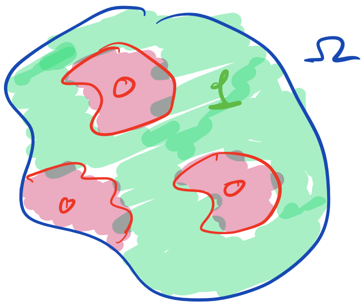
Study Variational Models for Phase Transitions

What unifies these models:

1) energies of $\neq$ dimensionality (bulk, surface, ...)
2) underlying maps have discontinuities

Energy $\rightsquigarrow$ want to minimize energy
(~ equilibrium state)



$\Omega \subset \mathbb{R}^N$ container

$u : \Omega \to \{0, 1\}$
fluid 0 ↗ ↖ fluid 1

Given $\displaystyle ⨍_\Omega u \, dx =: m$ volume fraction

average $\displaystyle \frac{1}{|\Omega|} \int_\Omega =: ⨍_\Omega$

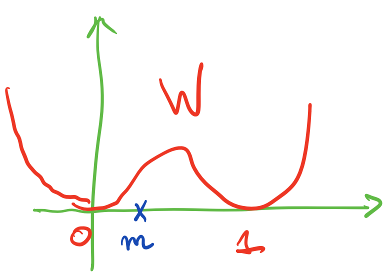
$|\Omega|$ ... $\mathcal{L}^N(\Omega)$ ... N-dim. Lebesgue measure

Energetically, what is the "best way" to arrange these 2 fluids ?

$$E(u) := \int_\Omega W(u(x)) \, dx \qquad W : \mathbb{R} \to [0, +\infty) \text{ continuous}$$

$u : \Omega \to \mathbb{R}$

↑ energy density



... double well potential

E is minimized iff $W(u(x)) = 0$ a.e.

$\iff u \in \{0,1\}$ a.e.

$m \in (0,1)$, so <u>no</u> pure phase

Prototype $W := u^2(1-u)^2$

If $A \subset \Omega$, $u_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$

$$m = ⨍_\Omega u_A(x)\,dx = \frac{|A|}{|\Omega|} \Rightarrow \boxed{|A| = m|\Omega|}$$

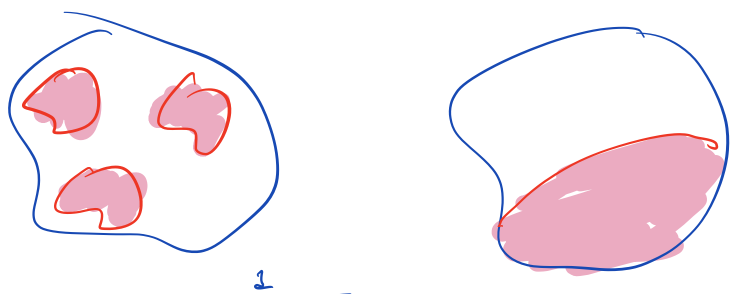
For any $A \subset \Omega$ measurable, $|A| = m|\Omega|$

$\Downarrow$

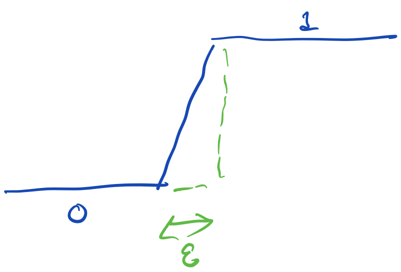
u_A minimizes E $\left(E(u_A) = 0 \right)$

Too much nonuniqueness!

Van der Waals, Cahn–Hilliard Theory of Phase Transitions



sharp interfaces



$\downarrow$ $\nabla u \sim O\left(\frac{1}{\varepsilon}\right)$

$$E_\varepsilon(u) := \underbrace{\int_\Omega W(u(x))\,dx}_{\sim O(\varepsilon)} + \cancel{\varepsilon^2} \overbrace{\int_\Omega |\nabla u|^2\,dx}^{\text{surface energy}} \sim O(\varepsilon)$$

Ω: $\sim \varepsilon$, $\quad |\nabla u|^2$: $\cancel{\frac{1}{\varepsilon^2}}$

Singular perturbation of the original energy

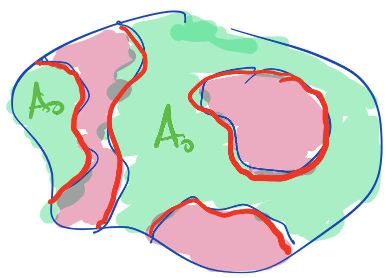
Let u_ε be a minimizer of E_ε (there exist!)

Goal: show
- $u_\varepsilon \to u_0$
- $u_0 \in \{0,1\}$
- u_0 is "special"

$$I_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u) = \int_\Omega \left(\frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u(x)|^2 \right) dx \sim O(1)$$

u_ε still minimizes I_ε !

Mort Gurtin conjectured: write $u_0 = u_{A_0}$, for some $A_0 \subset \Omega$ with $|A_0| = m|\Omega|$



A_0 minimizes surface area inside Ω

$= \mathrm{Per}(\{u=0\}; \Omega)$

$= \mathrm{Per}(\{u=1\}; \Omega)$ (perimeter inside Ω)

Claim: Assume sequence $v_\varepsilon : \Omega \to \mathbb{R}$ with

- $\sup_\varepsilon \frac{1}{\varepsilon} E_\varepsilon(v) = \sup_\varepsilon I_\varepsilon(v_\varepsilon) \left[= \int_\Omega \left(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \right] < +\infty$

$\Downarrow$ (up to a subsequence)

- $v_\varepsilon(x) \to v(x)$ pointwise a.e.

Then $v(x) \in \{0,1\}$ a.e.

Since $\sup_\varepsilon \frac{1}{\varepsilon} \int_\Omega W(v_\varepsilon(x))\, dx \leq M < +\infty$

(multiply through by ε, $\varepsilon \to 0^+$)

$$\Downarrow$$

$$\lim_{\varepsilon \to 0^+} \int_\Omega W(v_\varepsilon(x))\, dx = 0. \qquad (*)$$

Fatou's Lemma $f_\varepsilon \geq 0$

$f_\varepsilon(x) \to f(x)$ a.e.

then $$\boxed{\int_\Omega f(x)\,dx \le \underline{\lim} \int_\Omega f_\varepsilon(x)\,dx.}$$

W continuous, $v_\varepsilon(x) \to v(x)$ a.e.

$\Downarrow$ $W(v_\varepsilon(x)) \to W(v(x))$ a.e.

Apply Fatou's Lemma to (*): $f_\varepsilon(x) := W(v_\varepsilon(x))$

$f(x) := W(v(x))$

Then $$\int_\Omega W(v(x))\,dx \le \underline{\lim} \int_\Omega W(v_\varepsilon(x))\,dx = 0$$

$\uparrow$ by (*)

So $W(v(x)) = 0$ a.e.

$\iff$ $v(x) \in \{0,1\}$ a.e.

$\Rightarrow v = \chi_A$ some A; set of finite perimeter

Precisely, u_0 will minimize

$$I_0(u) := \sigma \,\mathrm{Per}(\{u=0\};\Omega) = \int_{(\partial\{u=0\})\cap\Omega} \sigma \, dH^{N-1}(x)$$

$\Updownarrow$ $\mathrm{Per}_\Omega(\{u=0\})$

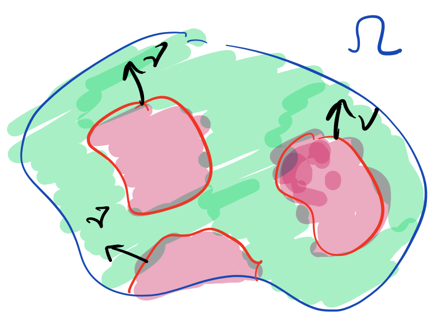
$(=\partial\{u=1\}\cap\Omega)$

constant

isotropic surface energy density

$N-1$ - dim. Hausdorff measure

$$\sigma := 2\int_0^1 \sqrt{W(t)}\,dt$$



ν ... normal to the interface

looking for anisotropic surface energy

$$\int_{(\partial\{u=0\})\cap\Omega} \sigma(\nu(x))\, dH^{N-1}(x)$$

Instead: $$\int_\Omega \underbrace{\frac{1}{\varepsilon} W(u(x))\,dx + \varepsilon f(\nabla u)}_{\frac{1}{\varepsilon} f(x,u,\varepsilon\nabla u)\ldots}$$ Bouchitté