Γ-Convergence and Applications to Phase Transitions

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Abstract: In this series of lectures, we will study the notion of Γ-convergence, as introduced by De Giorgi in 1975 (see [13]), to rigorously derive the classical model for phase transitions between two fluids as an asymptotic limit of a family of Cahn-Hilliard energies.

Introduction: Consider two fluids confined in a container $\Omega \subset \mathbb{R}^N$. In the classical Gibbs theory of phase transitions, it is assumed that, at equilibrium, it is energetically favored for the two fluids (or phases) to arrange themselves in order to minimize the area of the interface that separates them. Here, every configuration is described by a field $u : \Omega \to \{a, b\}$, with $a < b$, $u \equiv a$ on the set occupied by the first fluid, and $u \equiv b$ on the complement, i.e., the set occupied by the second fluid. The total mass is conserved, to be precise, $\int_\Omega u(x)\, dx = m\, \mathcal{L}^N(\Omega)$, for a fixed $m \in (a, b)$, where $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure. Writing $u(x) := a\chi_E(x) + b\chi_{\Omega\setminus E}(x)$, with $\mathcal{L}^N(E) = \frac{b-m}{b-a}\mathcal{L}^N(\Omega)$, it is then postulated that equilibrium configurations should minimize an energy of the form

$$F_0(u) := \sigma \mathcal{H}^{N-1}(\partial E \cap \Omega) \tag{1}$$

for an appropriate constant surface tension $\sigma \in (0, \infty)$. Here $\mathcal{H}^{N-1}$ is the $N-1$-dimensional Hausdorff measure.

In the van der Walls-Cahn-Hilliard theory of phase transitions, the transition is not given by a separating interface, but instead is a continuous transition occurring within a thin layer which, at the macroscopic level, we identify with an interface of width $\varepsilon > 0$. The family of energies under consideration is then (see [4, 21])

$$G_\varepsilon(u) := \int_\Omega W(u(x))\, dx + \varepsilon^2 \int_\Omega |\nabla u(x)|^2\, dx,$$

where $W : \mathbb{R} \to [0, \infty)$, the energy density per unit volume, is a double-well potential with $\{t \in \mathbb{R} : W(t) = 0\} = \{a, b\}$. When minimizing G_ε, the term $\int_\Omega W(u(x))\,dx$ favors those configurations that take values close to a and b, while the term $\varepsilon^2 \int_\Omega |\nabla u(x)|^2\,dx$ penalizes rapid changes of the density u.

If we had simply considered solutions to

$$\min\left\{\int_\Omega W(u(x))\,dx : \int_\Omega u(x)\,dx = m\right\}, \tag{2}$$

then we would have been faced with a striking nonuniqueness of solutions, since any u of the form $u(x) = a\chi_E(x) + b\chi_{\Omega\setminus E}(x)$, with $\mathcal{L}^N(E) = \frac{b-m}{b-a}\mathcal{L}^N(\Omega)$, would render this energy zero. By contrast, now stable density distributions are solutions to the minimization problem

$$\min\left\{G_\varepsilon(u) : \int_\Omega u(x)\,dx = m\right\}. \tag{3}$$

In 1993 Gurtin (see [16]) conjectured solutions u_ε to (3) converge, as $\varepsilon \to 0$, to solutions u_0 of (2) with minimal surface area, i.e., $u_0(x) = a\chi_{E_0}(x) + b\chi_{\Omega\setminus E_0}(x)$, with $\mathcal{H}^{N-1}(\partial E_0 \cap \Omega) \leq \mathcal{H}^{N-1}(\partial E \cap \Omega)$ for every measurable set $E \subset \Omega$ with $\mathcal{L}^N(E) = \frac{b-m}{b-a}\mathcal{L}^N(\Omega)$. Moreover, he also conjecture that

$$G_\varepsilon(u_\varepsilon) \sim \varepsilon\mathcal{H}^{N-1}(\partial E_0 \cap \Omega).$$

Using results of Modica and Mortola (see [18]), this conjecture was proved independently, for $N \geq 2$, by Modica (see [17]) and by Sternberg (see [22]) in the context of Γ-convergence. The one dimensional case $N = 1$ was studied by Carr, Gurtin, and Slemrod in [5].

In these lectures, and as time permits, we will :

- give a brief introduction to Γ-convergence,
- introduce the Cahn-Hilliard model for phase transitions,
- study optimal profile problem and the proof of the Modica-Mortola theorem.

Additional relevant references include [1–3, 6–12, 14, 15, 19, 20, 23, 24].

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