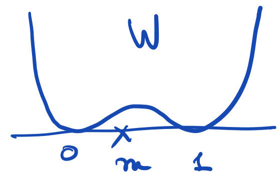
$$I_\varepsilon(u) := \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \right) dx$$



$\varepsilon_n \to 0^+$

$$⨍_\Omega u = m$$

Compactness If u_n admissible and if

$$M := \sup_n I_{\varepsilon_n}(u_n) < +\infty$$

To show: $\exists \; \{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$, $u \in BV(\Omega; \{0,1\})$

$$⨍_\Omega u = m$$

with $u_{n_k} \to u$ in L^1 strong

Obs 1 last time we "prove this" for u_n not necessarily minimizing I_{ε_n}

Obs 2 the "proof" did not considered the mass constraint

TODAY:
- $W: \mathbb{R} \to [0, +\infty)$ continuous
- $W(t) \geq c|t|$ for all $|t| > L$, some $L \gg 1$ for some $c > 0$.

claim 1 $\{u_n\}$ is bounded in $L^1(\Omega)$

$$\int_{\{|u_n| \geq L\}} |u_n| \, dx \leq \frac{1}{c} \int_{\{|u_n| \geq L\}} W(u_n) \, dx \leq \frac{M}{c} \underbrace{\varepsilon_n}_{\in (0,1)} \leq \frac{M}{c} \quad (1)$$

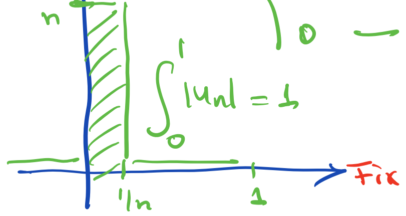
$$\int_\Omega |u_n| \, dx = \int_{\{|u_n| \geq L\}} |u_n| \, dx + \int_{\{|u_n| < L\}} |u_n| \, dx$$

$$\leq \frac{M}{c} + L|\Omega|, \quad \forall n \in \mathbb{N}$$

claim 2 $\{u_n\}$ is equi-integrable (i.e., uniformly small in energy on small sets)

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|E| \subset \Omega$ measurable, $|E| < \delta$, then

$u_n(x) = \,)^n \quad 0 < x < 1/n$ (p-equiintegrable $\sup \int_E |u_n|^p dx < \varepsilon$)



$\sup_n \int_E |u_n|\,dx < \varepsilon$ (no concentrations!)

Fix $\varepsilon>0$. let $\delta>0$ if $n \gg 1$ s.t. $\frac{1}{n}<\delta$ then $E:=(0,1/n)$ admissible, $\int_E |u_n| \leq \varepsilon$

$u_n \to \delta_0$

UPSHOT Then can use Vitali's Convergence Theorem:

if $|\Omega| < +\infty$, $1 \leq p < +\infty$ (we want $p=1$)

then $\boxed{v_n \to v \text{ in } L^p}$ iff (i) $v_n \to v$ in measure

i.e., $\forall \eta>0$ $|\{x\in\Omega : |v_n(x)-v(x)| > \eta\}| \xrightarrow[n\to+\infty]{} 0$

(ii) $\{v_n\}$ is p-equi-integrable

claim 2 addresses (ii) (we'll look @ (i) afterward)

Fix $\varepsilon>0$. Find $N_\varepsilon \gg 1$ s.t. $\forall\, n \geq N_\varepsilon$

$$\frac{M\varepsilon_n}{c} < \frac{1}{2}\varepsilon. \quad (\varepsilon_n \to 0^+) \qquad (2)$$

(1),
$$\int_{E\cap\{|u_n|\geq L\}} |u_n| \leq \frac{M}{c}\varepsilon_n \underset{\substack{(2)\\ n\geq N_\varepsilon}}{<} \frac{\varepsilon}{2}$$

$$\int_{E_n\{|u_n|<L\}} |u_n| \leq L|E| \underset{\text{want}}{<} \frac{\varepsilon}{2} \Rightarrow |E| < \frac{\varepsilon}{2L}$$

So
$$\begin{cases} n \geq N_\varepsilon \\ |E| < \frac{\varepsilon}{2L} \end{cases} \Rightarrow \int_E |u_n| < \varepsilon \qquad (3)$$

$\{u_1, \dots, u_{N_\varepsilon-1}\}$ finite family of L^1 integrable functions

let $\delta_1>0$ be such that $|E|<\delta_1$ then

$$\max_{i\in\{1,\dots,N_\varepsilon-1\}} \int_E |u_i|\,dx < \varepsilon \qquad (4)$$

By (3)+(4), $\delta := \min\{\delta_1, \frac{\varepsilon}{2L}\}$, if $|E|<\delta$ then

$\sup_n \int |u_n| < \varepsilon$

$\sup_n \int_E$

(ii) need to prove convergence in measure

Recall Egoroff's Thm

If $|\Omega| < +\infty$ and if $u_n(x) \to u(x)$ pointwise a.e.
then $\forall \varepsilon > 0$ $\exists$ $E \subset \Omega$ measurable, $|\Omega \setminus E| < \varepsilon$

$$\operatorname*{ess\,sup}_{x \in E} |u_n(x) - u(x)| \xrightarrow[n\to+\infty]{} 0 \quad \text{(uniform convergence)}$$

BUT: Pointwise convergence $\Rightarrow$ convergence in measure
(and if so, we "reduce" (i) to proving pointwise convergence!)

Why: Fix $\eta > 0$ To show: $\forall \varepsilon > 0$ $\exists$ N_ε $\forall n \geq N_\varepsilon$

$$|\{x \in \Omega : |u_n(x) - u(x)| > \eta\}| < \varepsilon$$

Egoroff's Thm: $\exists$ $E \subset \Omega$, $|\Omega \setminus E| < \varepsilon$ let $N_\varepsilon \gg 1$ $\forall n \geq N_\varepsilon$
Fix ε,

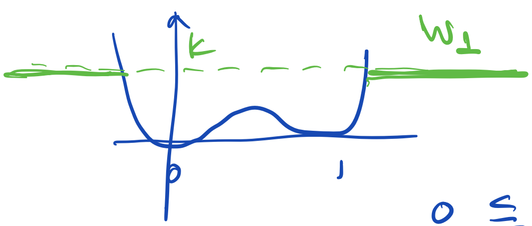
$$\operatorname*{ess\,sup}_{x \in E} |u_n(x) - u(x)| < \eta$$

So $\forall n \geq N_\varepsilon$

$$|\overbrace{\{x \in \Omega ; |u_n(x) - u(x)| > \eta\}}^{\subset \Omega \setminus E}| \leq |\Omega \setminus E| < \varepsilon.$$

Conclusion: Need to show that u_n converge pointwise a.e. to some u!

Fix $k > 0$ define $W_1(t) := \min\{W(t), k\}$



Define $\phi(t) := \int_0^t \sqrt{W_1(s)}\, ds$

$$0 \leq W_1 \leq W$$

$$\underset{(n \geq 1)}{M+1} \geq I_{\varepsilon_n}(u_n) = \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |\nabla u_n|^2 \right) dx \qquad a^2 + b^2 \geq 2ab$$

$$\geq 2 \int_\Omega \sqrt{W(u_n)}\, |\nabla u_n| \, dx$$

$W(x, \frac{x}{\varepsilon\alpha}, u)$

$$\geq 2 \int_\Omega \sqrt{W_n(u_n)}\, |\nabla u_n| \, dx$$

[Cristoferi, F., Hagerty cited...

$$= 2 \int_\Omega |\nabla(\phi \circ u_n)| \, dx$$

| m-wells / Sisto Baldo mid 90's

Hence $\sup_n \int_\Omega |\nabla(\phi \circ u_n)| \leq M+1$ (5)

$$\begin{cases} \|\phi'\|_\infty \leq \sqrt{k} \Rightarrow \phi \text{ Lipschitz} \\ \phi(0) = 0 \end{cases}$$

$$|\phi \circ u_n(x)| = |\phi \circ u_n(x) - \phi(0)|$$

$$\leq \sqrt{k}\, \underbrace{|u_n(x)|}_{\text{bounded in } L^1 \text{ (claim 1)}}$$

$\Downarrow$

$\{\phi \circ u_n\}$ bounded in L^1 (6)

Rellich-Kondrachov Thm: (5) + (6), i.e., sequence bounded $W^{1,1}$

$\Rightarrow \exists$ subsequence (not relabel) $\exists\, v \in BV(\Omega)$ st.

$$\phi \circ u_n \to v \text{ in } L^1(\Omega)$$

up to a further subsequence (not relabel)

$$w_n := \phi \circ u_n \to v(x) \quad \text{a.e. } x \in \Omega$$

ϕ is strictly increasing and continuous $\Rightarrow \phi^{-1}$ continuous

$$u_n(x) = \phi^{-1}(w_n(x)) \to \phi^{-1}(v(x)) =: u(x), \quad u \in BV$$

Using Vitali's $\Rightarrow \int_\Omega |u_n - u| \to 0 \Rightarrow \fint_\Omega u = \lim_n \fint_\Omega u_n = m$

lecture 1 $\Rightarrow$ $u_n \to u$ pointwise a.e.

$\Downarrow$ (Fatou's lemma) $u \in \{0,1\}$ a.e.

$u = 0\,\chi_E + 1\cdot\chi_{\Omega\setminus E}$, E set of finite perimeter

i.e. χ_E is a BV function. $\blacksquare$

Missing the existence of a Recovery Sequence:

if $u_0(x) = \begin{cases} 0 & x\in E, \\ 1 & x\in \Omega\setminus E, \end{cases}$ $E\subset\Omega$ meas., $\chi_E \in BV$

with $\fint_\Omega u_0 = m$

we want to find $u_\varepsilon \in W^{1,2}(\Omega)$, $\fint_\Omega u_\varepsilon(x)\,dx = m$ $\forall \varepsilon>0$

s.t.
- $u_\varepsilon \to u_0$ in L^1 strong
- $I_\varepsilon(u_\varepsilon) \to c_W \cdot$ "# jumps of u_0"

$c_W := 2\int_0^1 \sqrt{W(s)}\,ds$

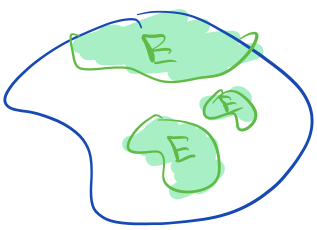
Step 1 $N=1$ $\Omega = (-\ell_1, \ell_2)$, $\ell_1,\ell_2>0$

$u_0(t) := \begin{cases} 0 & \text{if } -\ell_1<t<0 \\ 1 & \text{if } 0<t<\ell_2 \end{cases}$

jumps of u_0 = 1

Step 2 $N\geq 2$ $I_\varepsilon(u_\varepsilon) \to c_W\, \mathcal{H}^{N-1}(\partial E\cap\Omega)$

$\blacksquare$



Minimizing Movements [De Giorgi]

F., Fusco, Leoni and Morini