(u) + u^p = 0$$

with

$$u \approx \sum_{j=1}^{m} (V(P_j))^{\frac{1}{p-1}} w(\sqrt{V(P_j)}(\frac{x - P_j}{\varepsilon}))$$

New formulation

$$x = \varepsilon y$$

$$S(v) := \Delta v - V(\varepsilon y)v + v^p = 0 \qquad (0.11)$$

$$P_j = \varepsilon P_j', j = 1, \ldots, k$$

$$\lambda_j = V(P_j)$$

$$W_j = w_{\lambda_j}(x - P_j'), \lambda_j = V(P_j),$$

$$W = \sum_{j=1}^{k} W_j$$

Configuration space

$$\Lambda = \{\mathbf{P} = (P_1, P_2, \dots, P_k) \in \mathbb{R}^{kN} \ \mid \ \min_{j\neq l} |P_j - P_l| \gg \varepsilon\}$$

We look for a solution

$$v(x) = \sum_{j=1}^{k} W_j + \phi$$

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + E + N(\phi) = 0 \tag{0.12}$$

where

$$L[\phi] = \Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi$$

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W+\phi)^p - W^p - pW^{p-1}\phi.$$

$$E = \sum_j (\lambda_j - V(\varepsilon x)) W_j$$

$$+ (\sum_j W_j)^p - \sum_j W_j^p.$$

At each point $x = P_i$, $Z^i_j = \frac{\partial W_j}{\partial y_i}$ satisfies

$$\Delta Z^i_j - V(P_i) Z^i_j + p W_i^{p-1} Z^i_j = 0$$

so

$$L[Z^i_j] = \Delta Z^i_j - V(\varepsilon x) Z^i_j + p W^{p-1} Z^i_j$$

$$= (V(P_i) - V(P_i + \varepsilon y)) Z^i_j + p(W^{p-1} - W_i^{p-1}) Z^i_j \approx 0$$

Reduction Method

Step I: For any $\mathbf{P} \in \Lambda$, find a pair $(\phi_{\mathbf{P}}, c_{\mathbf{P}})$ such that

$$\begin{cases} S(W+\phi) = L[\phi] + E + N(\phi) = \sum_{i,j} c_j^i Z_j^i, \\ \langle \phi, Z_j^i \rangle = 0, \ \forall\, i = 1, \dots, N, j = 1, \dots, k \end{cases}$$

Step 2: solving the reduced problem

$$c_j^i(\mathbf{P}) = 0, \ \forall\, j = 1, \dots, N, i = 1, \dots, k$$

by adjusting $\mathbf{P}$ in the configuration space Λ.

Step I

To solve Step I, we first solve a linearized projected problem:

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \qquad (0.13)$$

The c_j^i's are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g) Z_{j_0}^{i_0} = \sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \dots, N$, $j_0 = 1, \dots, k$. The c_j^i's are indeed uniquely determined provided that $|P_l' - P_j'| > R_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is "nearly diagonal", which means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N}\int |\nabla W_j|^2 & \text{if } (i,j) = (i_0,j_0), \\ o(1) & \text{if not} \end{cases}$$

$$\sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0} = \int_{\mathbb{R}^N} (\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g) Z_{j_0}^{i_0}$$

$$= \int_{\mathbb{R}^N} (V(P_{j_0}) - V(\varepsilon x)) Z_{j_0}^{i_0}\phi + p(W^{p-1} - W_{j_0}^{p-1}) Z_{j_0}^{i_0}\phi + gZ_{j_0}^{i_0}$$

$$|c_{j_0}^{i_0}| \leq C \sum_{i,j} \int |\phi| [|\lambda_j - V| + p|W^{p-1} - W_j^{p-1}|] |Z_j^i| + \int |g||Z_j^i|$$

$$\leq C(\|\phi\|_\infty + \|g\|_\infty)$$

More precise estimates are needed.

If we rescale $x = P_j' + y$, we get

$$|(\lambda_j - V(\varepsilon x))Z_j^i| \leq |(V(P_j) - V(P_j + \varepsilon y))||\frac{\partial w_{\lambda_j}}{\partial y_i}| \leq C\varepsilon e^{-\frac{\sqrt{\alpha}}{2}|y|},$$

because $|\frac{\partial w_{\lambda_j}}{\partial y_i}| \leq Ce^{-|y|\sqrt{\lambda_j}}|y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1})Z_j^i| = |((1 - \sum_{l \neq j} \frac{W_l}{W_j})^{p-1} - 1)|W_j^{p-1}Z_j^i.$$

We estimate the interactions at each spike in two regions. Observe that if $|x - P_j'| < \delta_0 \min_{j_1 \neq j_2} |P_{j_1}' - P_{j_2}'|$, then

$$\frac{W_l(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_l}|x-P_l'|}}{e^{-\sqrt{\lambda_j}|x-P_j'|}} < \frac{e^{-\sqrt{\lambda_l}|x-P_l'|}}{e^{-\sqrt{\lambda_j}\delta_0 \min_{j_1 \neq j_2} |P_{j_1}'-P_{j_2}'|}}$$

If $\delta_0 \ll 1$ but fixed, we conclude that $e^{-\sqrt{\lambda_l}|P_j'-P_l'|+\delta_0(\sqrt{\lambda_l}-\sqrt{\lambda_j}) \min_{j_1 \neq j_2} |P_{j_1}'-P_{j_2}'|} < e^{-\rho \min_{j_1 \neq j_2} |P_{j_1}'-P_{j_2}'|} \ll 1$. Thus we conclude that if $|x - P_j'| < \delta_0 \min_{j_1 \neq j_2} |P_{j_1}' - P_{j_2}'|$ then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \leq e^{-\rho \min_{j_1 \neq j_2} |P_{j_1}'-P_{j_2}'|} e^{-\frac{\alpha}{2}|x-P_j'|}.$$

On the other hand if $|x - P'_j| > \delta_0 \min_{j_1 \neq j_2} |P'_{j_1} - P'_{j_2}|$, then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \leq C|Z_j^i| \leq Ce^{-\rho \min_{j_1 \neq j_2} |P'_{j_1} - P'_{j_2}|} e^{-\frac{\alpha}{2}|x - P'_j|}$$

As a conclusion we obtain the following estimate

$$|c_{j_0}^{i_0}| \leq C(\varepsilon + e^{-\rho \min_{j_1 \neq j_2} |P'_{j_1} - P'_{j_2}|})\|\phi\|_\infty + \|g\|_\infty \tag{0.14}$$

Main result on linearized projected problem

Main Result on the linearized projected problem: Given $k \geq 1$, there exist R_0, C_0, ε_0 such that for all points P'_j with $|P'_{j_1} - P'_{j_2}| > R_0$, $j = 1, \ldots, k$ and all $\varepsilon < \varepsilon_0$ then exist a unique solution ϕ to the linearized projected problem

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c^i_j Z^i_j, \qquad (0.15)$$

with

$$\|\phi\|_\infty \leq C_0 \|g\|_\infty.$$

The map $g \longrightarrow \phi$ is denoted $\phi = T[g]$.

We first prove the a priori estimate

$$\|\phi\|_\infty \leq C_0\|g\|_\infty.$$

If not there exist $\varepsilon_n \to 0$, $\|\phi_n\|_\infty = 1$, $\|g_n\|_\infty \to 0$, $P_j'^n$ with $\min_{j_1\neq j_2} |P_{j_1}'^n - P_{j_2}'^n| \to \infty$. We denote $W_n = \sum_j W_{j_n}$, and we have

$$\Delta\phi_n - V(\varepsilon_n x)\phi_n + pW_n^{p-1}\phi_n + g_n = \sum_{i,j}(c_j^i)_n(z_j^i)_n$$

Our first observation is that $(c_j^i)_n \to 0$ (which follows from the estimate for $c_{j_0}^{i_0}$).

$$|c_{j_0}^{i_0}| \leq C(\varepsilon + e^{-\rho \min_{j_1\neq j_2} |P'_{j_1} - P'_{j_2}|})\|\phi\|_\infty + \|g\|_\infty$$

Next we claim that $\forall R > 0$ $\|\phi_n\|_{L^\infty(B(P_j'^n, R))} \to 0$, $j = 1, \ldots, k$.

Inner Estimates

If not, there exist j_0 $\|\phi_n\|_{L^\infty(B(P'^n_{j_0},R))} \geq \gamma > 0$. We denote $\tilde{\phi}_n(y) := \phi_n(P'^n_{j_0} + y)$. We have $\|\tilde{\phi}_n\|_{L^\infty(B(0,R))} \geq \gamma > 0$. Since $|\Delta\tilde{\phi}_n| \leq C$, $\|\tilde{\phi}_n\|_\infty \leq 1$. This implies that $\|\nabla\tilde{\phi}_n\| \leq C$. Passing to a subsequence we may assume $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compacts sets. Observe that also $V(\varepsilon_n(P'^n_{j_0} + y)) = V(\varepsilon_n P'^n_{j_0}) + O(\varepsilon_n|y|) \to \lambda_{j_0}$ over compact sets and $W_n(P'^n_{j_0} + y) \to W_{\lambda_{j_0}}(y)$ uniformly on compact sets. This implies that $\tilde{\phi}$ is a solution of the problem

$$\Delta\tilde{\phi} - \lambda_{j_0}\tilde{\phi} + pw_{\lambda_0}^{p-1}\tilde{\phi} = 0, \quad \int \tilde{\phi}\frac{\partial W_{\lambda_{j_0}}}{\partial y_i}dy = 0, i = 1, \dots, N$$

Nondegeneracy of $w_{\lambda_{j_0}}$ implies that $\tilde{\phi} = \sum_i \alpha_i \frac{\partial w_{\lambda_{j_0}}}{\partial y_i}$. The orthogonality condition implies that $\alpha_i = 0$, $\forall i = 1, \dots, N$. This implies that $\tilde{\phi} = 0$ but $\|\tilde{\phi}\|_{L^\infty(B(0,R))} \geq \gamma > 0$, a contradiction.

Outer Estimate

Now we prove: $\|\phi_n\|_{L^\infty}(\mathbb{R}^N \setminus \cup_n B(P_j'^n, R)) \to 0$, provided that $R \gg 1$ and fixed so that $\phi_n \to 0$ in the sense of $\|\phi_n\|_\infty$ (again a contradiction). We will denote $\Omega_n = \mathbb{R}^N \setminus \cup_n B(P_j'^n, R)$. For $R \gg 1$ the equation for ϕ_n has the form

$$\Delta\phi_n - Q_n\phi_n + g_n = 0$$

where $Q_n = V(\varepsilon x) - pW_n^{p-1} \geq \frac{\alpha}{2} > 0$ for some R sufficiently large (but fixed).

What we know: $|\phi_n|_\infty = o(1)$ on $\partial(\cup_n B(P_j'^n, R))$; $|\phi_n|_\infty \leq 1$.

Let us take for $\sigma^2 < \alpha/2$ and we choose a barrier

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-P_j'^n|} + \mu_n.$$

where $\mu_n = \sum_j \|\phi_n\|_{L^\infty(B(\xi_j^n,R))} + C\|g_n\|_\infty$.
Easy to check that it is a barrier for $|y| > R \gg 1$:

$$-\Delta\bar{\phi} + Q_n\bar{\phi} - g_n > -\Delta\bar{\phi} + \frac{\alpha}{2}\bar{\phi} - \|g_n\|_\infty > \frac{\alpha}{2}\mu_n - \|g_n\|_\infty > 0$$

Maximum principle implies that $\phi_n(x) \leq \bar{\phi}$ for all $x \in \Omega_n$. Taking $\delta \to 0$ this implies that $\phi_n(x) \leq \mu_n$, for all $x \in \Omega_n$. It is also true that $|\phi_n(x)| \leq \mu_n$ for all $x \in \Omega_n^c$, and this implies that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$.

If in addition we have the following decay for the error

$$\theta_n = \|g_n \left(\sum_j e^{-\rho|x-P_j'^n|} \right)^{-1} \|_\infty \to 0$$

with $\rho < \alpha/2$, then we can use as a barrier function

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-P_j'^n|} + \mu_n \sum_j e^{-\rho|x-P_j'^n|}$$

with $\mu_n = e^{\rho R} \sum_j \|\phi_n\|_{L^\infty(B(P_j'^n, R))} + \theta_n$.

As a conclusion we also get the a priori estimate

$$\|\phi\|_* := \|\phi \left(\sum_{j=1}^k e^{-\rho|x-P_j'|} \right)^{-1} \|_\infty \le C\|g\|_{**} = C\|g \left(\sum_{j=1}^k e^{-\rho|x-P_j'|} \right)^{-1}$$

provided that $0 \le \rho < \alpha/2$, $|P_{j_1}' - P_{j_2}'| > R_0 \gg 1$, $\varepsilon < \varepsilon_0$.

This extra decay estimate will be helpful for the compactness argument.

Existence

Let $R >> 1$ and $g \in L^2(B_R) \cap L^\infty$. The weak formulation for

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \quad \int \phi Z_j^i = 0, \forall i,j \tag{0.16}$$

is to find $\phi \in X_R = \{\phi \in H_0^1(B_R) : \int \phi Z_j^i = 0, \forall i,j\}$ such that

$$\int_{\mathbb{R}^N} \nabla\phi\nabla\psi + V\phi\psi - pw^{p-1}\phi\psi - g\psi = 0, \quad \forall\psi \in X_R. \tag{0.17}$$

Let Π be the orthogonal projection of g onto the space spanned by the Z_j^i's. By density argument it follows that ϕ solves in weak sense

$$-\Delta\phi + V\phi - pW^{p-1}\phi - g = \Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] \tag{0.18}$$

and $\Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] = \sum_{i,j} c_i^j Z_i j$. Therefore by definition ϕ solves (0.17) implies that ϕ solves (0.18). Classical regularity gives that this weak solution is solution of (0.18) in strong sense, in particular $\phi \in L^\infty$ so that

$$\|\phi\|_\infty \leq C\|g\|_\infty. \tag{0.19}$$

Now we give the proof of existence for (0.16). We take $g \in L^2(B_R) \cap L^\infty$. By Riesz theorem, equation (0.18) can be written in the following way:

$$\langle \phi, \psi \rangle_{H^1} + \langle B[\phi], \psi \rangle_{H^1} = \langle \tilde{g}, \psi \rangle_{H^1} \qquad (0.20)$$

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$, where $B := (-\Delta + V)^{-1}[pW^{p-1}\phi]$ is a compact operator.

Because of the decaying property of W, it is easy to see that B is a compact operator.

Now we prove existence with the aid of Fredholm alternative. Problem (0.16) is solvable if for $\tilde{g} = 0$ the only solution to (0.17) is $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (0.16)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Thus for $g \in L^\infty \cap L^2$ we can first solve $g_R = g\chi_{B_R(0)}$ to get

$$\|\phi_R\|_\infty \le \|g_R\|_\infty \tag{0.21}$$

Taking $R \to \infty$ then along a subsequence $\phi_R \to \phi$ uniform over compacts we obtain a solution to (0.16).

Lipschitz regularity of the map

Next we want to study the dependence and regularity of the solution with respect to the parameters. Let $g \in L^\infty$. We denote $\phi = T_{\mathbf{P}'}[g]$, where $\mathbf{P}' = (P_1', \dots, P_k')$. We want to analyze derivatives $\partial_{P_{ji}'} T_{P'}[g]$. We know that $\|T_{P'}[g]\|_\infty \leq C_0 \|g\|_\infty$. First we make a formal differentiation with respect to P_j'. We denote $\Phi = \frac{\partial \phi}{\partial P_{i_0 j_0}'}$.

We have $\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i$ and $\int \phi Z_j^i = 0$, for all i, j. Formal differentiation yields

$$\Delta\Phi - V\Phi + pW^{p-1}\Phi + +\partial_{\xi_{i_0 j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0 j_0}} Z_i^j = \sum_{i,j} \tilde{c}_j^i Z_j^i \tag{0.22}$$

where formally $\tilde{c}_i^j = \partial_{\xi_{i_0 j_0}} c_i^j$.

The orthogonality conditions is reduced to

$$\int_{\mathbb{R}^N} \Phi Z_j^i = \begin{cases} 0 & \text{if } j \neq j_0 \\ -\int \phi \partial_{\xi_{i_0 j_0}} Z_{j_0}^i & \text{if } j = j_0 \end{cases} \tag{0.23}$$

Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j} Z_j^i$. We want $\int \tilde{\Phi} Z_j^i = 0$, for all i, j. We need

$$\sum_{i,j} \alpha_{i,j} \int Z_j^i Z_{\bar{j}}^{\bar{i}} = \begin{cases} 0 & \text{if } \bar{j} \neq j_0 \\ -\int \phi \partial_{\xi_{i_0 j_0}} Z_{j_0}^i & \text{if } \bar{j} = j_0 \end{cases} \tag{0.24}$$

The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \tilde{\Phi} Z_j^i = 0$, for all i, j. We add to the equation the term $\sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z_j^i$, so $\tilde{\Phi}$ satisfies the equation$\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i$

$$\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{i_0 j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0 j_0}} Z_i^j \tag{0.25}$$
$$= \sum_{i,j} \tilde{c}_j^i Z_j^i - \sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z_j^i$$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$ and hence $\|\Phi\| \leq C\|g\|_\infty$.

The above formal procedure can be made rigorous by performing the analysis discretely, namely we consider solutions corresponding to P and $P+h$ respectively. Then we consider the quotient and pass the limit in h. In conclusion the map $P \to \partial_P \phi$ is well defined and continuous (into L^∞). Besides we also have $\|\partial_\xi \phi\|_\infty \leq C\|g\|_\infty$, and this implies

$$\|\partial_P T_P[\phi]\| \leq C\|g\| \tag{0.26}$$

Nonlinear projected problem

Consider now the nonlinear projected problem

$$\Delta\phi - V\phi + pw^{p-1}\phi + E + N(\phi) = \sum_{i,j} c_i^j Z_j^i, \quad \int \phi Z_i^j = 0, \ \forall i,j \tag{0.27}$$

We solve this by fixed point. We have $\phi = T(E + N(\phi)) =: M(\phi)$. We define $\Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(R^N) : \|\phi\|_* \le M\|E\|_{**}\}$. Remember that $E = \sum_i (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p$. Observe that

$$|E| \le \varepsilon \sum_i e^{-\sigma|x-P_j'|} + ce^{-\delta_0 \min_{j_1 \ne j_2} |P_{j_1}' - P_{j_2}'|} \sum_j e^{-\sigma|x-P_j'|} \tag{0.28}$$

so, for existence we have $\|E\|_{**} \le C[\varepsilon + e^{-\delta_0 \min_{j_1 \ne j_2} |P_{j_1}' - P_{j_2}'|}] =: \rho$ (see that ρ is small). Contraction mapping implies there exists a unique solution $\phi = \Phi(P)$ and $\|\Phi(P)\| \le M\rho$. The proof is standard and hence omitted.

Differentiability in P' of $\Phi(P')$

As before the solutions obtained for the nonlinear projected problem has more regularity. In fact we can write the equation for Φ as

$$\Phi - T'_P(E'_P + N'_P(\phi)) = A(\Phi, P') = 0 \qquad (0.29)$$

If $(D_\Phi A)(\Phi(P'), P')$ is invertible in L^∞, then $\Phi(P')$ turns out to be of class C^1. This is a consequence of the fixed point characterization, i.e., $D_\Phi A(\Phi(P'), P') = I + o(1)$ (the order $o(1)$ is a direct consequence of fixed point characterization). Then it is invertible. Contraction mapping theorem yields the existence of C^1 derivative of $A(\Phi, P')$ in (ϕ, P'). This implies $\Phi(P')$ is C^1. With a little bit of more work we can show that $\|D'_P\Phi(P')\| \leq C\rho$ (just using the derivative given by the implicit function theorem).

Step 2: Solving the reduced problem:Direct Method

By (0.27), to solve (0.12), we need to find $\mathbf{P}'$ such that the reduced problem

$$c_j^i = 0, \forall i, j \tag{0.30}$$

to get a solution to the original problem (0.11). There are two ways to solve the reduced problem (0.30): the first one is the direct method, and the second one is the variational reduction method. We describe the first method first by proving the following

Assume that $P_j^0, j = 1, ..., k$ are k distinct non-degenerate critical points of V. Then there exist a solution u_ε to the original problem with

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} w_{V(P_j^\varepsilon)}(x - P_j^\varepsilon/\varepsilon), \quad P_j^\varepsilon \to P_j^0$$

To solve the problem (0.30) we first obtain the asymptotic formula for c_j^i. To this end we multiply the equation (0.27) by $Z_{j_0}^{i_0}$ and integrate by parts. We obtain

$$\int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0} c_j^i = \int_{\mathbb{R}^N} (V(P_j + \epsilon x) - V(P_j)) w_{\lambda_j} Z_{j_0}^{i_0} + O(\epsilon^2)$$

and thus

$$c_{j_0}^{i_0} \sim \partial_{i_0} V(P_j^0) + O(\epsilon)$$

The nondegeneracy of the critical point $\nabla V(P_j^0)$ and implicit function theorem yields the existence of $P_j = P_j^0 + O(\epsilon)$ such that (0.30) holds.

Solving the reduced problem: variational reduction

If the problem concerned has a variational structure, it is more appropriate to use a variational reduction method to solve (0.30). This method gives much stronger results under very weak assumptions.

We now describe the procedure that we call Variational Reduction in which the problem of finding ξ' with $c_j^i = 0$, for all i, j, is equivalent to finding a critical point of a reduced functional of ξ'.

Define an energy functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(\varepsilon x)v^2 - \frac{1}{p+1}\int_{\mathbb{R}^{N+1}} v_+^{p+1} \qquad (0.31)$$

where $v \in H^1(\mathbb{R}^N)$ and $1 < p < \frac{N+2}{N-2}$. Since p is subcritical, by standard elliptic regularity arguments and Maximum Principle v is a solution of the problem

$$\Delta v - Vv + v^p = 0, v \to 0 \qquad (0.32)$$

if and only if $v \in H^1(\mathbb{R}^N)$ and $J'(v) = 0$. Observe that $\langle J'(v), \varphi\rangle = \int \nabla v \nabla \varphi + Vv\varphi - v_+^p \varphi$.

We will prove the following Variational Reduction Principle

Lemma: $v = W_{P'_*} + \phi(P')$ is a solution of the original problem (for $\rho \ll 1$) if and only if

$$\partial_{P'} J(W_{P'} + \phi(P'))|_{P'=P'_*} = 0. \qquad (0.33)$$

Indeed, observe that $v(P') := W_{P'} + \phi(P')$ solves the problem $\Delta v(P') - V(\varepsilon x)v(P') + v(P')^p = \sum_{i,j} c^i_j Z^i_j$ and also that

$$\partial_{P'_{j_0 i_0}} J(v(P')) = \langle J'(v(P')), \partial_{P'_{j_0 i_0}} v(P') \rangle = -\sum_{j,i} c^i_j \int Z^i_j \partial_{P'_{j_0 i_0}} v \tag{0.34}$$

$$= -\sum_{i,j} c^i_j \int Z^j_i (\partial_{P'_{j_0 i_0}} W_{P'} + \partial_{P'_{j_0 i_0}} \phi(P')).$$

Recall that $W_{P'} = \sum_{j=1}^k w_{\lambda_j}(x - P'_j)$ and $\int Z^j_i \phi(P') = 0$ we have

$$\int Z^j_i \partial_{P'_{j_0 i_0}} \phi(P') = -\int \phi(P') \partial_{P'_{j_0 i_0}} Z^j_i = O(\rho)$$

Finally, observe that

$$-\int Z^i_j (\partial_{\xi'_{j_0 i_0}} W'_P + \partial_{P'_{j_0 i_0}} \phi) = \int Z^i_j Z^{i_0}_{j_0} + O(\rho) \tag{0.35}$$

The matrix of these numbers is invertible provided $\rho \ll 1$.

We now discuss several applications of the reduction principle.

Result 1: (del Pino and Felmer (1996)) Assume that there exists an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{\partial\Lambda} V > \inf_{\Lambda} V, \tag{0.36}$$

then there exist a solution to the original problem, v_ε with $v_\varepsilon(x) = w_{V(P_\varepsilon)}((x - P_\varepsilon)/\varepsilon) + o(1)$ and $V(P_\varepsilon) \to \min_\Lambda V$, $P = P_\varepsilon$.

Result 2: (del Pino-Felmer (1998)) Assume that $\Lambda_1, \ldots, \Lambda_k$ are disjoint bounded sets with

$$\inf_{\Lambda_j} V < \inf_{\partial \Lambda_j} V, j = 1, \cdots, k.$$

Then there exist a solution u_ε to the original problem with

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} w_{V(P_j^\varepsilon)}(x - P_j^\varepsilon/\varepsilon), \quad P_j^\varepsilon \in \Lambda_j$$

and $V(P_j^\varepsilon) \to \inf_{\Lambda_j} V$. The same result holds if the minimum is replaced by maximum.

Result 3: (Kang-Wei (2000)) Let Γ be a bounded open set such that

$$\max_{\Gamma} V(x) > \max_{\partial\Gamma} V(x)$$

Then for any positive integer K there exists a solution u_ϵ such that

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} w_{V(P_j^\varepsilon)}(x - P_j^\varepsilon/\varepsilon), \quad P_j^\varepsilon \in \Lambda, V(P_j^\varepsilon) \to \max_{\Lambda} V(x)$$

Assume that $j = 1$ first so that $v(P') = W_{P'} + \phi(P')$. Then we can compute the reduced energy as follows:

$$J(v(P')) = J(W_{P'} + \phi(P')) + \langle J'(W'_P + \phi), -\phi \rangle + \frac{1}{2} J''(W'_P + (1-t)\phi) \tag{0.37}$$

Observe that $\langle J'(W'_P + \phi), -\phi \rangle = \sum_{i,j} c^i_j \int Z^j_i \phi = 0$ and also that

$$J''(W'_P + (1-t)\phi)[\phi]^2$$
$$= \int |\nabla \phi|^2 + V(\varepsilon x)\phi^2 - p(W'_P + (1-t)\phi)\phi^2 = O(\varepsilon^2)$$

uniformly on P' because $\nabla \phi, \phi = O(\varepsilon e^{-\delta |x - P'|})$.

We call $\Phi(P) := J(v(P')) = J(W_{P'}) + O(\varepsilon^2)$, and

$$J(W_{P'}) = \frac{1}{2}\int |\nabla W_{P'}|^2 + V(P)W_{P'}^2 - \frac{1}{p+1}\int W_{P'}^{p+1}$$
$$+ \int (V(\varepsilon x) - V(P'))W_{P'}^2$$

Taking $\lambda = V(P)$, we have that

$$\frac{1}{2}\int |\nabla W_{P'}|^2 + V(P^{'})W_{P'}^2 - \frac{1}{p+1}\int W_{P'}^{p+1} = V(P)^{p+1/p-1-N/2}c_{p,N}$$

and we also have

$$\int (V(\varepsilon x) - V(P))w_\lambda(x-P')^2 = O(\varepsilon) \qquad (0.38)$$

uniformly in P'.

In summary we have the following asymptotic expansion of the reduced energy

$$\Phi(P) = J(v(P')) = V(P)^{p+1/p-1-N/2}c_{p,N} + O(\varepsilon) \qquad (0.39)$$

To prove Result 1, we observe that $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall \varepsilon \ll 1$ we have

$$\inf_{P \in \Lambda} \Phi(P) < \inf_{P \in \partial\Lambda} \Phi(P) \tag{0.40}$$

and therefore Φ has a local minimum $P_\varepsilon \in \Lambda$ and $V(P_\varepsilon) \to \min_\Lambda V$. The same procedure also works for local maximums.

For several separated local minimums, the proof is similar. In fact when $|P_{j_1} - P_{j_2}| > \delta$, for all $j_1 \neq j_2$, we have $\rho = e^{-\delta_0 \min_{j_1 \neq j_2} |P'_{j_1} - P'_{j_2}|} + \varepsilon \leq e^{-\delta_0 \delta/\varepsilon} + \varepsilon < 2\varepsilon$. So we obtain

$$|\nabla_x \phi(P')| + |\phi(P')| \leq C\varepsilon \sum_j e^{-\delta_0 |x - P'_j|} \tag{0.41}$$

Now we get

$$J(v(P')) = \sum_j V(\varepsilon P'_j)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon) \tag{0.42}$$

$\varepsilon P' = (P_1, \ldots, P_k)$ implies for several minimal points on the Λ_j we have the result desired.

Finally we prove the existence of multiply interacting spikes. The computations are little bit involved since we have to measure precisely the interactions. The reduced energy functional takes the following form:

$$J(v(P')) = \sum_j V(\varepsilon P_j)^{p+1/p-1-N/2}(c_{p,N} + o(1))$$
$$-(1+o(1)) \sum_{i \neq j} e^{-\min_{i\neq j}(\sqrt{V(P_i),V(P_j)})|P'-P'_j|}.$$

We shall take the following configuration space

$$\Sigma = \{(P_1, ..., P_k) \mid P_i \in \Lambda, \min_{i\neq j} |P_i - P_j| > \rho\epsilon \log \frac{1}{\epsilon}\}$$

and prove that the following maximization problem attains a solution in the interior part of the set Σ:

$$\max_{(P_1,...,P_k)\in\Sigma} J(v(P'))$$

Key takeaways

- finite-dimensional reduction method works for variational or non-variational problems
- The first ingredient needed is the nondegeneracy of the building block
- The key argument is the a priori estimates for the linearized projected problem.
- A priori estimates have two parts: inner estimates and outer estimates. Inner-outer gluing scheme grows out of these considerations.
- To solve the reduced problem, many new techniques from critical point theory, integrable system, etc can be used.

Other reduction problems

- Problem I: Lin-Ni-Takagi type problem

$$\epsilon^2\Delta u - u + u^p = 0 \text{ in } \Omega,\ \partial_\nu u = 0 \text{ on} \partial\Omega$$

Wei-Winter (1998), Lin-Ni-Wei (2007), Ao-Wei-Zeng (2007)
Intermediate Reduction Method: del Pino-Wei-Yao (2012), Ao-Wei (2012)

- Problem II: Brezis-Nirenberg type problems

$$\Delta u + \lambda u + u^{\frac{N+2}{N-2}} = 0$$

Building Block: Talenti bubbles

$$U[z,\lambda](x) := (N(N-2))^{\frac{N-2}{4}} \left(\frac{\lambda}{1+\lambda^2|x-z|^2} \right)^{\frac{N-2}{2}}$$

Bahri-Li-Rey (1995), Rey (1992), del Pino-Felmer-Musso (2003), Li-Wei-Xu (2017)

- Problem III: Liouville type problems

$$\Delta u + \varepsilon^2 e^u = 0 \text{ in } \mathbb{R}^2$$

del Pino-Kowalczyk-Musso (2005)

- Problem IV: Ginzburg-Landau type problems

$$\varepsilon^2 \Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^2$$

del Pino-Kowalczyk-Musso (2006)

Applications of
Finite-dimensional Reduction Methods

Application I: Klein-Gordon Equation

$$(KG) \quad \Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2, u \in H^1(\mathbb{R}^2)$$

Does the finite energy assumption or the decaying condition

$$\lim_{|x| \to +\infty} u(x) = 0$$

impose some kind of symmetry on the solutions?

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$$\lim_{|x| \to +\infty} u(x) = 0$$

impose some kind of symmetry on the solutions?
Yes, if $u > 0$. Gidas-Ni-Nirenberg 1979

Surprisingly, the answer to this question is negative for sign-changing solutions. In fact, we prove the :

Theorem (Ao-Musso-Pacard-Wei 2012): *There exist infinitely many solutions of (KG) which have finite energy but whose group of symmetry reduces to the identity.*

Idea of Proofs

The construction is by Liapunov-Schmidt finite-dimensional reduction method.

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The construction is by Liapunov-Schmidt finite-dimensional reduction method.

What is the perturbation parameter?

We use the number of spikes as the perturbation parameter.
(Wei-Yan 2010)

The idea of the construction is to start with two finite sets of points

$Z^+ := \{z_j^+ \in \mathbf{C} \,:\, j = 1, \dots, n^+\}$, $Z^- := \{z_j^- \in \mathbf{C} \,:\, j = 1, \dots, n^-\}$,

and define an approximate solution to (KG) by simply adding copies of $+w$ centered at the points z_j^+ and copies of $-w$ centered at the points z_j^-. More precisely, we define an approximate solution $\tilde{u}$ by the formula

$$\tilde{u} := \sum_{z \in Z^+} w(\cdot - z) - \sum_{z' \in Z^-} w(\cdot - z').$$

We set

$$Z := Z^+ \cup Z^-,$$

and let

$$\ell := \min_{z \neq z' \in Z} |z - z'|.$$

Since the solution w is exponentially decreasing to 0 at infinity, $\tilde{u}$ is a fairly good approximate solution of (KG) as ℓ tends to infinity. Indeed, if

$$\tilde{E} := \Delta \tilde{u} - \tilde{u} + \tilde{u}^3,$$

it is not hard to check that

$$\|\tilde{E}\|_{L^\infty(\mathbf{C})} \leq C\, e^{-\ell}\, \ell^{-1/2}.$$

for some constant $C > 0$ which does not depend on $\ell \gg 1$.

The natural idea is then to let ℓ tend to infinity and to look for a solution u of (KG) as a (small) perturbation of $\tilde{u}$. Writing $u = \tilde{u} + v$, this amounts to solve a nonlinear problem of the form

$$\tilde{L}v + \tilde{E} + \tilde{N}(v) = 0, \tag{0.43}$$

where

$$\tilde{L} := \Delta - 1 + 3\,\tilde{u}^2,$$

is the linearized operator about $\tilde{u}$ and where

$$\tilde{N}(v) := v^3 + 3\,\tilde{u}\,v^2,$$

collects all the nonlinear terms.

The linear operator L has $2\,(n^+ + n^-)$ eigenfunctions of $\tilde{L}$ which are associated to small eigenvalues which in addition tend to 0 as ℓ tends to infinity (in fact, in absolute value, these small eigenvalues can be seen to tend to 0 exponentially fast as ℓ tends to infinity).

Reduction

We perform the usual reduction method:
Step I: For $z \in Z$ there exists $u := \tilde{u} + v$ (where v is a small function) solution of

$$\Delta u - u + u^3 = \sum_{z \in Z} \langle F_z, \nabla w(\cdot - z) \rangle_{\mathbf{C}}, \tag{0.44}$$

where, for each $z \in Z$, the complex number $F_z \in \mathbf{C}$ depends on all the coordinates of the points of Z.
Step 2: find the set of points Z (which become parameters of the construction) in such a way that

$$F_z = 0, \quad \text{for all} \quad z \in Z. \tag{0.45}$$

The asymptotic behavior of F_z as ℓ, the minimum of the distances between the points of Z, tends to infinity, is

$$F_z \sim \sum_{z' \in Z - \{z\}} \eta_z \, \eta_{z'} \, \Upsilon(|z' - z|) \, \frac{z' - z}{|z' - z|}, \qquad (0.46)$$

where the interaction function Υ is explicitly known and is known to satisfy

$$\Upsilon(s) \sim e^{-s} \, s^{-1/2},$$

as s tends to infinity and where $\eta_z = +1$ if, in the definition of $\tilde{u}$, there is a positive copy of w centered at the point z and $\eta_z = -1$ if, in the definition of $\tilde{u}$, there is negative copy of w centered at the point z.

At this stage, even if we assume that ℓ is large, finding the sets of points of Z in such a way that $F_z = 0$ for all $z \in Z$ seems to be a rather difficult and even hopeless task. However, in view of the asymptotic behavior of Υ, one quickly realizes that, in the expression of F_z given by (0.46), only the closest neighbors of z in Z are of interest since the influence of the other points will be of higher order and hence, will be negligible. This suggests that we should restrict our attention to the sets of points Z satisfying the following condition :

There exists $C > 0$ and $\delta > 0$ such that, if $z \neq z' \in Z$, then

$$\text{either} \quad \ell \leq |z' - z| \leq \ell + C, \text{or} \quad |z' - z| \geq (1+\delta)\,\ell. \tag{0.47}$$

Under this condition, we define, for all $z \in Z$

$$N_z := \{z' \in Z - \{z\} \, : \, |z' - z| \leq \ell + C\},$$

to be the set of closest neighbors of z in Z and, for each $z' \in N_z$, we define $\lambda_{zz'} \in \mathbf{R}$ by

$$|z' - z| = \ell - \lambda_{zz'}.$$

Under condition (0.47) and using these notations, we find that, at main order

$$F_z \sim e^{-\ell} \, \ell^{-1/2} \sum_{z' \in N_z} \eta_z \, \eta_{z'} \, e^{\lambda_{zz'}} \, \frac{z' - z}{|z' - z|}.$$

Therefore, to find a set Z satisfying (0.45), it is reasonable to look for a set of points which are perturbations of a set Z for which

$$\sum_{z' \in N_z} a_{zz'} \frac{z' - z}{|z' - z|} = 0, \tag{0.48}$$

for all $z \in Z$, where we have defined

$$a_{zz'} := \eta_z \, \eta_{z'} \, e^{\lambda_{zz'}} \in \mathbf{R} - \{0\}.$$

In other words, the question reduces now to be able to find a set of points Z as well as parameters $a_{zz'} \in \mathbf{R} - \{0\}$ for each $z, z' \in Z$ such that $z' \in N_z$, in such a way that (0.48) holds. But, we also need to require that

$$|z' - z| = \ell - \ln |a_{zz'}|, \tag{0.49}$$

for all $z \neq z' \in Z$ such that $z' \in N_z$.

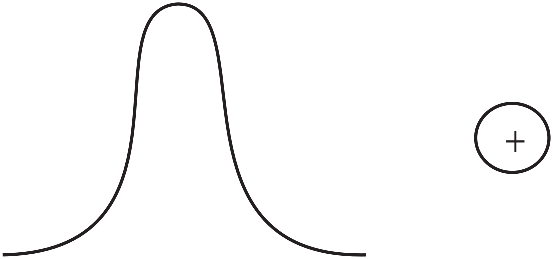
The question is now to find points $z \in \mathbb{R}^2$ such that the following two requirements are satisfied

$$\sum_{z' \in N_z} a_{zz'} \frac{z' - z}{|z' - z|} = 0, \tag{0.50}$$

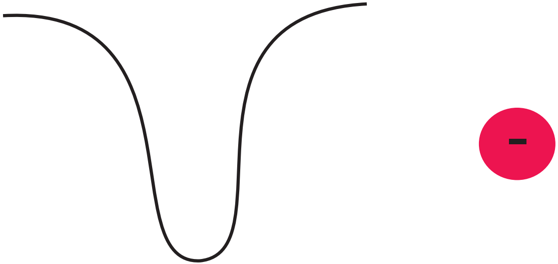
$$|z' - z| = \ell - \ln |a_{zz'}|, \tag{0.51}$$

Furthermore, these points are also needed to form Flexible,closable and balanced networks

Positive Bump: w



Negative Bump: w



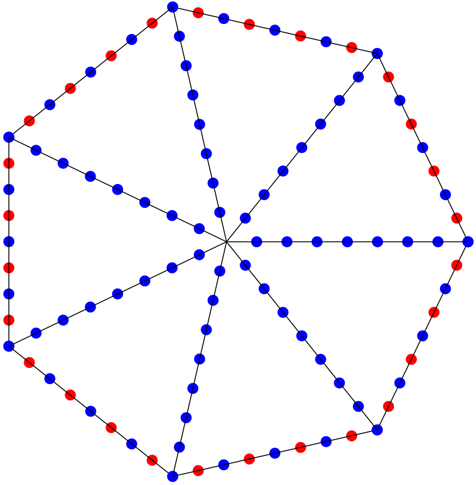


Figure: The location of the bumps. Here $k = 7$, $m = 8$, $n = 4$.

Let m be the number of positive bumps, n be the number of negative bumps, and l be the distance between positive bumps. Then the following relation hold:

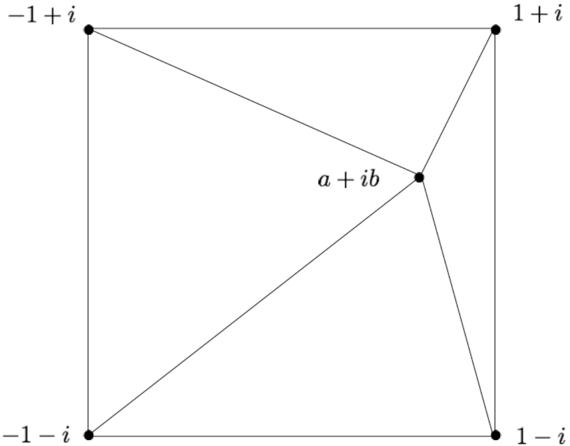
$$\frac{2n-1}{m} = 2\sin\frac{\pi}{k}\,\left(1 - \ln\left(2\sin\frac{\pi}{k}\right)\,\ell^{-1} + O(\ell^{-2})\right)\,. \tag{0.52}$$

$$l \approx \frac{\frac{2n-1}{m} - 2\sin\frac{\pi}{k}}{-\ln\left(2\sin\frac{\pi}{k}\right)} \tag{0.53}$$

$$m_i, n_i \to +\infty$$

$$l_i \to +\infty$$

Nonsymmetric network



Application 2: Optimal quantitative estimates of Struwe's decomposition

Sobolev inequality: for any $N \geq 3$ and any $u \in D^{1,2}\left(\mathbb{R}^N\right)$, it holds that

$$S_N \|u\|_{L^{\frac{2N}{N-2}}} \leq \|\nabla u\|_{L^2}. \qquad (0.54)$$

It is well-known that the Euler-Lagrange equation, up to scaling, is given by

$$\Delta u + u|u|^{\frac{4}{N-2}} = 0 \qquad \text{in } \mathbb{R}^N. \qquad (0.55)$$

Biancho-Engnell estimate

It is known that all the positive solutions are Talenti bubles, i.e.

$$U[z,\lambda](x) := (N(N-2))^{\frac{N-2}{4}} \left(\frac{\lambda}{1+\lambda^2|x-z|^2} \right)^{\frac{N-2}{2}}. \quad (0.56)$$

These are all the minimizers of the Sobolev inequality, up to scaling.

Bianchi-Engnell 1993: quantitative estimate near the minimizers, that is

$$\inf_{z\in\mathbb{R}^N,\lambda>0,\alpha\in\mathbb{R}} \|\nabla(u-\alpha U[z,\lambda])\|_{L^2}^2 \le C(n)\left(\|\nabla u\|_{L^2}^2 - S_N^2\|u\|_{L^{2^*}}\right). \quad (0.57)$$

Struwe Decomposition

A natural and more challenging question is that whether a function u that almost solves

$$\Delta u + |u|^{\frac{4}{N-2}} u \approx 0$$

must be quantitatively close Talenti bubbles.

Struwe Decomposition: Let $N \geq 3$ and $\nu \geq 1$ be positive integers. Let $(u_k)_{k \in \mathbb{N}} \subseteq \dot{H}^1(\mathbb{R}^N)$ be a sequence of nonnegative functions such that $\left(\nu - \frac{1}{2}\right) S_N \leq \int_{\mathbb{R}^N} |\nabla u_k|^2 \leq \left(\nu + \frac{1}{2}\right) S_N$, and assume that

$$\Gamma(u) := \left\| \Delta u_k + u_k^{\frac{n+2}{n-2}} \right\|_{H^{-1}} \to 0 \quad \text{as } k \to \infty.$$

Then

$$\delta(u) := \inf_{\substack{(z_1, \cdots, z_\nu) \in \mathbb{R}^n \\ l_1, \cdots, l_\nu > 0}} \left\| \nabla u - \nabla \left(\sum_{i=1}^{\nu} U\left[\bar{z}_i, \bar{\lambda}_i\right] \right) \right\|_{L^2} \to 0$$

Figalli and Glaudo (2020): for $3 \leq N \leq 5$

$$\delta(u) \lesssim \Gamma(u)$$

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Theorem (Deng-Sun-Wei (2021.March.28)):

$$\delta(u) \leq C \begin{cases} \Gamma(u) \left|\log \Gamma(u)\right|^{\frac{1}{2}} & \text{if } N = 6, \\ \left|\Gamma(u)\right|^{\frac{N+2}{2(N-2)}} & \text{if } N \geq 7. \end{cases}$$

Furthermore, we show that this inequality is optimal.

Let

$$\delta(u) := \inf_{\substack{(z_1,\cdots,z_\nu)\in\mathbb{R}^n \\ l_1,\cdots,l_\nu>0}} \left\| \nabla u - \nabla\left(\sum_{i=1}^{\nu} U\left[\bar{z}_i, \bar{\lambda}_i\right]\right)\right\|_{L^2}.$$

be achieved by some

$$\sigma := \sum_{i=1}^{\nu} U\left[z_i, \lambda_i\right]. \tag{0.58}$$

Let us denote $U_i := U\left[z_i, \lambda_i\right]$.

Let $\rho := u - \sum_i U_i$ be the difference between the original function and the best approximation. Then ρ satisfies the equation

$$\Delta(\sum_i U_i + \rho) + (\sum_i U_i + \rho)^{\frac{N-2}{N-2}} = f \tag{0.59}$$

where $f = \Delta u + u|u|^{\frac{4}{N-2}}$.

Moreover, ρ also satisfies the following orthogonal conditions

$$\int_{\mathbb{R}^n} \nabla\rho\nabla Z_i^a = 0 \;\text{ for any } 1 \le i \le \nu, 1 \le a \le n+1, \tag{0.60}$$

where Z_i^a are the derivatives of $U[z_i, l_i]$ with respect to z_i and l_i.

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Question: what is the relation between

$$\|\nabla \rho\|_{L^2}$$

and $\|f\|_{H^{-1}}$?

The key idea is to obtain the precise behavior of first approximation of ρ by standard finite-dimensional reduction method: given a family of $(U_i)_{1\leq i\leq \nu}$ which is weak-interacting, we can find a function ρ_0 and a family of scalars $\left(c_a^i\right)$ such that we can solve

$$\begin{cases} \Delta(\rho_0+\sum_j U_j)+|\rho_0+\sum_j U_j|^{\frac{4}{N-2}}(\rho+\sum_j U_j)=\sum_{i=1}^{\nu}\sum_{a=1}^{N+1} c_a^i U_i^{\frac{4}{N-2}} Z_i^a \\ \int \nabla\rho_0\nabla Z_i^a=0, \quad i=1,\cdots,\nu;\ a=1,\cdots,N+1. \end{cases} \tag{0.61}$$

It turns out that $\rho\approx\rho_0$.

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Key Question: what is the asymptotic behavior of ρ?

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Key Question: what is the asymptotic behavior of ρ?
Main Problem: bubble towers, bubble clusters, bubble towers/clusters interwinned.

Linear Projected Problem for critical exponent problem: One bubble Case

One bubble problem: denote $U_0 = U[0;1], p = \frac{N+2}{N-2}$.

$$\begin{cases} \Delta\phi + pU_0^{p-1}\phi + g = \sum_{i=1}^{N+1} c_i U_0^{p-1} Z^i \\ \int U_0^{p-1} Z^i \phi = 0 \end{cases} \tag{0.62}$$

$$Z^i = \frac{\partial U_0}{\partial y_i}, Z^{N+1} = U_0 + \frac{N-2}{2} y \nabla U_0$$

$$\|\phi\|_* = \sup(1+|y|)^\sigma |\phi|$$

$$\|g\|_{**} = \sup(1+|y|)^{2+\sigma} |g|$$

For $\sigma \in (0, N-2)$

$$\|\phi\|_* \lesssim \|g\|_{**}$$

Reduction Method for two bubbling clusters

Let $U_1 := U[-Re_1, 1]$ and $U_2 = U[Re_1, 1]$. By reduction theorem, one can prove the existence of ρ such that

$$\begin{cases} \Delta(U_1 + U_2 + \rho) + (U_1 + U_2 + \rho)^p = \sum_{i,a} \tilde{c}_a^i U_i^{p-1} Z_i^a = 0, \\ \int \tilde{U}_i^{p-1} Z_i^a \rho = 0. \end{cases} \tag{0.63}$$

Estimates for ρ:

$\rho \sim R^{2-n}(< x - Re_1 >^{-2} \chi_{|x-Re_1|<R} + < x + Re_1 >^{-2} \chi_{|x+Re_1|<R})$

(interior estimate)

$$+R^{-2}|x|^{2-N} \log \frac{|x|}{R} \chi_{|x-Re_1|>R, |x+Re_1|>R}$$

(outer estimate)