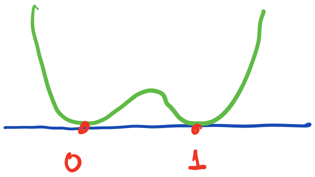
Problem on phase transitions – mixture of 2 fluids



$u=0$ phase 1

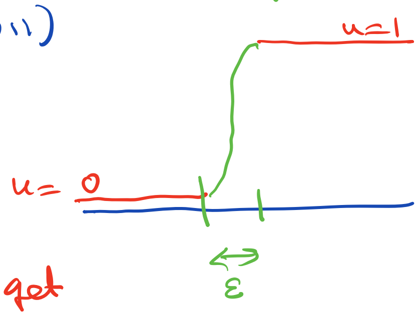
$u=1$ phase 2

$W \geq 0$, $\{W=0\} = \{0,1\}$, W continuous

$$I_\varepsilon(u) := \int_\Omega \Big[\frac{1}{\varepsilon} W(u(x)) + \underbrace{\varepsilon |\nabla u(x)|^2}_{\text{penalizing interface}} \Big] dx \qquad \varepsilon > 0$$

$I_\varepsilon(u) = O(1)$

$$\fint_\Omega u(x)\,dx = m \in (0,1)$$



Question : $I_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} ?$ Do we get a selection criterium

$$I(u) := \int_\Omega W(u(x))\,dx = 0 \quad \text{if} \quad u(x) = \chi_E(x) = \begin{cases} 1 & x \in E, \\ 0 & \text{—} \end{cases}$$

$|E| = m|\Omega|$

$\Omega \subset \mathbb{R}^N$ open, bounded domain

Road Map:

1. $\forall \varepsilon > 0$ I_ε admits a minimizer, u_ε
2. (up to a subsequence) $u_\varepsilon \to u$ ($\Rightarrow$ $u_\varepsilon(x) \to u(x)$ a.e. $x \in \Omega$)
3. $u(x) \in \{0,1\}$ a.e.

Already saw: $\left\{ \begin{array}{l} u_\varepsilon(x) \to u(x) \quad \text{a.e. } x \in \Omega \\ \sup_\varepsilon I_\varepsilon(u_\varepsilon) =: M < +\infty \end{array} \right]$ $\boxed{\text{COMPACTNESS}}$

$\Downarrow$ (Fatou's lemma)

$u(x) \in \{0,1\}$ a.e.

4. Characterize what makes this u special!

To show: • $I_\varepsilon \xrightarrow[\varepsilon \to 0^+]{\Gamma} I$ (Γ-converges) w.r.t. some topology τ

- Get a representation for I

✓ (For free)

- u minimizes I

FACT : If
- $I_\varepsilon \xrightarrow{\Gamma} I$ w.r.t. τ
- u_ε minimizes I
- $u_\varepsilon \xrightarrow{\tau} u$

$$\Downarrow$$

u minimizes I

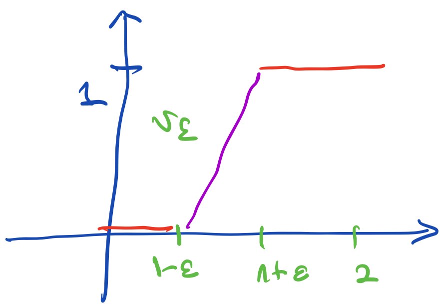
Assume ①, i.e., u_ε minimizer of I_ε

claim $\quad \sup_\varepsilon I_\varepsilon(u_\varepsilon) =: M < +\infty$

Particular setting : $\quad \Omega := (0,2)$

$$W(t) := t^2(1-t)^2$$

mass constraint $\quad \fint_\Omega u(x)\,dx = \frac{1}{2} =: m$



$$v_\varepsilon(x) := \begin{cases} 0 & x \in (0, 1-\varepsilon), \\ \frac{\varepsilon-1}{2\varepsilon} + \frac{1}{2\varepsilon}x & x \in (1-\varepsilon, 1+\varepsilon), \\ 1 & x \in (1+\varepsilon, 2) \end{cases}$$

v_ε is admissible:

$$\fint_\Omega v_\varepsilon(x)\,dx = \frac{1}{2}\int_0^2 v_\varepsilon(x)\,dx$$

$$= \frac{1}{2}\left[\int_{1-\varepsilon}^{1+\varepsilon}\left(\frac{\varepsilon-1}{2\varepsilon} + \frac{1}{2\varepsilon}x\right)dx + (1-\varepsilon)\right]$$

$$= \cdots = \frac{1}{2} = m$$

$$I_\varepsilon(v_\varepsilon) = \int_0^2 \left[\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon |v_\varepsilon'|^2\right] dx$$

$$= \int_{1-\varepsilon}^{1+\varepsilon} \frac{1}{\varepsilon} \underbrace{\left(\frac{\varepsilon-1}{2\varepsilon} + \frac{1}{2\varepsilon}x\right)^2}_{\leq 1} \underbrace{\left(1 - \frac{\varepsilon-1}{2\varepsilon} - \frac{1}{2\varepsilon}x\right)^2}_{\leq 1} dx$$

$$+ \int_{1-\varepsilon}^{1+\varepsilon} \varepsilon \left(\frac{1}{2\varepsilon}\right)^2 dx$$

$$\leq \frac{1}{\varepsilon} 2\varepsilon + \frac{1}{2} = \frac{5}{2}$$

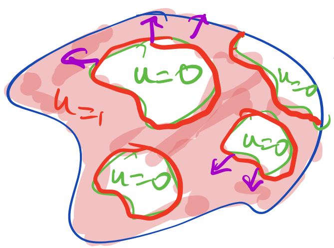
Since u_ε minimizes I_ε $\Rightarrow$ $I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(v_\varepsilon)$

$$\Rightarrow \sup_\varepsilon I_\varepsilon(u_\varepsilon) =: M \leq 5/2 < +\infty .$$

④ We will see that $\int_{\partial\{u=0\}\cap\Omega} \sigma(\nu(x))\, d\mathcal{H}^{N-1}(x)$

$$I(u) := \begin{cases} \sigma \operatorname{Per}(\{u=0\}; \Omega), & u \in \{0,1\} \text{ a.e., } ⨍_\Omega u = m \\ +\infty & \text{—} \end{cases}$$

$$\operatorname{Per}(\{u=0\};\Omega) = \operatorname{Per}(\{u=1\};\Omega) = \mathcal{H}^{N-1}(\partial\{u=0\}\cap\Omega)$$



Ω

$N-1$-dimensional Hausdorff measure.

σ (isotropic) surface energy density — does not depend on the normal

$$\sigma = 2\int_0^1 \sqrt{W(s)}\, ds$$

$u = \chi_E$, $|E| = m|\Omega|$, E ... set of finite perimeter, i.e., $\chi_E \in BV$ (function of bounded variation)

De Giorgi's Γ-convergence

- I_ε ... family of functionals, $I_\varepsilon : X \to (-\infty, +\infty]$

(X, τ) topological space

• $I : X \to [-\infty, +\infty]$

Note: in our case $X = BV(\Omega)$

$$I_\varepsilon(u) := \begin{cases} \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right] dx & \text{if } u \in H^1(\Omega) = W^{1,2}(\Omega) \text{ and } \fint_\Omega u = m \\ +\infty & \text{———} \end{cases}$$

$$I_\varepsilon \xrightarrow{\Gamma(\tau)} I, \quad \text{if :}$$

(i) Γ-$\underline{\lim}$: if $v_\varepsilon \xrightarrow{\tau} v$ then

$$I(v) \leq \varliminf_{\varepsilon \to 0} I_\varepsilon(v_\varepsilon)$$

(ii) Γ-$\overline{\lim}$: $\forall\ v \in X$ $\exists\ w_\varepsilon$ s.t.

(discontinuous) (continuous)

$w_\varepsilon \xrightarrow{\tau} v$

$$I(v) \geq \varlimsup_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon)$$

So

$$I(v) \underset{(ii)}{\geq} \varlimsup_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon) \geq \varliminf_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon) \underset{(i)}{\geq} I(v)$$

UPSHOT:

$$\boxed{I(v) = \lim_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon)}$$

recovery sequence for v

FUNDAMENTAL PROPERTY OF Γ-converge

If $\begin{cases} I_\varepsilon \xrightarrow{\Gamma(\tau)} I \\ u_\varepsilon \text{ minimizes } I_\varepsilon \\ u_\varepsilon \xrightarrow{\tau} u \end{cases} \implies \boxed{u \text{ minimizes } I}$

$$u_\varepsilon \xrightarrow{\tau} u \overset{(i)}{\implies} I(u) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \qquad (*)$$

Claim 1 $\quad \forall\, v \in X \qquad I(u) \leq I(v)$

Fix v arbitrary. By (ii) $\exists\, \{w_\varepsilon\}$ recovery sequence for v:

$$\begin{cases} w_\varepsilon \xrightarrow{\tau} v \\ I(v) = \lim\limits_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon) \end{cases}$$

$$I(v) = \lim_{\varepsilon \to 0} I_\varepsilon(w_\varepsilon) \geq \limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon)$$

u_ε minimizes I_ε

$$\geq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \underset{(*)}{\geq} I(u).$$

Claim 2 $\quad$ Can do better than $(*)$:

$$I(u) = \lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon)$$

why: let $\{z_\varepsilon\}$ recovery sequence for u:

$$I(u) \underset{(ii)}{=} \lim_{\varepsilon \to 0} I_\varepsilon(z_\varepsilon) \geq \limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon)$$

u_ε minimizes I_ε

$$\geq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \underset{(*)}{\geq} I(u)$$

Hence $\quad I(u) = \lim\limits_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon)$

Remark Minimizing sequences are recovery sequences for the limit u

BUT there maybe recovery sequences for u, which are not minimizing sequences

NEXT: Prove (1) ... using the **DIRECT METHOD OF THE CALCULUS OF VARIATIONS**