LORENTZ ESTIMATES FOR WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC EQUATIONS WITH SINGULAR DIVERGENCE-FREE DRIFTS

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Abstract. This paper investigates regularity in Lorentz spaces for weak solutions of a class of divergence form quasi-linear parabolic equations with singular divergence-free drifts. In this class of equations, the principal terms are vector field functions which are measurable in (x,t)-variable, and nonlinearly dependent on both unknown solutions and their gradients. Interior, local boundary, and global regularity estimates in Lorentz spaces for gradients of weak solutions are established assuming that the solutions are in BMO space, the Jonh-Nirenberg space. The results are even new when the drifts are identically zero because they do not require solutions to be bounded as in the available literatures. In the linear setting, the results of the paper also improve the standard Calderón-Zygmund regularity theory to the critical borderline case. When the principal term in the equation does not depend on the solution as its variable, our results recover and sharpen known, available results. The approach is based on the perturbation technique introduced by Caffarelli-Peral together with a "double-scaling parameter" technique, and the maximal function free approach introduced by Acerbi-Mingione.

1. Introduction

This paper establishes local interior, local boundary, and global regularity estimates in Lorentz spaces for gradients of weak solutions of the following class of quasi-linear parabolic equations with singular divergence-free drifts, and with conformal boundary condition

$$\left\{\begin{array}{rcll} u_t - \operatorname{div}\Big[\mathbf{A}(x,t,u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)\Big] & = & f(x,t) & (x,t) \in \Omega\times(0,T), \\ \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{\nu}\rangle & = & 0 & (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) & = & u_0(x), & x \in \Omega. \end{array}\right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\vec{\nu}$ is the unit outward normal vector on $\partial\Omega$, $f : \Omega\times(0,T) \to \mathbb{R}$ is a given measurable function, $\mathbf{F}, \mathbf{b} : \Omega\times(0,T) \to \mathbb{R}^n$ are given vector field functions, and u is an unknown solution with a given initial condition u_0 for which we do not require any regularity. Moreover, T is a given fixed positive number, and the principal term

$$\mathbf{A} = \mathbf{A}(x,t,s,\xi) : \Omega\times(0,T)\times\mathbb{K}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is a given vector field. We assume that $\mathbf{A}(\cdot,\cdot,s,\xi)$ is measurable in $\Omega_T = \Omega\times(0,T)$ for every $(s,\xi)\in\mathbb{K}\times\mathbb{R}^n$; $\mathbf{A}(x,t,\cdot,\xi)$ Hölder continuous in $\mathbb{K}$ for a.e. $(x,t)\in\Omega_T$ and for all $\xi\in\mathbb{R}^n$; and $\mathbf{A}(x,t,s,\cdot)$ differentiable in $\mathbb{R}^n$ for each $s\in\mathbb{K}$ and for a.e. $(x,t)\in\Omega_T$. Here, $\mathbb{K}$ is an open interval in $\mathbb{R}$, which could be the same as $\mathbb{R}$. We assume in addition that there exist constants $\Lambda > 0$ and $\alpha_0 \in (0,1]$ such that $\mathbf{A}$ satisfies the following natural growth conditions

$$\langle \mathbf{A}(x,t,s,\eta) - \mathbf{A}(x,t,s,\xi), \eta-\xi\rangle \geq \Lambda^{-1}|\eta-\xi|^2, \quad \text{for a.e. } (x,t)\in\Omega_T, \quad \forall\, s\in\mathbb{K}, \quad \forall\, \xi,\eta\in\mathbb{R}^n, \tag{1.2}$$

$$|\mathbf{A}(x,t,s,\xi)| + |\xi||\partial_\xi \mathbf{A}(x,t,s,\xi)| \leq \Lambda|\xi|, \quad \text{for a.e. } (x,t)\in\Omega_T, \quad \forall\, s\in\mathbb{K}, \quad \forall\, \xi\in\mathbb{R}^n, \tag{1.3}$$

$$|\mathbf{A}(x,t,s_1,\xi) - \mathbf{A}(x,t,s_2,\xi)| \leq \Lambda|\xi||s_1-s_2|^{\alpha_0} \quad \forall\, s_1,s_2\in\mathbb{K}, \quad \text{for a.e. } (x,t)\in\Omega_T, \quad \forall \xi\in\mathbb{R}^n. \tag{1.4}$$

Under the conditions (1.2)–(1.4) and with $\mathbf{F} = \mathbf{b} = 0$, the class of equations (1.1) contains the well-known quasi-linear parabolic equations with zero-flux boundary condition. If $\mathbf{F} = 0$, but $\mathbf{b} \neq 0$, the equation (1.1) is the standard nonlinear advection-diffusion equations. The drift term $\mathbf{b}$ considered in this paper could be singular. Due to its relevance in many applications such as in fluid dynamics and mathematical biology

Date: January 19, 2018.

(see [4, 11, 27, 43, 44, 47] for examples), we are particularly interested in the case that $\mathbf{b}$ is divergence-free, i.e.

$$\text{div}\,[\mathbf{b}(\cdot, t)] = 0, \quad \text{in the sense of distributions in } \Omega, \text{ for a.e. } \quad t \in (0, T). \tag{1.5}$$

On one hand, when $\mathbf{b} = 0$ and $\mathbf{F}, f$ are sufficiently regular, the $C^{1,\alpha}$-regularity theory for *bounded, weak solutions* of this class of equations (1.1) has been investigated extensively in the classical work, see for example [22, 23, 31, 32, 45], assuming some regularity of $\mathbf{A}$ in $(x, t, s, \xi) \in \Omega_T \times \mathbb{K} \times \mathbb{R}^n$. On the other hand, when $\mathbf{b}, \mathbf{F}, f$ are not so regular or when $\mathbf{A}$ is discontinuous in (x, t), one does not expect those mentioned Schauder's type estimates for weak solutions of (1.1) to hold. It is therefore mathematically interesting, and essentially important to search for regularity estimates of Calderón-Zygmund type for gradients of weak solutions in Lebesgue spaces. In particular, in these situations, this kind of Calderón-Zygmund regularity estimates is vital in studying many questions in nonlinear equations and systems of equations, see [27] for example. In this perspective, it is known that to establish the Calderón-Zygmund theory, the class of considered equations must be invariant under the scalings and dilations, see [46] for more geometric intuition of this issue. However, due to the fact that the nonlinearity of the principal term $\mathbf{A}$ depends on u as its variable, the class of this equations (1.1) is not invariant under the scalings and dilations

$$u \mapsto u/\lambda, \quad \text{and} \quad u(x,t) \mapsto \frac{u(rx, r^2 t)}{r}, \quad \text{for all positive numbers} \quad r, \lambda. \tag{1.6}$$

Due the lack of this homogeneity, Calderón-Zygmund type regularity theory for weak solutions of (1.1) becomes delicate, and is still not completely understood. In a simpler case when $\mathbf{A}$ is independent on the variable $s \in \mathbb{K}$, and $\mathbf{b} = f = 0$, the equation (1.1) is reduced to

$$u_t - \text{div}\,[\mathbf{A}(x, t, \nabla u)] = \text{div}\,[\mathbf{F}] \quad \text{in} \quad \Omega_T, \tag{1.7}$$

and the $W^{1,q}$-regularity estimate for weak solutions of equations (1.7) has been non-trivially and extensively developed by many authors for both elliptic, parabolic settings and also for p-Laplacian type equations, for example see [6–8, 10, 12, 14, 17, 18, 30, 34, 37].

In the recent work [27, 38, 39], the $W^{1,q}$-regularity estimates for weak solutions of equations (1.1) with $\mathbf{b} = 0$ is addressed, and the $W^{1,q}$-regularity estimates are established for bounded weak solutions. To overcome the loss of homogeneity that we mentioned, in [27, 38, 39], we introduced some "double-scaling parameter" technique. Essentially, we study an enlarged class of "double-scaling parameter" equations of the type (1.1). Then, by some compactness argument, we successfully applied the perturbation method in [10] to tackle the problem. Careful analysis is required to ensure that all intermediate steps in the perturbation process are uniform with respect to the scaling parameters. See also a very recent work [9] for further implementation of this idea for which global regularity theory for bounded weak solutions of some class of degenerate elliptic equations is obtained. In all mentioned papers [9, 27, 38, 39], the boundedness assumption on the solutions is essential to start the investigation of $W^{1,q}$-theory. This is because the approach uses maximum principle for the unperturbed equations to implement the perturbation technique. We would like to refer also to [5] for which the $W^{1,q}$-theory for parabolic p-Laplacian type equations of the form (1.1) is also achieved, but only for continuous weak solutions plus other assumptions on $\mathbf{A}$.

In this paper, we establish regularity estimates in Lorentz spaces for gradients of weak solutions of (1.1) by assuming that the solutions are in the BMO space, i.e. the critical borderline case, and including the singular drifts $\mathbf{b} \neq 0$. We achieve this in Theorem 1.1 and Theorem 1.2, Theorem 1.3 below. Our paper therefore generalizes the results in [5, 9, 27, 39] for (1.1) by relaxing the boundedness assumption on solutions, and putting into the context of Lorentz space setting. Even in the linear case, and with $f = 0$, our results are also stronger than the classical Calderón-Zygmund results. Precisely, in this case, (1.1) is reduced to

$$u_t - \text{div}\,[\mathbf{A}_0(x, t)\nabla u] = \text{div}\,[\mathbf{b}u + \mathbf{F}] \tag{1.8}$$

and the classical Calderón-Zygmund theory gives

$$\|\nabla u\|_{L^p} \leq C[\|\mathbf{F}\|_{L^p} + \|u\|_{L^\infty} \|\mathbf{b}\|_{L^p}].$$

Our results in Theorem 1.1, Theorem 1.2, Theorem 1.3 below improve this estimate by replacing $\|u\|_{L^\infty}$ by its borderline case $[[u]]_{\mathrm{BMO}}$. See also [43] for some similar results in this direction for linear equations and with more regularity assumptions on $\mathbf{b}$. At this point, we also would like to note that when $\mathbf{b}, \mathbf{F}, f$ satisfies some certain regularity conditions, weak solutions of (1.8) are proved in [44, 47] to be in C^α, with some $\alpha \in (0,1)$. The results in this paper therefore can be considered as the Sobolev counter part of this result, but for more general nonlinear equations.

Unlike [9, 27, 38, 39] which use "double-scaling parameter", we only use "single scaling parameter" in the class of our equations (see [41, 42]). Precisely, we will investigate the following class of equations

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}\left[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)\right] & = & f(x,t), & \text{in} \quad \Omega_T, \\ \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{\nu}\rangle & = & 0, & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot,0) & = & u_0(\cdot), & \text{in} \quad \Omega, \end{array}\right. \tag{1.9}$$

with the scaling parameter $\lambda \geq 0$. As we will see in Subsection 2.1, this class of equations is the smallest one that is invariant with respect to the scalings and dilations (1.6), and that contains the class of equations (1.1). When $\lambda = 0$, $f = 0$, and $\mathbf{b} = 0$, the equation (1.9) clearly becomes the equation (1.7). This paper therefore recovers all known results for (1.7) such as [8, 27, 43].

In this paper, $B_R(y)$ denotes the ball in $\mathbb{R}^n$ with radius $R > 0$ and centered at $y \in \mathbb{R}^n$. If $y = 0$, we write $B_R = B_R(0)$. Also, for each $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, we write

$$Q_R(z_0) = B_R(x_0) \times \Gamma_R(t_0), \quad \text{with} \quad \Gamma_R(t_0) = (t_0 - R^2, t_0 + R^2).$$

When $z_0 = 0$, we write

$$Q_R = Q_R(0,0), \quad \Gamma_R = \Gamma_R(0).$$

For a measurable set $U \subset \mathbb{R}^{n+1}$, for some $\rho_0 > 0$, and for a locally integrable $f : U \to \mathbb{R}^n$, the bounded mean oscillation semi-norm of f is defined by

$$[[f]]_{\mathrm{BMO}(U,\rho_0)} = \sup_{\substack{z_0=(x_0,t_0)\in U\\ 0<\rho<\rho_0}} \frac{1}{|Q_\rho(z_0)\cap U|}\int_{Q_\rho(z_0)\cap U} |f(x,t) - \bar{f}_{Q_\rho(z_0)\cap U}|\,dxdt, \quad \text{where}$$

$$\bar{f}_{Q_\rho(z_0)\cap U} = \frac{1}{|Q_\rho(z_0)\cap U|}\int_{Q_\rho(z_0)\cap U} f(x,t)\,dxdt.$$

For each $p > 0$ and $q \in (0,\infty]$, the Lorentz quasi-norm of f on U is defined by

$$\|f\|_{L^{p,q}(U)} = \left\{ \begin{array}{ll} \left\{p\int_0^\infty s^q \Big|\{(x,t)\in U : |f(x,t)| > s\}\Big|^{q/p}\frac{ds}{s}\right\}^{1/q}, & \text{if} \quad q < \infty, \\ \sup_{s>0} s\Big|\{(x,t)\in U : |f(x,t)| > s\}\Big|^{1/p}, & \text{if} \quad q = \infty. \end{array}\right. \tag{1.10}$$

The set of all measurable functions f defined on U so that $\|f\|_{L^{p,q}(U)} < \infty$ is denoted by $L^{p,q}(U)$ and called Lorentz space with indices p and q. It is clear that $L^{p,p}(U) = L^p(U)$ - the usual Lebesgue space. Moreover, $L^{p,q}(U) \subset L^{p,r}$ for all $p > 0$ and $0 < q < r \leq \infty$. When $q = \infty$, the space $L^{p,\infty}(U)$ is usually called "weak-$L^p(U)$" space or Lorentz-Marcinkiewicz space. See [24, Chapter 1.4], for example, for more details on Lorentz spaces.

Our first main result is the interior regularity estimates for the gradients of solutions of (1.1).

Theorem 1.1. *Let $\Lambda > 0, M > 0, p > 2, q \in (0,\infty]$, and $\alpha_0 \in (0,1]$. Then, there exists a sufficiently small constant $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ such that the following statement holds. For every $R > 0$, let $\mathbf{A} : Q_{2R}\times\mathbb{K}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Carathéodory map satisfying* (1.2)-(1.4) *on $Q_{2R}\times\mathbb{K}\times\mathbb{R}^n$ for some $R > 0$ and some open interval $\mathbb{K}\subset\mathbb{R}$, and*

$$[\mathbf{A}]_{\mathrm{BMO}(Q_R,R)} := \sup_{\substack{z_0=(x_0,t_0)\in Q_R,\\ 0<\rho\leq R}} \frac{1}{|Q_\rho(z_0)|}\int_{Q_\rho(z_0)}\Big[\sup_{\substack{\xi\in\mathbb{R}^n\setminus\{0\},\\ s\in\mathbb{K}}} \frac{|\mathbf{A}(x,t,s,\xi) - \bar{\mathbf{A}}_{B_\rho(x_0)}(t,s,\xi)|}{|\xi|}\Big]\,dxdt \leq \delta. \tag{1.11}$$

Then, if $\mathbf{F} \in L^{p,q}(Q_{2R}, \mathbb{R}^n)$*,* $f \in L^{n_* p, n_* q}(Q_{2R})$*, and* u *is a weak solution of*

$$u_t - \operatorname{div}[\mathbf{A}(x, , t, \lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}] = f(x,t) \quad in \quad Q_{2R},$$

with $[[\lambda u]]_{\mathrm{BMO}(Q_R,R)} \leq M$ *for some* $\lambda \geq 0$*, and* $[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b} \in L^{p,q}(Q_{2R}, \mathbb{R}^n)$ *for some given divergence-free vector field* $\mathbf{b}$ *defined on* Q_{2R}*, there holds*

$$\begin{aligned} \|\nabla u\|_{L^{p,q}(Q_R)} \leq C\Big[& \|\mathbf{F}\|_{L^{p,q}(Q_{2R})} + R|Q_{2R}|^{\frac{1}{p} - \frac{1}{pn_*}} \|f\|_{L^{n_* p, n_* q}(Q_{2R})} \\ & + \left\|[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b}\right\|_{L^{p,q}(Q_{2R})} + |Q_{2R}|^{\frac{1}{p} - \frac{1}{2}} \|\nabla u\|_{L^2(Q_{2R})}\Big], \end{aligned} \tag{1.12}$$

where $n_* = \frac{n+2}{n+4}$*, and* C *is a constant depending only on* q*,* p*,* n*,* Λ, α_0*,* $M, \mathbb{K}$*.*

Local regularity estimates near the boundary are not only interesting by themselves, but also important in many problems because they only require local information on data. Our next result is the local regularity estimate on the boundary $\partial\Omega$ for weak solutions u of (1.1). In this theorem, for $z = (y,t) \in \Omega \times \mathbb{R}$, and $R > 0$, we write

$$\Omega_R(y) = \Omega \cap B_R(y), \quad K_R(z) = \Omega_R(y) \times \Gamma_R(t), \quad T_R(z_0) = (\partial\Omega \cap B_R(y)) \times \Gamma_R(t)$$

When $z = (0,0)$, we write

$$\Omega_R = \Omega_R(0), \quad K_R = K_R(0,0), \quad T_R = T_R(0,0).$$

For each $\hat{x} \in \partial\Omega$, we assume that $\operatorname{div}[\mathbf{b}] = 0$ in $\Omega_{2R}(\hat{x})$ and $\langle \mathbf{b}, \vec{\nu} \rangle = 0$ on $B_{2R}(\hat{x}) \cap \Omega$ in the sense that

$$\int_{\Omega_{2R}(\hat{x})} \langle \mathbf{b}(x,t), \nabla\varphi(x)\rangle dx = 0, \quad \forall\ \varphi \in C_0^\infty(B_{2R}(\hat{x})), \quad \text{for a.e. } t \in (0,T). \tag{1.13}$$

Theorem 1.2. *Let* $M > 0, \Lambda > 0, p > 2, q \in (0,\infty]$*, and* $\alpha_0 \in (0,1]$*. Then, there exists a sufficiently small constant* $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ *such that the following statement holds true. Suppose that* $0 \in \partial\Omega$ *and for some* $R > 0$*,* $\partial\Omega \cap B_{2R}$ *is* C^1*, and suppose that* $\mathbf{A} : K_{2R} \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a Carathéodory map satisfying* (1.2)-(1.4) *on* $K_{2R} \times \mathbb{K} \times \mathbb{R}^n$ *for some* $R > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$*, and*

$$[\mathbf{A}]_{\mathrm{BMO}(K_R,R)} := \sup_{\substack{z_0=(x_0,t_0)\in K_R,\\ 0<\rho\leq R}} \frac{1}{|K_\rho(z_0)|} \int_{K_\rho(z_0)} \Big[\sup_{\substack{\xi\in\mathbb{R}^n\setminus\{0\},\\ s\in\mathbb{K}}} \frac{|\mathbf{A}(x,t,s,\xi) - \bar{\mathbf{A}}_{\Omega_\rho(x_0)}(t,s,\xi)|}{|\xi|} \Big] dxdt \leq \delta. \tag{1.14}$$

Then, for every $\mathbf{F} \in L^{p,q}(K_{2R}, \mathbb{R}^n)$*,* $f \in L^{n_* p, n_* q}(K_{2R})$*, if* u *is a weak solution of*

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}] & = & f(x,t), & in \quad K_{2R}, \\ \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu}\rangle & = & 0, & on \quad T_{2R}, \end{array}\right. \tag{1.15}$$

satisfying $[[\lambda u]]_{\mathrm{BMO}(K_R,R)} \leq M$*, and* $[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b} \in L^{p,q}(K_{2R}, \mathbb{R}^n)$ *with some* $\lambda \geq 0$ *and some given divergence-free vector field* $\mathbf{b}$ *defined on* K_{2R} *and satisfying* (1.13) *at* $\hat{x} = 0$*, there holds*

$$\begin{aligned} \|\nabla u\|_{L^{p,q}(K_R)} \leq C\Big[& \|\mathbf{F}\|_{L^{p,q}(K_{2R})} + R|K_{2R}|^{\frac{1}{p} - \frac{1}{pn_*}} \|f\|_{L^{n_* p, n_* q}(K_{2R})} \\ & + \left\|[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b}\right\|_{L^{p,q}(K_{2R})} + |K_{2R}|^{\frac{1}{p} - \frac{1}{2}} \|\nabla u\|_{L^2(K_{2R})}\Big], \end{aligned} \tag{1.16}$$

where $n_* = \frac{n+2}{n+4}$*, and* C *is a constant depending only on* q*,* p*,* n*,* Λ, α_0*,* $M, \mathbb{K}$*.*

Theorem 1.1, Theorem 1.3 are still valid if we replace $Q_\rho(z_0)$ by $\hat{Q}_\rho(z_0) = B_\rho(x_0) \times (t_0 - r^2, t_0]$ and $K_\rho(z_0)$ by $\hat{K}_\rho(z_0) = \Omega_\rho(x_0) \times (t_0 - \rho^2, t_0]$. As a consequence, the following global regularity estimates in Lorentz space for gradients of weak solutions of (1.9) can be obtained.

Theorem 1.3. *Let* $M > 0, \Lambda > 0, p > 2, q \in (0,\infty]$*, and* $\alpha_0 \in (0,1]$*. Then, there exists a sufficiently small constant* $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ *such that the following statement holds true. Suppose that* $\partial\Omega$ *is* C^1*,* *and suppose that* $\mathbf{A} : \Omega_T \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a Carathéodory map satisfying* (1.2)-(1.4) *on* $\Omega_T \times \mathbb{R} \times \mathbb{R}^n$ *for some* $T > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$, *and*

$$\sup_{\substack{z_0=(x_0,t_0)\in\Omega\times(\bar{t},T),\\ 0<\rho\le r}} \frac{1}{|\hat{Q}_\rho(z_0)\cap(\Omega\times(\bar{t},T))|}\int_{\hat{Q}_\rho(z_0)\cap\Omega_T}\Big[\sup_{\substack{\xi\in\mathbb{R}^n\setminus\{0\},\\ s\in\mathbb{K}}}\frac{|\mathbf{A}(x,t,s,\xi)-\bar{\mathbf{A}}_{\Omega_\rho(x_0)}(t,s,\xi)|}{|\xi|}\Big]\,dxdt \le \delta,$$

for some $r > 0, \bar{t} \in (0,T)$. *Then, for every* $\mathbf{F} \in L^{p,q}(\Omega_T,\mathbb{R}^n)$, $f \in L^{n_*p,n_*q}(\Omega_T)$, *if* u *is a weak solution of* (1.9) *satisfying* $[[\lambda u]]_{\mathrm{BMO}(\Omega_T,r)} \le M$ *and* $[[u]]_{\mathrm{BMO}(\Omega_T,r)}\mathbf{b} \in L^{p,q}(\Omega_T,\mathbb{R}^n)$ *with some* $\lambda \ge 0$ *and some given vector field* $\mathbf{b}$ *satisfying* (1.13) *at every* $\hat{x} \in \partial\Omega$, *there holds*

$$\|\nabla u\|_{L^{p,q}(\Omega\times(\bar{t},T))} \le C\left[\|\mathbf{F}\|_{L^{p,q}(\Omega_T)} + \|f\|_{L^{n_*p,n_*q}(\Omega_T)} + \left\|[[u]]_{\mathrm{BMO}(\Omega_T,r)}\mathbf{b}\right\|_{L^{p,q}(\Omega_T)}\right],$$

where $n_* = \frac{n+2}{n+4}$, *and* C *is a constant depending only on* q, p, n, Λ, α_0, $M, \mathbb{K}, r, \Omega, \bar{t}, T$.

Several remarks are worth mentioning regarding Theorem 1.1, Theorem 1.2, and Theorem 1.3 . Firstly, we reinforce that the most important improvement in Theorem 1.1, Theorem 1.2, and Theorem 1.3 is that they relax and do not requires the solutions to be bounded as in the known work [5, 9, 27, 38, 39]. This is completely new even for the case $\mathbf{b} = 0$ and $f = 0$, in comparison to the known work that we already mentioned for both the Schauder's regularity theory and the Sobolev's one regarding weak solutions of equations (1.1). To overcome the loss of boundedness from the assumption, instead of applying maximum principle during the approximation process, we directly derive and carefully use some delicate analysis estimates and Hölder's regularity estimates for solutions of the corresponding homogeneous equations, see the estimates (3.5) and (3.18) for examples. These estimates are first observed in the work [41, 42] but for elliptic equations. In a related context, interested readers may see [15, 33] for other study on C^α-regularity of weak, BMO solutions. Secondly, we also note that due to the availability of f, which is scaled differently compared to $\mathbf{F}$ and ∇u, the approach based on Hardy-Littlewood maximal function, and harmonic analysis used in [7, 8, 10, 41, 42, 46] does not seem to produce our desired estimates here. Instead, we use the maximal-function free approach introduced in [1], and also used in [2, 5, 6]. This paper seems to be the first one that treat the equations (1.1) with in-homogeneuous f in the Lorentz space setting. In addition, this paper also treats quasi-linear equations with non-homogeneous singular drifts $\mathbf{b}$, which has not done before. As one will find in the proof, to deal with $\mathbf{b}$, we introduce the function $\mathbf{G}(x,t) \approx [[u]]_{\mathrm{BMO}}\mathbf{b}(x,t)$ which has the same scaling properties as $\mathbf{F}, \nabla u$. This key fact plays an essential role in the proof. Thirdly, we note that when $\lambda = 0$, $f = 0$, and $\mathbf{b} = 0$, Theorem 1.1, Theorem 1.2 and Theorem 1.3 recover and sharpen results in [6–8, 10, 14, 17, 18, 27, 30, 34, 37, 43] when restricting to the class of equations (1.1) in which $\mathbf{A}$ is independent on $u \in \mathbb{K}$, see Remark 1.4 for more details on this. See also [16, 29, 40] for some other related work with more regular $f, \mathbf{F}$. This paper therefore not only unifies both $W^{1,q}$-theories for (1.1) and (1.7) but also extends the theory to the Lorentz regularity estimate setting. Lastly, observe that all papers such as [6, 7, 9, 37], to cite a few, regarding the $W^{1,q}$-regularity estimates in non-smooth domains only establish globally regularity estimates. Our paper provides the regularity estimates locally for both the interior and the boundary one. Our Theorem 1.1, Theorem 1.2 can be considered as some high regularity estimates of Caccioppoli type which are important for many practical purposes for which local information is available and required. Certainly, our local regularity estimates imply the global ones as Theorem 1.3. However, it is generally impossible to derive local estimates directly from the global ones in [6–9, 37].

Remark 1.4. *Two important points are worth pointing out.*

(i) *This paper does not require any regularity assumption on the initial data* u_0 *in* (1.1), *compared to [6, 8, 37] in which it is assumed that* $u_0 = 0$. *Moreover,* M *is not required to be small. Note also that the condition* $[[u]]_{\mathrm{BMO}}\mathbf{b} \in L^{p,q}$ *is trivial if* $\mathbf{b} = 0$. *Similarly, the condition* $[[\lambda u]]_{\mathrm{BMO}} \le M$ *is always satisfied when* $\lambda = 0$.

(ii) *If $\lambda = 0, \mathbf{b} = 0$ and $f = 0$, known results for* (1.7) *such as [5, 6, 8, 37] provide the estimates of the form*

$$\|\nabla u\|_{L^p} \leq C\Big[\, \|\mathbf{F}\|_{L^p} + 1\Big]. \tag{1.17}$$

Our estimates in Theorems 1.1-1.3 are invariant under the scalings and dilations, and they do not contains the inhomogeneous constant, i.e. the number 1 *in the right hand side of* (1.17)*. Our results are natural, and sharp.*

We now conclude this section by outlining the organization of this paper. Section 2 reviews some definitions, and proves preliminaries needed in the paper. Perturbation arguments, and approximation estimates are given in Section 3. Section 4 establishes estimates of level sets of gradients of solutions. The proofs of main theorems, Theorems 1.1-1.3 are given in Section 5. The paper concludes with an appendix, Appendix A, giving proofs for some reverse Hölder's inequalities needed in the paper.

2. Definitions, and preliminaries

2.1. **Invariant properties, and definitions of weak solutions.** Let $\lambda' \geq 0$, and $Q_{2R} \subset \mathbb{R}^{n+1}$ be the parabolic cylinder of radius $2R$. Let us consider a weak solution u of

$$u_t - \operatorname{div}[\mathbf{A}(x, t, \lambda' u, \nabla u) - \mathbf{b}u(x,t) - \mathbf{F}(x,t)] = f(x,t) \quad \text{in} \quad Q_{2R}$$

Then it is simple to check that for some fixed $\lambda > 0$, the rescaled function

$$v(x,t) = \frac{u(x,t)}{\lambda} \quad \text{for} \quad (x,t) \in Q_{2R}, \tag{2.1}$$

is a weak solution of

$$v_t - \operatorname{div}\Big[\hat{\mathbf{A}}(x, \hat{\lambda} v, \nabla v) - \mathbf{b}(x,t)v(x,t) - \hat{\mathbf{F}}(x,t)\Big] = \hat{f} \quad \text{in} \quad Q_{2R}$$

for $\hat{\lambda} = \lambda\lambda' \geq 0$, $\hat{\mathbf{A}} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\hat{\mathbf{A}}(x,t,s,\xi) = \frac{\mathbf{A}(x,t,s,\lambda\xi)}{\lambda}, \quad \text{and} \quad \hat{f}(x,t) = \frac{f(x,t)}{\lambda}, \quad \hat{\mathbf{F}}(x,t) = \frac{\mathbf{F}(x,t)}{\lambda}, \quad (x,t) \in Q_{2R}. \tag{2.2}$$

Moreover, let $\tilde{\lambda} = R\lambda'$,

$$\begin{aligned} &\tilde{v}(x,t) = \frac{u(Rx, R^2 t)}{R}, \quad \tilde{\mathbf{A}}(x,t,s,\xi) = \mathbf{A}(Rx, R^2 t, s, \xi) \quad (x,t) \in Q_2,\ s \in \mathbb{K}, \quad \xi \in \mathbb{R}^n, \quad \text{and} \\ &\tilde{\mathbf{F}}(x,t) = \mathbf{F}(Rx, R^2 t), \quad \tilde{f}(x,t) = Rf(Rx, R^2 t), \quad \tilde{\mathbf{b}}(x,t) = R\mathbf{b}(Rx, R^2 t), \quad (x,t) \in Q_2. \end{aligned} \tag{2.3}$$

Then, $\tilde{v}$ is a weak solution of

$$\tilde{v}_t - \operatorname{div}[\tilde{\mathbf{A}}(x, t, \tilde{\lambda}\tilde{v}, \nabla\tilde{v}) - \tilde{\mathbf{b}}\tilde{v} - \tilde{\mathbf{F}}] = \tilde{f}, \quad \text{in} \quad Q_2.$$

This is the main reason that we study the class of equation (1.9) with a parameter λ, instead of (1.1).

Remark 2.1. *It is not too hard to see that if $\mathbf{A} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies conditions* (1.2)–(1.4) *on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n$, then the rescaled vector field $\hat{\mathbf{A}} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined in* (2.2) *also satisfies the conditions* (1.2)–(1.4) *on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the same constants Λ, α_0. The same conclusion also holds for $\tilde{\mathbf{A}} : Q_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined in* (2.3)*. Moreover,*

$$[\hat{\mathbf{A}}]_{\mathrm{BMO}(Q_R,R)} = [\tilde{\mathbf{A}}]_{\mathrm{BMO}(Q_1,1)} = [\mathbf{A}]_{\mathrm{BMO}(Q_R,R)}, \quad \textit{and}$$
$$[[\hat{\lambda} v]]_{\mathrm{BMO}(Q_R,R)} = [[\tilde{\lambda}\tilde{v}]]_{\mathrm{BMO}(Q_1,1)} = [[\lambda' u]]_{\mathrm{BMO}(Q_R,R)}.$$

With respect to the scalings and dilations, the following remark follows directly from (1.10), see also [24, Remark 1.4.7].

Remark 2.2. *For all $0 < p, r < \infty$ and for all $0 < q \leq \infty$, if f is a measurable function defined on a measurable set $U \subset \mathbb{R}^{n+1}$, then*
$$\left\| |f|^r \right\|_{L^{p,q}(U)} = \|f\|^r_{L^{rp,rq}(U)}.$$
Moreover, for a measurable function f defined on Q_R with some $R > 0$, then
$$\left\| \tilde{f} \right\|_{L^{p,q}(Q_1)} = R^{-(n+2)/p} \|f\|_{L^{p,q}(Q_R)},$$
where
$$\tilde{f}(x,t) = f(Rx, R^2 t), \quad (x,t) \in Q_1.$$

Let us now give the precise definition of weak solutions that is used throughout the paper.

Definition 2.3. *Let $\mathbb{K} \subset \mathbb{R}$ be an interval, let $\Lambda > 0, \alpha_0 \in (0,1]$ and $\alpha > 2$. Also, let $\Omega \subset \mathbb{R}^n$ be open, and bounded domain with boundary $\partial\Omega$, and $\mathbf{A} : \Omega_T \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy conditions* (1.2)–(1.4) *on Ω_T. For each $\mathbf{F} \in L^2(\Omega_T; \mathbb{R}^n)$, $f \in L^{\frac{2(n+2)}{n+4}}(\Omega_T)$ and $\lambda \geq 0$, a function u is called weak solution of*
$$\begin{cases} u_t - \operatorname{div}\Big[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}\Big] &= f(x,t), \qquad \text{in} \quad \Omega_T, \\ \langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu} \rangle &= 0, \qquad \text{on} \quad \partial\Omega \times (0,T), \end{cases}$$
if $\lambda u(x,t) \in \mathbb{K}$ for a.e. $(x,t) \in \Omega_T$, $u \in L^\infty((0,T), L^2(\Omega)) \cap L^2((0,T), W^{1,2}(\Omega))$, $[[u]]_{\mathrm{BMO}(\Omega_T)}\mathbf{b} \in L^\alpha_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$, and for all $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi(\cdot, 0) = \varphi(\cdot, T) = 0$
$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} \Big\langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}, \nabla\varphi \Big\rangle dxdt = \int_{\Omega_T} f(x,t)\varphi(x,t) dxdt.$$
Here, $L^p(U, \mathbb{R}^n)$ for $1 \leq p < \infty$ is the Lebesgue space consists all measurable functions $f : U \to \mathbb{R}^n$ such that $|f|^p$ is integrable on U, and $W^{1,p}(U)$ is the standard Sobolev space on U. Moreover, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$.

Remark 2.4. *When $\mathbf{b} \neq 0$, we require that the solution $u \in \mathrm{BMO}(\Omega_T)$ to insure that $\int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi \rangle dz$ is well-defined for a singular vector field $\mathbf{b}$. Indeed, for $\mathbf{b} \in L^\alpha_{\mathrm{loc}}(\Omega_T)$ with some $\alpha > 2$, if $\varphi \in C^\infty_0(Q)$ with some cube $Q \subset \Omega_T$, since $\operatorname{div}[\mathbf{b}(\cdot,t)] = 0$, we can write*
$$\int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi \rangle dz = \int_Q \langle \mathbf{b}(u - \bar{u}_Q), \nabla\varphi \rangle dz.$$
Then, it follows from the Hölder's inequality that
$$\left| \int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi \rangle dz \right| \leq \left(\int_Q |\mathbf{b}|^\alpha dz \right)^{1/\alpha} \left(\int_Q |u - \bar{u}_Q|^{\alpha'} dz \right)^{\alpha'} \left(\int_Q |\nabla\varphi|^2 dz \right)^{1/2} < \infty,$$
where α' is defined as
$$\frac{1}{\alpha} + \frac{1}{\alpha'} + \frac{1}{2} = 1. \tag{2.4}$$

2.2. **Some technical lemmas.** Several technical, analysis lemmas are needed in the paper. Our first lemma is a standard iteration lemma which can be found, for example, in [25, Lemma 4.3] or [23, Lemma 6.1].

Lemma 2.5. *Let $\phi : [r, R]$ be a bounded, non-negative function. Assume that for all $r < s < t \leq R$,*
$$\phi(t) \leq \theta\phi(s) + \frac{A}{(t-s)^\kappa} + B$$
where $A, B \geq 0$, $\kappa > 0$ and $\theta \in (0,1)$. Then,
$$\phi(r) \leq C(\kappa, \theta) \left[\frac{A}{(R-r)^\kappa} + B \right].$$

Our next lemma is the classical Hardy's inequality, which can be found, for example, in [26, Theorem 330], [2, Lemma 3.4], and [28].

Lemma 2.6. *Let $h : [0,\infty) \to [0,\infty)$ be a measurable function such that*

$$\int_0^\infty h(\lambda)d\lambda < \infty.$$

Then, for every $\kappa \geq 1$, and for every $r > 0$, there holds

$$\int_0^\infty \lambda^r \left(\int_\lambda^\infty h(\mu)d\mu\right)^\kappa \frac{d\lambda}{\lambda} \leq \left(\frac{\kappa}{r}\right)^\kappa \int_0^\infty \lambda^r \Big[\lambda h(\lambda)\Big]^\kappa \frac{d\lambda}{\lambda}.$$

The following variant of reverse-Hölder's inequality can be found in [2, Lemma 3.5] and it will be useful for the paper.

Lemma 2.7. *Let $h : [0,\infty) \to [0,\infty)$ be a non-increasing, measurable function, and let $\kappa \in [1,\infty), r > 0$. Then, there is $C > 0$ such that*

$$\left(\int_\lambda^\infty \Big[t^r h(t)\Big]^\kappa \frac{dt}{t}\right)^{1/\kappa} \leq \lambda^r h(\lambda) + C\int_\lambda^\infty t^r h(t)\frac{dt}{t}, \quad \textit{for any} \quad \lambda \geq 0.$$

2.3. **Hölder regularity of weak solutions of homogeneous equations.** We recall some results on Hölder's regularity for weak solutions of homogeneous equations that will be needed in the paper. Those results are indeed consequences of the well-known, classical De Giorgi-Nash-Möser theory. Our first lemma is about the interior Hölder's regularity estimate, whose proof, for example, can be found in [31, Theorem 1.1, p. 419 and Theorem 2.1 p. 425], and also in [3, Theorem 2, Theorem 4] and [45, Theorem 2.2].

Lemma 2.8. *Let $\Lambda > 0$, and let $\mathbb{A}_0 : Q_r \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and satisfy* (1.2)-(1.3) *on Q_{2r} with some $r > 0$. If v is a weak solution of the equation*

$$v_t - \operatorname{div}[\mathbb{A}_0(x,t,\nabla v)] = 0, \quad \textit{in} \quad Q_r.$$

Then, there exists $C_0 > 0$ depending only on Λ, n such that

$$\|v\|_{L^\infty(Q_{5r/6})} \leq C_0 \left(\fint_{Q_r} |v|^2 dz\right)^{1/2}.$$

Moreover, there exists $\beta_0 \in (0,1)$ depending on Λ, n and $\|v\|_{L^\infty(Q_{5r/6})}$ such that

$$|v(z) - v(z')| \leq C_0 \|v\|_{L^\infty(Q_{5r/6})} \left[\frac{|x-x'| + |t-t'|^{1/2}}{r}\right]^{\beta_0}, \quad \forall\, z = (x,t),\ z' = (x',t') \in \overline{Q}_{2r/3}.$$

To state the boundary regularity, we recall that for some domain $\Omega \subset \mathbb{R}^n$, and for each $r > 0$, $z_0 = (x_0,t_0) \in \partial\Omega \times \mathbb{R}$, we define

$$\Omega_r(x_0) = \Omega \cap B_r(x_0), \quad \Omega_r = \Omega_r(0), \quad K_r(z_0) = \Omega_r(x_0) \times \Gamma_r(t_0), \quad K_r = K_r(0,0).$$

Moreover, we also write

$$T_r(z_0) = (\partial\Omega \cap B_r(x_0)) \times \Gamma_r(t_0), \quad T_r = T_r(0,0).$$

The following classical boundary Hölder's regularity result can be found in [31, Theorem 1.1, p. 419 and Theorem 2.1 p. 425], and [45, Theorem 4.2].

Lemma 2.9. *Let $\Lambda > 0$ be fixed and let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\partial\Omega \in C^1$. Assume that $\mathbb{A}_0 : K_r \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and satisfy* (1.2)-(1.3) *on $K_r \times \mathbb{R}^n$ for some $r > 0$. Assume also that $T_r \neq \emptyset$ and v is a weak solution of the equation*

$$\begin{cases} v_t - \operatorname{div}[\mathbb{A}_0(x,t,\nabla v)] &= 0, \quad \textit{in} \quad K_r, \\ \langle \mathbb{A}_0(x,t,\nabla v), \vec{\nu} \rangle &= 0, \quad \textit{on} \quad T_r, \end{cases}$$

then there exists $C_0 > 0$ depending only on Λ, n such that

$$\|v\|_{L^\infty(K_{5r/6})} \leq C_0 \left(\fint_{K_r} |v|^2 dz\right)^{1/2}.$$

Moreover, there exists a constant β_0 depending only on Λ, n and $\|v\|_{L^\infty(K_{5r/6})}$ such that $v \in C^{\beta_0}(\overline{K}_{5r/6})$, and

$$|v(z) - v(z')| \leq C_0 \, \|v\|_{L^\infty(K_{5r/6})} \left[\frac{|x-x'| + |t-t'|^{1/2}}{r} \right]^{\beta_0}, \quad \forall \, z = (x,t), \; z' = (x',t') \in \overline{K}_{2r/3}.$$

2.4. **Self-improving regularity estimates of Meyers-Gehring's type.** We need to establish two higher regularity estimates of Meyers-Gehring's type, see [19–21, 35, 36, 44], for weak solutions of (1.1). To begin, let us introduce the following notation with will be used frequently in the paper. For each function f defined on $U \subset \mathbb{R}^{n+1}$, we write

$$\mathcal{G}_U(f) = \left(\int_U |f|^{2n_*} dz \right)^{\frac{1-n_*}{2n_*}}, \quad \text{with} \quad n_* = \frac{n+2}{n+4}. \tag{2.5}$$

Our first lemma is the interior one.

Lemma 2.10. *Let $\Lambda > 0$. Then, there exists $\epsilon_0 = \epsilon_0(\Lambda, n) > 2$ such that the following statement holds. Suppose that $\mathbf{A} : Q_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.3) *on Q_2. If u is a weak solution of the equation*

$$u_t - \operatorname{div}\left[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}\right] = f(x,t), \quad in \quad Q_2,$$

with some $\lambda \geq 0$. Then for every $p \in [2, 2+\epsilon_0]$ and $\gamma_0 > 0$, there exists a constant $C = C(\Lambda, p, n) > 0$ such that

$$\left(\fint_{Q_r(z_0)} |\nabla u|^p dz \right)^{1/p} \leq C \left[\left(\fint_{Q_{2r}(z_0)} |\nabla u|^2 dx \right)^{1/2} + \left(\fint_{Q_{2r}(z_0)} |\mathbf{F}|^p dx \right)^{1/p} + \left(\fint_{Q_{2r}(z_0)} |\mathbf{G}|^{p(1+\gamma_0)} dz \right)^{\frac{1}{(1+\gamma_0)p}} \right.$$
$$\left. + \mathcal{G}_{Q_2}(f) \left(\fint_{Q_{2r}(z_0)} |f|^{n_* p} dx \right)^{1/p} \right],$$

where $z_0 = (x_0, t_0) \in Q_1$, $r \in (0, 1/2)$, $\mathbf{G}(x,t) = \hat{C}_0(n,\gamma_0)[[u]]_{\mathrm{BMO}(Q_1,1)}\mathbf{b}$ with $\hat{C}_0(n,\gamma_0)$ is some definite constant.

The next lemma is a self-improving regularity estimate on the boundary.

Lemma 2.11. *For every $\Lambda > 0$, there exists $\epsilon_0 = \epsilon_0(\Lambda, n) > 2$ such that the following statement holds. Suppose that $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega \in C^1$. Suppose that $\mathbf{A} : K_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.3) *on $K_2 \times \mathbb{K} \times \mathbb{R}^n$ and* (1.13) *holds on Ω_2 with $T_2 \neq \emptyset$. Suppose also that u is a weak solution of the equation*

$$\begin{cases} u_t - \operatorname{div}\left[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}\right] &= f(x,t), \quad in \quad K_2, \\ \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu} \rangle &= 0, \quad on \quad T_2, \end{cases}$$

with some $\lambda \geq 0$. Then, for every $p \in [2, 2+\epsilon_0]$, and $\gamma_0 > 0$, there exists a constant $C = C(\Lambda, p, \gamma_0, n) > 0$ such that

$$\left(\fint_{K_r(z_0)} |\nabla u|^p dz \right)^{1/p} \leq C \left[\left(\fint_{K_{2r}(z_0)} |\nabla u|^2 dz \right)^{1/2} + \left(\fint_{K_{2r}(z_0)} |\mathbf{G}|^{p(1+\gamma_0)} dz \right)^{\frac{1}{p(1+\gamma_0)}} + \left(\fint_{K_{2r}(z_0)} |\mathbf{F}(x,t)|^p dz \right)^{1/p} \right.$$
$$\left. + \mathcal{G}_{K_2}(f) \left(\fint_{K_{2r}(z_0)} |f(x,t)|^{n_* p} dz \right)^{1/p} \right],$$

for every $z_0 = (z_0, t_0) \in T_1$, $r \in (0, 1/2)$, $\mathbf{G}(x,t) = \hat{C}_0(n,\gamma_0)[[u]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}$, $n_* = \frac{n+2}{n+4}$, and with $\hat{C}_0(n,\gamma_0)$ is some definite constant.

Remark 2.12. *Two remarks on Lemma 2.10 and Lemma 2.11 are in ordered.*

(i) *Observe that when $\mathbf{b} \in L^q(Q)$ and $u \in \mathrm{BMO}$, it does not follows that $u\mathbf{b} \in L^q(Q)$. Therefore, the above self-improving regularity estimates are new and could not directly deduced from the known self-improving regularity estimates.*

(ii) *If* $\mathbf{b} \in L^\infty(\mathrm{BMO}^{-1})$ *and* $\mathbf{F} = f = 0$*, a similar self-improving regularity estimate as in Lemma 2.10 for linear equations is established in [44].*

From Remark 2.12, proofs of Lemma 2.10 and Lemma 2.11 are needed. We follow the standard approach using Caccioppoli's estimates as in [19,20]. Details will be given in the appendix at the end of the paper.

3. Approximation estimates

3.1. **Interior approximation estimates.** In this section, let $\mathbf{A} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1.2)–(1.4) on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some $R > 0$. We also recall that $\partial_p Q_R$ is the parabolic boundary of Q_R. We study a weak solution u of the class of equations

$$u_t - \mathrm{div}[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}(x,t)u - \mathbf{F}] = f(x,t), \quad \text{in} \quad Q_{2R}, \tag{3.1}$$

with the parameter $\lambda \geq 0$. The following number is used frequently in the paper

$$n_* = \frac{n+2}{n+4}. \tag{3.2}$$

In the sequel, for each $\alpha > 2$, let $\alpha' > 2$ be the number such that

$$\frac{1}{\alpha'} + \frac{1}{\alpha} = \frac{1}{2}, \quad \text{i.e} \quad \alpha' = \frac{2\alpha}{\alpha - 2}.$$

Moreover, if u is a weak solution of (3.1), we define

$$\mathbf{G}(x,t) = \hat{C}_0(n,\alpha)[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b}(x,t), \quad (x,t) \in Q_{2R}, \quad \text{with some definite constant} \quad \hat{C}_0(n,\alpha),$$

In our first step, we freeze u in $\mathbf{A}$, and then approximate the solution u of (3.1) by a solution of the corresponding homogeneous equations with frozen u. See also in [5,9,27,38,41,42] for some similar approaches.

Lemma 3.1. *Let* $\Lambda, \alpha > 2$ *be fixed. Assume that* $\mathbf{A} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies* (1.2)–(1.4)*, and assume that* $\mathbf{F} \in L^2(Q_{2R}, \mathbb{R}^n)$*,* $f \in L^{2n_*}(Q_{2R})$ *and* $\mathbf{G} \in L^\alpha(Q_{2R})$*. Assume also that* $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ *is a weak solution of* (3.1) *with some* $\lambda \geq 0$*. Then, for each* $z_0 = (x_0, t_0) \in Q_R$*,* $r \in (0,R)$*,*

$$\begin{aligned} \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dx \leq C(\Lambda, n) \Bigg[\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2 \left(\fint_{Q_r(z_0)} |f|^{2n_*} dz \right)^{1/n_*} \\ + \left(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dz \right)^{\frac{2}{\alpha'}} \left(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^\alpha dz \right)^{\frac{2}{\alpha}} \Bigg], \end{aligned} \tag{3.3}$$

where $v \in C(\Gamma_r(t_0), L^2(B_r(x_0))) \cap L^2(\Gamma_r(t_0), W^{1,2}(B_r(x_0)))$ *is the weak solution of*

$$\begin{cases} v_t - \mathrm{div}\,[\mathbf{A}(x,t,\lambda u, \nabla v)] &= 0, \quad \text{in} \quad Q_r(z_0), \\ v &= u, \quad \text{on} \quad \partial_p Q_r(z_0). \end{cases} \tag{3.4}$$

Moreover, it also holds that

$$\begin{aligned} &\left(\fint_{Q_r(z_0)} |v - \bar{u}_{Q_r(z_0)}|^2 dz \right)^{1/2} \\ &\leq C(n,p) \left[r \left(\fint_{Q_r(z_0)} |\nabla v - \nabla u|^2 dx \right)^{1/2} + \left(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^2 dz \right)^{1/2} \right]. \end{aligned} \tag{3.5}$$

Proof. Though, the proof is similar and simpler than that of Lemma 3.5 below. We give the proof for the sake of clarity and completeness. Observe that for a given weak solution $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ of (3.1), by taking $\mathbf{A}_0(x,t,\xi) := \mathbf{A}(x,t,\lambda u(x,t), \xi)$, we see that $\mathbf{A}_0 : Q_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ is independent on the variable $s \in \mathbb{K}$, and it satisfies all assumptions in (1.2)–(1.3). Therefore, the existence of weak solution $v \in C(\Gamma_{2r}, L^2(B_{2r})) \cap L^2(\Gamma_{2r}, W^{1,2}(B_{2r}))$ of (3.4) can be obtained using the Galerkin's method, [31, p. 466-475]. It remains to prove the estimates (3.3), and (3.5). Through the procedure using the Skelov's average (see [5, 13, 38], for examples), we can formally use $v - u$ as a test function for the equation (3.4), and the equation (3.1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{B_{r(x_0)}} |v-u|^2 dx + \int_{B_r(x_0)} \langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla v - \nabla u\rangle dx \\ &= \int_{B_r(x_0)} \langle \mathbf{b}u + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{B_r(x_0)} f(x,t)(v-u)dx. \end{aligned} \tag{3.6}$$

Also, because div $[b(\cdot,t)] = 0$, it follows that

$$\begin{aligned}&\left|\int_{B_r(x_0)} \langle \mathbf{b}(x,t)u(x,t), \nabla u - \nabla v\rangle dx\right| \\ &= \left|\int_{B_r(x_0)} \langle \mathbf{b}(x,t)[u(x,t) - \bar{u}_{Q_r(z_0)}], \nabla u - \nabla v\rangle dx\right| \\ &\leq \left(\int_{B_r(x_0)} |\nabla u - \nabla v|^2 dx\right)^{1/2} \left(\int_{B_r(x_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dx\right)^{1/\alpha'} \left(\int_{B_r(x_0)} |\mathbf{b}(x,t)|^{\alpha}\right)^{1/\alpha}.\end{aligned}$$

Then, it follows from an integration in time, Remark 2.1, (3.6), and the Young's inequality that

$$\begin{aligned}&\frac{1}{2}\sup_{\Gamma_r(t_0)}\int_{B_{r(x_0)}} |v-u|^2 dx + \int_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz \\ &\leq C(\Lambda)\left[\frac{1}{2}\sup_{\Gamma_r(t_0)}\int_{B_{r(x_0)}} |v-u|^2 dx + \int_{Q_r(z_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla v - \nabla u\rangle dz\right] \\ &\leq C(\Lambda)\left[\int_{Q_r(x_0)} |\langle \mathbf{b}u + \mathbf{F}, \nabla u - \nabla v\rangle| dz + \int_{Q_r(z_0)} |f||v-u| dz\right] \\ &\leq \frac{1}{2}\int_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz + C(\Lambda)\left\{\int_{Q_r(z_0)} |\mathbf{F}|^2 dz + \int_{Q_r(z_0)} |f||v-u| dz \right. \\ &\qquad \left. + \left(\int_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dxdt\right)^{2/\alpha'} \left(\int_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt\right)^{2/\alpha}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{\Gamma_r(t_0)} r^{-2}\int_{B_{r(x_0)}} |v-u|^2 dx + \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz \\ &\leq C(\Lambda)\left\{\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + 2\fint_{Q_r(z_0)} |f||v-u| dz \right. \\ &\qquad \left. + \left(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dxdt\right)^{2/\alpha'} \left(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt\right)^{2/\alpha}\right\}.\end{aligned} \tag{3.7}$$

Now, let us denote $p_0 = 2n_* > 1$ where n_* is the number defined in (3.2). Also, let p_0' be the number such that $\frac{1}{p_0} + \frac{1}{p_0'} = 1$, i.e. $p_0' = \frac{2(n+2)}{n}$. It follows from Hölder's inequality, and the parabolic Sobolev imbedding (see [31, eqn (3.2), p. 74] or [13, Proposition 3.1, p. 7]), and Young's inequality

$$
\begin{aligned}
\fint_{Q_r(z_0)} |f||v-u|dz &\leq \left(\fint_{Q_r(z_0)} |v-u|^{p_0'}dz\right)^{1/p_0'} \left(\fint_{Q_r(z_0)} |f|^{p_0}dz\right)^{1/p_0} \\
&\leq C(n) r \left(\fint_{Q_r(z_0)} |\nabla v - \nabla u|^2 dz\right)^{1/p_0'} \left(\sup_{\Gamma_r(t_0)} r^{-2} \fint_{B_r(x_0)} |v-u|^2 dx\right)^{\frac{2}{np_0'}} \left(\fint_{Q_r(z_0)} |f|^{p_0}dz\right)^{1/p_0} \\
&\leq \frac{1}{4} \fint_{Q_r(z_0)} |\nabla v - \nabla u)|^2 dz + \frac{1}{4}\sup_{\Gamma_r(t_0)} r^{-2} \fint_{B_r(x_0)} |v-u|^2 dx + C(n) r^2 \left(\fint_{Q_r(z_0)} |f|^{p_0}dz\right)^{2/p_0}.
\end{aligned}
\tag{3.8}
$$

The estimate (3.8), and (3.7) imply that

$$
\begin{aligned}
& \sup_{\Gamma_r(t_0)} r^{-2} \int_{B_{r(x_0)}} |v-u|^2 dx + \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz \\
& \leq C_0(\Lambda, n) \left[\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2 \left(\fint_{Q_r(z_0)} |f|^{p_0} dz\right)^{2/p_0} \right. \\
& \qquad \left. + \left(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dxdt\right)^{\frac{2}{\alpha'}} \left(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt\right)^{\frac{2}{\alpha}} \right].
\end{aligned}
$$

This proves the estimate (3.3). Also, by the Poincaré's inequality, we see that

$$
\begin{aligned}
\left(\fint_{Q_r(z_0)} |v - \bar{u}_{Q_r(z_0)}|^2 dz\right)^{1/2} &\leq \left[\left(\fint_{Q_r(z_0)} |v-u|^2 dz\right)^{1/2} + \left(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^2 dz\right)^{1/2} \right] \\
&\leq \left[C(n,p) r \left(\fint_{Q_r(z_0)} |\nabla v - \nabla u|^2 dz\right)^{1/2} + \left(\fint_{Q_r(x_0)} |u - \bar{u}_{Q_r(z_0)}|^2 dz\right)^{1/2} \right].
\end{aligned}
$$

This proves (3.5) and completes the proof of the lemma. □

The next step is to approximate the solution u in $Q_{\kappa r}(z_0)$ by the solution w of

$$
\left\{
\begin{array}{rcll}
w_t - \operatorname{div}[\mathbf{A}(x,t,\lambda \bar{u}_{Q_{\kappa r}(z_0)}, \nabla w)] & = & 0, & \text{in } Q_{\kappa r}(z_0), \\
w & = & v, & \text{on } \partial_p Q_{\kappa r}(z_0),
\end{array}
\right.
\tag{3.9}
$$

where in (3.9) v is the weak solution of (3.4), and $\kappa \in (0, 1/3)$ is some sufficiently small constant which will be determined.

Lemma 3.2. *Let $\Lambda, M > 0, \alpha > 2$, and $\alpha_0 \in (0,1]$ be fixed, and let $\epsilon \in (0,1)$. There exist positive, sufficiently small numbers $\bar{\kappa} = \bar{\kappa}(\Lambda, M, n, \alpha_0, \epsilon)$ and $\delta_1 = \delta_1(\epsilon, \Lambda, M, n, \alpha_0) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (1.2)–(1.4)*, and assume that $\mathbf{F} \in L^2(Q_{2R}, \mathbb{R}^n), \mathbf{G} \in L^{\alpha}(Q_{2R}, \mathbb{R}^n)$, $f \in L^{2n_*}(Q_{2R})$ and*

$$
\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2 \left(\fint_{Q_r(z_0)} |f|^{2n_*} dz\right)^{1/n_*} + \left(\fint_{Q_r(z_0)} |\mathbf{G}(x,t)|^{\alpha} dz\right)^{\frac{2}{\alpha}} \leq \delta_1^2,
$$

for some $z_0 = (x_0, t_0) \in Q_R$ and some $r \in (0, R)$. Then, for every $\lambda > 0$, if $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^p(\Gamma_{2R}, W^{1,2}(B_{2R}))$ is a weak solution of (3.1) *satisfying*

$$
\fint_{Q_{2\bar{\kappa} r}(z_0)} |\nabla u|^2 dz \leq 1, \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(Q_R, R)} \leq M,
$$

it holds that

$$
\fint_{Q_{\bar{\kappa} r}(z_0)} |\nabla v - \nabla w|^2 dz \leq \epsilon^2,
$$

where w is the weak solution of (3.9).

Proof. The proof is similar that of Lemma 3.6 in the next subsection using Lemma 2.8 and (3.5) instead of Lemma 2.9 and (3.18) respectively. We therefore skip the proof. □

The next lemma is a general result which particularly compares the solution w of (3.9) with a solution of the corresponding constant coefficient equation.

Lemma 3.3. *Let $\Lambda > 0$ be fixed, then there is some $\gamma = \gamma(\Lambda, n) > 2$ such that the following statement holds. For some $z_0 = (x_0, t_0) \in Q_R$, assume that $\mathbf{A}_0 : Q_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.3) *hold for some $R > 0$. Assume also for some $\rho \in (0, R/2)$, w is a weak solution of*

$$w_t - \operatorname{div}[\mathbf{A}_0(x, t, \nabla w)] = 0, \quad \textit{in} \quad Q_{2\rho}(z_0).$$

Then, there is some function $h \in L^\infty((\Gamma_{7\rho/4}(t_0), L^2(B_{7\rho/4}(x_0)) \cap L^2(\Gamma_{7\rho/4}(t_0), W^{1,2}(B_{7\rho/4}(x_0)))$ such that

$$\left(\frac{1}{|Q_{7\rho/4}(z_0)|}\int_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \leq C(\Lambda, n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_{2R},R)} \left(\frac{1}{|Q_{2\rho}(z_0)|}\int_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\right)^{1/2}.$$

Moreover,

$$\|\nabla h\|_{L^\infty(Q_{3\rho/2}(z_0))} \leq C(\Lambda, n)\left(\frac{1}{|Q_{2\rho}(z_0)|}\int_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\right)^{1/2}.$$

Proof. The proof is simple, and we give it here for the sake of completeness. Observe that from Lemma 2.11, there is $p_1 = p_1(\Lambda, n) > 2$ such that

$$\left(\fint_{Q_{7\rho/4}} |\nabla w|^{p_1} dz\right)^{1/p_1} \leq C(\Lambda, p, n)\left(\fint_{Q_{2\rho}} |\nabla w|^2 dz\right)^{1/2}. \tag{3.10}$$

Let us denote

$$\mathbf{a}(t, \xi) = \fint_{B_{7\rho/4}(x_0)} \mathbf{A}_0(x, t, \xi)dx, \quad \Theta_{\mathbf{A}_0, B_{7\rho/4}(x_0)} = \frac{|\mathbf{A}_0(x, t, \xi) - \mathbf{a}(t, \xi)|}{|\xi|}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then, let h be the weak solution of

$$\begin{cases} h_t - \operatorname{div}[\mathbf{a}(t, \nabla v)] &= 0, \quad \text{in} \quad Q_{7\rho/4}(z_0), \\ h &= w, \quad \text{on} \quad \partial_p Q_{7\rho/4}(z_0). \end{cases} \tag{3.11}$$

Observe that the existence of h can be obtained by a standard method using Galerkin's approximation. Also, from the Skelov's average as in [5, 13], we can formally use $w - h$ as a test function for both the equations of w and of h to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{B_{7\rho/4}(x_0)} |w - h|^2 dx + \int_{B_{7\rho/4}(x_0)} \langle \mathbf{a}(t, \nabla w) - \mathbf{a}(t, \nabla h), \nabla w - \nabla h\rangle dx \\ &= \int_{B_{7\rho/4}(x_0)} \langle \mathbf{a}(t, \nabla w) - \mathbf{A}_0(x, t, \nabla h), \nabla w - \nabla h\rangle dx. \end{aligned}$$

This and (1.2) imply that

$$\begin{aligned} &\frac{1}{2}\sup_{t\in\Gamma_{7\rho/4}(t_0)} \int_{B_{7\rho/4}(x_0)} |w - h|^2 dx + \int_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz \\ &\leq C(\Lambda)\int_{Q_{7\rho/4}(z_0)} |\mathbf{a}(t, \nabla w) - \mathbf{A}_0(x, t, \nabla w)||\nabla w - \nabla h| dz \\ &\leq C(\Lambda)\int_{Q_{7\rho/4}(z_0)} \Theta_{\mathbf{A}_0, B_{7\rho/4}(x_0)}(x, t)|\nabla w||\nabla w - \nabla h| dz \end{aligned}$$

Let us now denote $\gamma > 2$ be a number satisfying

$$\frac{1}{\gamma} + \frac{1}{p_1} = \frac{1}{2}.$$

Then, by using Hölder's inequality and (3.10), we see that

$$
\begin{aligned}
\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz &\leq C(\Lambda)\left(\fint_{Q_{7\rho/4}(z_0)} |\Theta_{\mathbf{A}_0,\Omega_{7\rho/4}(x_0)}(x,t)|^\gamma dz\right)^{1/\gamma}\left(\fint_{Q_{7\rho/4}(z_0)} |\nabla w|^p dz\right)^{1/p} \times \\
&\qquad \times \left(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \\
&\leq C(\Lambda, n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_R,R)}\left(\fint_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\right)^{1/2}\left(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2}.
\end{aligned}
$$

Hence,

$$
\left(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \leq C(\Lambda, n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_R,R)}\left(\fint_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\right)^{1/2}.
$$

and this proves the first assertion of the lemma. To prove the last estimate of Lemma 3.3, we can use standard regularity theory for equation (3.11) to obtain

$$
\|\nabla h\|_{L^\infty(Q_{3\rho/2}(z_0))} \leq C(\Lambda, n)\left(\fint_{Q_{7\rho/4}(z_0)} |\nabla h|^2 dz\right)^{1/2}.
$$

This, together with the fact that $[\mathbf{A}_0]_{\mathrm{BMO}(Q_R,R)} \leq C(\Lambda, n)$, triangle inequality, and (3.33) imply (3.11). The proof of the lemma is then complete. □

Our next result is the main result of the section.

Proposition 3.4. *Let $\Lambda > 0, \alpha > 2$ and $\alpha_0 \in (0,1]$ be fixed. Then, for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\bar{\kappa} = \bar{\kappa}(\Lambda, M, n, \alpha_0, \epsilon) \in (0, 1/3)$ and $\delta_0 = \delta_0(\epsilon, \Lambda, M, n, \alpha_0) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (1.2)–(1.4) *and* (1.11) *hold with δ replaced by δ_0. Assume also that*

$$
\fint_{Q_{2r}(z_0)} |\mathbf{F}|^2 dz + r^2\left(\fint_{Q_{2r}(z_0)} |f|^{2n_*} dz\right)^{1/n_*} + \left(\fint_{Q_{2r}(z_0)} |\mathbf{G}(x,t)|^\alpha dz\right)^{\frac{2}{\alpha}} \leq \delta_0^2,
$$

for some $z_0 = (x_0, t_0) \in \overline{Q}_R$ and some $r \in (0, R/2)$. Then, for every $\lambda \geq 0$, if $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ is a weak solution of (3.1) *satisfying*

$$
\fint_{Q_{4\bar{\kappa}r}(z_0)} |\nabla u|^2 dz \leq 1, \quad \textit{and} \quad [[\lambda u]]_{\mathrm{BMO}(Q_R,R)} \leq M,
$$

then there is $h \in C(\Gamma_{7\bar{\kappa}r/2}(t_0), L^2(B_{7\bar{\kappa}r/2}(x_0))) \cap L^2(\Gamma_{7\bar{\kappa}r/2}(t_0), W^{1,2}(B_{7\bar{\kappa}r/2}(x_0)))$ such that the following estimate holds

$$
\frac{1}{|Q_{7\bar{\kappa}r/2}(z_0)|}\int_{Q_{7\bar{\kappa}r/2}(z_0)} |\nabla u - \nabla h|^2 dz \leq \epsilon^2, \quad \|\nabla h\|_{L^\infty(Q_{3\bar{\kappa}r}(z_0))} \leq C(\Lambda, n). \tag{3.12}
$$

Proof. The proposition follows directly by applying Lemma 3.2 with r replaced by $2r$, and Lemma 3.3 with $\mathbf{A}_0(x,t,\xi) = \mathbf{A}(x,t,\lambda\bar{u}_{Q_{4\bar{\kappa}r}(z_0)}, \xi)$ and $\rho = 2\bar{\kappa}r$, and the triangle inequality. □

3.2. **Boundary approximation estimates.** To be convenient for the readers and self-contained, we recall some frequently used notation. For each $R > 0$, we write $B_R = B_R(0)$ the ball in $\mathbb{R}^n$ centered at the origin with radius R. Moreover, for an open set $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$, we write

$$
\Omega_R = B_R \cap \Omega, \quad K_R = \Omega_R \times \Gamma_R, \quad \text{and} \quad T_R = (\partial\Omega \cap B_R) \times \Gamma_R.
$$

We also denote $\partial_p K_R$ the parabolic boundary of K_R, moreover, for each $z_0 = (x_0, t_0)$, it is denoted that

$$
\Omega_R(x_0) = x_0 + \Omega_R, K_R(z_0) = z_0 + K_R, \quad T_R(z_0) = z_0 + T_R.
$$

We can assume that $T_{2R} \neq \emptyset$, and we investigate weak solutions u of the equation

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}\left[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)\right] & = & f(x,t), & \text{in} \quad K_{2R}, \\ \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{\nu}\rangle & = & 0, & \text{on} \quad T_{2R}. \end{array} \right. \tag{3.13}$$

with the parameter $\lambda \geq 0$. By weak solutions of (3.13), we mean any $u \in C(\Gamma_{2R}, L^2(\Omega_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(\Omega_{2R}))$ such that $\mathbf{G} \in L^\alpha(K_{2R})$ for some $\alpha > 2$, $\lambda u(x,t) \in \mathbb{K}$ for a.e. $(x,t) \in K_{2R}$, and

$$-\int_{K_{2R}} u(x,t)\partial_t \varphi(x,t)dz + \int_{K_{2R}} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}, \nabla \varphi\rangle dz = \int_{K_{2R}} f(x,t)\varphi(x,t)dz,$$

for all $\varphi \in C_0^\infty(\overline{Q}_{2R})$. Here,

$$\mathbf{G}(x,t) = \hat{C}_0(n,\alpha)[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b}(x,t), \quad z = (x,t) \in K_{2R}, \quad \text{for some definite constant} \quad \hat{C}_0(n,\alpha). \tag{3.14}$$

As before, for each $\alpha > 2$, let α' be the number satisfying

$$\frac{1}{\alpha'} + \frac{1}{\alpha} = \frac{1}{2}, \quad \text{i.e.} \quad \alpha' = \frac{2\alpha}{\alpha - 2}. \tag{3.15}$$

Recall that the number n_* is defined in (3.2). Our first step is to approximate u by the solution v of the homogeneous equation with frozen u in $\mathbf{A}$.

Lemma 3.5. *Let $\Lambda, \alpha > 2$ be fixed. Assume that $\mathbf{A} : K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (1.2)–(1.4). *Assume that $\mathbf{G} \in L^\alpha(K_R)$, and u is a weak solution of* (3.13) *for some $\lambda \geq 0$, then for each $z_0 = (x_0,t_0) \in K_R$ and each $r \in (0,R)$, it holds that*

$$\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \leq C_0(\Lambda,n)\left[\fint_{K_r(z_0)} |\mathbf{F}|^2 dz + \left(\fint_{K_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\right)^{\frac{2}{\alpha}} + r^2\left(\fint_{K_r(z_0)} |f|^{2n_*}dz\right)^{1/n_*}\right], \tag{3.16}$$

where $v \in C(\Gamma_r(t_0), L^2(\Omega_r(x_0))) \cap L^2(\Gamma_r(t_0), W^{1,2}(\Omega_r(x_0)))$ is the weak solution of

$$\left\{ \begin{array}{rcll} v_t - \operatorname{div}\left[\mathbf{A}(x,t,\lambda u,\nabla v)\right] & = & 0, & \text{in} \quad K_r(z_0), \\ v & = & u, & \text{on} \quad \partial_p K_r(z_0) \setminus T_r(z_0), \\ \langle \mathbf{A}(x,t,\lambda u, \nabla v), \vec{\nu}\rangle & = & 0, & \text{on} \quad T_r(z_0), \quad \text{if} \quad T_r(z_0) \neq \emptyset. \end{array} \right. \tag{3.17}$$

Moreover,

$$\left(\fint_{K_r(z_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2} \leq C(n)\left[r\left(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + \left(\fint_{K_r(z_0)} |u - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2}\right]. \tag{3.18}$$

Proof. If $T_r(z_0) = \emptyset$, this lemma follows directly from Lemma 3.1. Therefore, we only consider the case that $T_r(z_0) \neq \emptyset$. The proof is similar to that of Lemma 3.1 with some modification dealing with the boundary. Observe that since $\partial\Omega \in C^1$, $\Omega_r(x_0)$ is a Lipschitz domain. Therefore, $W^{1,2}(\Omega_r(x_0))$ is well-defined with all imbdedding and compact imbedding properties. Therefore, the existence of the solution v of (3.18) can be obtained by the Galerkin's method, see [31, p. 466-475]. It then remains to prove the estimates (3.16) and (3.18). By proceeding with the Steklov's average (see [5, 13, 38]), we can formally use $v - u$ as a test function for the equations (3.17), and (3.13) to infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_r(x_0)} |v-u|^2 dx + \int_{\Omega_r(x_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\ &= \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)u + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{\Omega_r(x_0)} f(v-u)dx. \end{aligned}$$

Due to the fact that $\operatorname{div}\mathbf{b} = 0$ on Ω_{2R} and $\langle \mathbf{b}, \vec{\nu}\rangle = 0$ on $B_{2R} \cap \partial\Omega$ in the sense that

$$\int_{\Omega_r(x_0)} \mathbf{b}(x,t)\cdot\nabla\varphi(x)dx = 0, \quad \text{for all} \quad \varphi \in C_0^\infty(B_r(x_0)), \quad \text{for a.e.} \quad t \in \Gamma_R,$$

we see that

$$\int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)u, \nabla u - \nabla v\rangle dx = \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)[u - \bar{u}_{K_r(z_0)}], \nabla u - \nabla v\rangle dx.$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_r(x_0)} |v-u|^2 dx + \int_{\Omega_r(x_0)} \langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\
&= \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)[u - \bar{u}_{K_r(z_0)}] + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{\Omega_r(x_0)} f(v-u) dx.
\end{aligned}$$

From this, and the condition (1.2)–(1.4), we infer that

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in\Gamma_r(t_0)} \int_{\Omega_r(x_0)} |v-u|^2 dx + \int_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\leq \frac{1}{2}\sup_{t\in\Gamma_r(t_0)} \int_{\Omega(x_0)} |v-u|^2 dx + C(\Lambda)\left[\int_{K_r(z_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla u - \nabla v\rangle dz\right] \\
&\leq C(\Lambda)\left[\int_{K_r(z_0)} \Big(|\mathbf{F}| + |\mathbf{b}||u - \bar{u}_{K_r(z_0)}|\Big)|\nabla u - \nabla v| + \int_{K_r(z_0)} |f||v-u| dz\right] \\
&\leq \frac{1}{2}\int_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\qquad + C(\Lambda)\left[\int_{K_r(z_0)} \Big(|\mathbf{F}|^2 + |\mathbf{b}|^2|u - \bar{u}_{K_r(z_0)}|^2\Big) dz + \int_{K_r(z_0)} |f||v-u| dz\right].
\end{aligned}$$

Then,

$$\begin{aligned}
&\sup_{t\in\Gamma_r(t_0)} r^{-2} \fint_{\Omega_r(x_0)} |v-u|^2 dx + \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\leq C(\Lambda, n)\left[\fint_{K_r(z_0)} \Big(|\mathbf{F}|^2 + |\mathbf{b}|^2|u - \bar{u}_{K_r(z_0)}|^2\Big) dz + \fint_{K_r(z_0)} |f||v-u| dz\right].
\end{aligned}$$

Now now control the last two terms in the right hand side of the previous estimate. Observe that from (3.15), the John-Nirenberg's theorem, and the Hölder's inequality it follows that

$$\begin{aligned}
&\fint_{K_r(z_0)} |\mathbf{b}|^2 |u - \bar{u}_{K_r(z_0)}|^2\Big) dz \\
&\leq C(n)\left(\fint_{K_r(z_0)} |\mathbf{b}|^\alpha dz\right)^{2/\alpha} \left(\frac{1}{|Q_r(z_0)|}\int_{K_r(z_0)} |u - \bar{u}_{K_r(z_0)}|^{\alpha'} dz\right)^{2/\alpha'} \\
&\leq \hat{C}_0(n,\alpha)[[u]]^2_{\mathrm{BMO}(K_R,R)} \left(\fint_{K_r(z_0)} |\mathbf{b}|^\alpha dz\right)^{2/\alpha} = \left(\fint_{K_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\right)^{2/\alpha}.
\end{aligned}$$

On the other hand, as in (3.8), we denote $p_0 = 2n_*$ and p_0' such that $1/p_0 + 1/p_0' = 1$, i.e. $p_0' = \frac{2n}{n+2}$. From Hölder's inequality, the parabolic Sobolev imbedding (see [31, eqn (3.2), p. 74] or [13, Proposition 3.1, p. 7]), and Young's inequality, it follows that

$$\begin{aligned}
&\fint_{K_r(z_0)} |f||v-u| dz \leq \left(\fint_{K_r(z_0)} |v-u|^{p_0'} dz\right)^{1/p_0'} \left(\fint_{K_r(z_0)} |f|^{p_0} dz\right)^{1/p_0} \\
&\leq \frac{1}{2}\fint_{K_r(z_0)} |\nabla v - \nabla u|^2 dz + \frac{1}{4}\sup_{\Gamma_r(t_0)} r^{-2}\fint_{B_r(x_0)} |v-u|^2 dx + C(n) r^2 \left(\fint_{K_r(z_0)} |f|^{p_0} dz\right)^{2/p_0}.
\end{aligned} \tag{3.19}$$

Therefore,

$$
\begin{aligned}
&\sup_{t\in\Gamma_r(t_0)} r^{-2} \fint_{\Omega_r(x_0)} |v-u|^2 dx + \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\leq C(\Lambda, n)\left[\fint_{K_r(z_0)} |\mathbf{F}|^2 + \left(\fint_{K_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\right)^{2/\alpha} + r^2 \left(\fint_{K_r(z_0)} |f(x,t)|^{p_0} dz\right)^{2/p_0}\right]
\end{aligned}
\tag{3.20}
$$

and (3.16) follows. Lastly, we prove (3.18). Observe that the triangle inequality gives

$$
\fint_{K_r(x_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz \leq C\left[\fint_{K_r(z_0)} |v-u|^2 dz + \fint_{K_r(z_0)} |u - \bar{u}_{K_r(z_0)}|^2 dx\right].
$$

Then, using the Poincaré's inequality for the first term in the right hand side of the above inequality, we see that

$$
\left(\fint_{K_r(z_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2} \leq C(n)\left[r\left(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + \left(\fint_{K_r(z_0)} |u - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2}\right].
$$

This proves (3.18) and also completes the proof. ◻

Lemma 3.6. *Let $\Lambda, M > 0, \alpha > 2$ and $\alpha_0 \in (0,1)$ be fixed. Then, for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon) \in (0, 1/3)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha_0) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.4) *hold with some $R > 0$ and some open set $\mathbb{K} \subset \mathbb{R}$, and assume that*

$$
\fint_{K_r(z_0)} |\mathbf{F}|^2 dz + \left(\fint_{K_r(z_0)} |\mathbf{G}|^\alpha dz\right)^{\frac{2}{\alpha}} + r^2\left(\fint_{K_r(z_0)} |f|^{2n_*} dz\right)^{1/n_*} \leq \delta_2^2,
\tag{3.21}
$$

for some $z_0 = (x_0, t_0) \in K_R$ and some $r \in (0, R)$. Then, for every $\lambda \geq 0$, if u is a weak solution of (3.13) *satisfying*

$$
\fint_{K_{2\kappa r}(z_0)} |\nabla u|^2 dx \leq 1, \quad \textit{and} \quad [[\lambda u]]_{\mathrm{BMO}(K_R, R)} \leq M,
$$

then there is a weak solution w of

$$
\begin{cases}
w_t - \operatorname{div}[\mathbf{A}(x,t,\lambda\bar{u}_{K_{\kappa r}(z_0)}, \nabla w)] = 0, & \textit{in} \quad K_{\kappa r}(z_0), \\
w = v, & \textit{on} \quad \partial_p K_{\kappa r}(z_0) \setminus T_{\kappa r}(z_0), \\
\langle \mathbf{A}(x,t,\lambda\bar{u}_{K_{\kappa r}(z_0)}, \nabla w), \vec{\nu}\rangle = 0, & \textit{on} \quad T_{\kappa r}(z_0), \quad \textit{if} \quad T_{\kappa r}(z_0) \neq \emptyset,
\end{cases}
\tag{3.22}
$$

such that the following estimate holds

$$
\left(\fint_{K_{\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\right)^{1/2} \leq \epsilon, \quad \textit{and} \quad \left(\fint_{K_{\kappa r}(z_0)} |\nabla w|^2 dz\right)^{1/2} \leq 1 + 2^{\frac{n+2}{2}}.
\tag{3.23}
$$

where in (3.22) *the function v is defined as in Lemma 3.5.*

Proof. For a given sufficiently small $\epsilon > 0$, let $\epsilon' \in (0, \epsilon/2)$ and $\kappa \in (0, 1/3)$ both sufficiently small depending on $\epsilon, \Lambda, M, n, \alpha_0$ which will be determined. Now, by Lemma 3.5 with ϵ', we can find $\delta_2 = \delta_2(\epsilon', \Lambda, \kappa) > 0$ sufficiently small such that if (3.21) holds, then

$$
\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \leq (\epsilon')^2 \kappa^{n+2}, \quad \text{and} \quad \lambda\left(\fint_{K_r(z_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2} \leq C(n,p)[r\epsilon'\kappa^{\frac{n+2}{2}}\lambda + M],
\tag{3.24}
$$

where v is the solution of (3.17). Observe that the first inequality in (3.24) and the fact that $\epsilon', \kappa \in (0,1)$ imply

$$
\begin{aligned}
\left(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\right)^{1/2} &\leq \left(\fint_{K_{2\kappa r}(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + \left(\fint_{K_{2\kappa r}(z_0)} |\nabla u|^2 dz\right)^{1/2} \\
&\leq \left(\frac{1}{(2\kappa)^{n+2}} \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + 1 \leq 2.
\end{aligned}
\tag{3.25}
$$

Note that when $\lambda = 0$, $w = v$. The lemma is then trivial with every $\kappa \in (0, 1/3)$. Therefore, we only need to consider the case $\lambda > 0$. From the standard Caccioppli's type estimate for the solution v of (3.17), and (3.24), we also see that

$$
\begin{aligned}
\left(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\right)^{1/2} &\leq \frac{C(\Lambda, n)}{(1-2\kappa)\kappa^{\frac{n+2}{2}} r} \left(\fint_{K_r(z_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2} \\
&\leq C(\Lambda, n)\Big[\epsilon' + M(\lambda \kappa^{\frac{n+2}{2}} r)^{-1}\Big],
\end{aligned} \tag{3.26}
$$

where in the last estimate we have used the fact that $\kappa \in (0, 1/3)$ to control the factor $1 - 2\kappa$. Now, let w be the weak solution of (3.22), whose existence can be obtained by a standard method using Galerkin's method, see [31, p. 466-475]. It remains to prove the estimate (3.23). By using the Skelov's average as in [5, 13], we can formally take $v - w$ as a test function for the equation (3.22) and the equation (3.17) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx + \int_{\Omega_{\kappa r}(x_0)} \langle \mathbf{A}(x, t, \lambda u, \nabla v), \nabla w - \nabla v\rangle dx \\
&= \int_{\Omega_{\kappa r}(x_0)} \langle \mathbf{A}(x, t, \lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla w - \nabla v\rangle dx
\end{aligned} \tag{3.27}
$$

From this, it follows that

$$
\begin{aligned}
&\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx + \int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \\
&\leq C(\Lambda)\Bigg[\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx \\
&\qquad + \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x, t, \lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x, t, \lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla v - \nabla w\rangle dz\Bigg] \\
&\leq C(\Lambda)\Bigg[\int_{K_{\kappa r}(z_0)} \Big|\langle \mathbf{A}(x, t, \lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla v), \nabla v - \nabla w\rangle\Big| dz \\
&\qquad + \int_{\mathbb{K}_{\kappa r}(z_0)} \Big|\langle \mathbf{A}(x, t, \lambda u, \nabla v), \nabla v - \nabla w\rangle\Big| dz\Bigg] \\
&\leq C(\Lambda)\int_{K_{\kappa r}(z_0)} |\nabla v||\nabla v - \nabla w| dz \\
&\leq C(\Lambda)\int_{K_{\kappa r}(z_0)} |\nabla v|^2 dz + \frac{1}{2}\int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz.
\end{aligned}
$$

This last estimate together with (3.26) imply that

$$
\begin{aligned}
\left(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz\right)^{1/2} &\leq C(\Lambda, n)\left(\fint_{K_{\kappa r}(z_0)} |\nabla v|^2 dz\right)^{1/2} \\
&\leq C(\Lambda, n)\left(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\right)^{1/2} \leq C_1(\Lambda, n)\Big[\epsilon' + M(r\kappa^{\frac{n+2}{2}}\lambda)^{-1}\Big].
\end{aligned}
$$

Hence, if $MC_1(\Lambda, n)(\lambda\kappa^{\frac{n+2}{2}} r)^{-1} \leq \frac{\epsilon}{4}$, we can choose ϵ' sufficiently small such that

$$
4C_1(\Lambda, n)\epsilon' < \epsilon. \tag{3.28}
$$

From these choices, it follows that

$$
\left(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz\right)^{1/2} \leq \epsilon/2.
$$

This estimate, the triangle inequality, and the first estimate in (3.24) gives

$$\begin{aligned}\left(\fint_{K_{\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\right)^{1/2} &\leq \left(\fint_{K_{\kappa r}(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + \left(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz\right)^{1/2} \\ &\leq \left(\frac{1}{\kappa^{n+2}} \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\right)^{1/2} + \epsilon/2 \leq \epsilon' + \epsilon/2 \leq \epsilon\end{aligned}$$

which is the first estimate in (3.23). It therefore remains to consider the case

$$\lambda r \kappa^{\frac{n+2}{2}} \epsilon \leq 2C_1(\Lambda, n) M. \tag{3.29}$$

In this case, we note that from our choice that $\epsilon' \leq \epsilon$, we particularly have

$$\lambda \kappa^{\frac{n+2}{2}} \epsilon' r \leq C(\Lambda, M, n).$$

Then, it follows from (3.24) that

$$\lambda \left(\fint_{K_r(x_0)} |v - \bar{u}_{K_r(z_0)}|^2 dz\right)^{1/2} \leq C(\Lambda, M, n).$$

From this and the equation (3.17), we can apply the Hölder's regularity theory in Lemma 2.8 and Lemma 2.9 for the function

$$\tilde{v}(x,t) := \lambda[v(x - x_0, t - t_0) - \bar{u}_{K_r(z_0)}], \quad (x,t) \in K_r,$$

to find that there is $\beta_0 \in (0,1)$ depending only on Λ, n such that $\tilde{v} \in C^{\beta_0}(\bar{K}_{2r/3})$. Then, by scaling back, we obtain the following estimates

$$\begin{cases} \lambda \left\| v - \bar{u}_{K_r(z_0)} \right\|_{L^\infty(K_{5r/6}(z_0))} \leq C(\Lambda, M, n), \quad \text{and} \\ \lambda |v(z) - v(z')| \leq C(\Lambda, M, n) \left[\dfrac{|x - x'| + |t - t'|^{1/2}}{r}\right]^{\beta_0} \leq \kappa^{\beta_0}, \quad \forall\, z = (x,t), z' = (x', t') \in \overline{K}_{\kappa r}(z_0). \end{cases} \tag{3.30}$$

From now on, for simplicity, we write $\hat{u} = u - \bar{u}_{K_{\kappa r}(z_0)}$. We can use (3.27) again to obtain

$$\begin{aligned}&\frac{1}{2} \sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx + \int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \\ &\leq C(\Lambda) \Bigg[\frac{1}{2} \sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx \\ &\qquad + \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x,t,\lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x,t,\lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla v - \nabla w\rangle dz\Bigg] \\ &\leq C(\Lambda) \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x,t,\lambda \bar{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x,t,\lambda u, \nabla v), \nabla v - \nabla w\rangle dz \\ &\leq C(\Lambda) \int_{K_{\kappa r}(z_0)} [\lambda \hat{u}]^{\alpha_0} |\nabla v| |\nabla v - \nabla w| dz \\ &\leq \frac{1}{2} \int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz + C(\Lambda) \int_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^{2\alpha_0} |\nabla v|^2 dz.\end{aligned}$$

Hence,

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \leq C(\Lambda) \fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^{2\alpha_0} |\nabla v|^2 dz.$$

For $p_1 > 2$ and sufficiently close to 2 depending only on Λ, n, we write $q = \frac{2\alpha_0 p_1}{p_1 - 2} > 2$. Then, using the Hölder's inequality, and the self-improving regularity estimate, Lemma 2.11, we obtain

$$\begin{aligned}\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz &\leq C(\Lambda) \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^q \right)^{\frac{p_1-2}{p_1}} \left(\fint_{K_{\kappa r}(z_0)} |\nabla v|^{p_1} dz \right)^{\frac{2}{p_1}} \\ &\leq C(\Lambda, n) \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^q dz \right)^{\frac{p_1-2}{p_1}} \left(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz \right)\end{aligned}$$

We further write

$$\begin{aligned}\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^q dz &= \fint_{K_{\kappa r}(z_0)} |\hat{u}| |\lambda \hat{u}|^{q-1} dz \leq \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{1/2} \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^{2q-2} dz \right)^{1/2} \\ &\leq C(n, \alpha_0) [[\lambda u]]^{q-1}_{\mathrm{BMO}(K_R, R)} \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{1/2} \leq C(n, M, \alpha_0) \left(\fint_{K_{\kappa r}(z_0)} \left|\lambda \hat{u}\right|^2 dz \right)^{1/2}.\end{aligned}$$

Therefore,

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \leq C(\Lambda, M, n, \alpha_0) \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{\frac{p_1-2}{2p_1}} \left(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz \right).$$

This, and (3.25) imply that

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \leq C(\Lambda, M, n, \alpha_0) \left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{\frac{p_1-2}{2p_1}}. \tag{3.31}$$

On the other hand, we also write

$$\begin{aligned}\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz &\leq C \Bigg[\fint_{K_{\kappa r}(z_0)} |\lambda (u - v)|^2 dz + \fint_{K_{\kappa r}(z_0)} |\lambda (v - \bar{v}_{K_{\kappa r}(z_0)})|^2 dz \\ &\quad + \fint_{K_{\kappa r}(z_0)} |\lambda (\bar{u}_{K_{\kappa r}(z_0)} - \bar{v}_{K_{\kappa r}(z_0)})|^2 dz \\ &\leq C(n) \left[\kappa^{-(n+2)} \fint_{K_r(z_0)} |\lambda (u - v)|^2 dz + \fint_{K_{\kappa r}(z_0)} |\lambda (v - \bar{v}_{K_{\kappa r}(z_0)})|^2 dz \right].\end{aligned}$$

Then, by using the Poincaré's inequality for the first term on the right hand side of the last estimate, we obtain

$$\left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{1/2} \leq C(\Lambda, n) \left[\lambda r \kappa^{-\frac{n+2}{2}} \left(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \right)^{1/2} + \lambda \sup_{x, y \in \overline{K}_{\kappa r}(z_0)} |v(z) - v(z')| \right],$$

This, (3.24), and (3.30) imply that

$$\left(\fint_{K_{\kappa r}(z_0)} |\lambda \hat{u}|^2 dz \right)^{1/2} \leq C(\Lambda, n) \Big[(\lambda r) \epsilon' + \kappa^{\beta_0} \Big].$$

From this last estimate, the estimate (3.31) can be written as

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \leq C(\Lambda, M, n, \alpha_0) \left(\lambda r \epsilon' + \kappa^{\beta_0} \right)^{\frac{p_1-2}{p_1}}.$$

This, (3.29), we can further imply that

$$\left(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \right)^{1/2} \leq C_2(\Lambda, M, \alpha_0, n) \Big[\frac{\epsilon'}{\epsilon \kappa^{\frac{n+2}{2}}} + \kappa^{\beta_0} \Big]^{\frac{p_1-2}{2p_1}}.$$

Now, we choose κ sufficiently small depending only on Λ, M, n, α_0 and ϵ such that

$$\kappa^{\beta_0} \leq \frac{1}{2}\Big[\frac{\epsilon}{2C_2(\Lambda, M, \alpha_0, n)}\Big]^{\frac{2p_1}{p_1-2}}.$$

Then, we choose $\epsilon' > 0$ sufficiently small depending only on Λ, n, α_0 and ϵ such that

$$\epsilon' \leq \frac{\epsilon\kappa^{\frac{n+2}{2}}}{2}\Big[\frac{\epsilon}{2C_2(\Lambda, M, \alpha_0, n)}\Big]^{\frac{2p_1}{p_1-2}}$$

From these choices, we obtain

$$\left(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz\right)^{1/2} \leq \epsilon/2.$$

Then, we use the first estimate in (3.24), the triangle inequality again to obtain the first estimate in (3.23). It now remains to prove the second estimate in (3.23). By the triangle inequality, the assumption in this lemma and since $\epsilon \in (0,1)$, we see that

$$\begin{aligned}\left(\fint_{K_{\kappa r}(z_0)} |\nabla w|^2 dz\right)^{1/2} &\leq \left(\fint_{K_{\kappa r}(z_0)} |\nabla - \nabla u|^2 dz\right)^{1/2} + \left(\fint_{K_{\kappa r}(z_0)} |\nabla u|^2 dz\right)^{1/2} \\ &\leq \epsilon + \left(2^{n+2}\fint_{K_{2\kappa r}(z_0)} |\nabla w|^2 dz\right)^{1/2} \leq 1 + 2^{\frac{n+2}{2}}\end{aligned}$$

as desired. The proof is therefore complete. □

Our next result is a standard approximation which particularly gives a comparison of the solution w of (3.22) with a constant coefficient solution.

Lemma 3.7. *Let $\Lambda > 0$ be fixed. Then, for every $\epsilon \in (0,1)$, there exists a constant $\delta' = \delta'(\Lambda, n, \epsilon) > 0$ and sufficiently small such that the following statement holds. Assume that Ω is be an open bounded domain with boundary $\partial\Omega \in C^1$. Assume also that $\mathbf{A}_0 : K_{2R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying* (1.2)–(1.3) *and* $[\mathbf{A}_0]_{\mathrm{BMO}(K_R,R)} \leq \delta'$ *for some $R > 0$. Suppose that w is a weak solution of*

$$\begin{cases} w_t - \operatorname{div}[\mathbf{A}_0(x,t,\nabla w)] = 0, & \text{in} \quad K_{4\rho}(z_0), \\ \langle \mathbf{A}_0(x,t,\nabla w), \vec{\nu}\rangle = 0, & \text{on} \quad T_{4\rho}(z_0), \quad \text{if} \quad T_{4\rho}(z_0) \neq \emptyset, \end{cases} \tag{3.32}$$

and it satisfies

$$\fint_{K_{4\rho}} |\nabla w|^2 dz \leq 1,$$

with some $0 < \rho < R/4$, and some $z_0 = (x_0, t_0) \in K_R$. Then, there is some function h such that

$$\left(\fint_{K_{2\rho}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \leq \epsilon^2, \quad \text{and} \quad \|\nabla h\|_{L^\infty(K_\rho(z_0))} \leq C(\Lambda, n). \tag{3.33}$$

Proof. The proof can be done exactly the same as that of Lemma 3.3. Since $\partial\Omega$ is C^1, the Lipschitz regularity estimates for weak solutions of the corresponding homogeneous equation with frozen coefficient holds true if $T_{2\rho}(z_0) \neq \emptyset$. Alternatively, since $\partial\Omega$ is C^1, it is sufficiently flat in the sense of Reifenberg's. Therefore, this lemma follows from [8, Lemma 6 and Corollary 1], see also [6]. One can flatten the boundary as in [27] and prove a similar approximation in the upper-half cube Q_r^+ as Lemma 3.3. □

Finally, we state and prove the main result of the section.

Proposition 3.8. *Let $\Lambda, M > 0, \alpha > 2$ and $\alpha_0 \in (0,1)$ be fixed. Then, for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon) \in (0, 1/3)$ and $\delta = \delta(\epsilon, \Lambda, M, n, \alpha_0) \in (0, \epsilon)$ such that the following holds. Assume $\partial\Omega \in C^1$, and $\mathbf{A} : K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.14) *hold, and assume that*

$$\fint_{K_{8r}(z_0)} |\mathbf{F}|^2 dz + \left(\fint_{K_{8r}(z_0)} |\mathbf{G}|^\alpha dz\right)^{\frac{2}{\alpha}} + (8r)^2\left(\fint_{K_{8r}(z_0)} |f|^{2n_*} dz\right)^{1/n_*} \leq \delta^2,$$

for some $z_0 = (x_0, t_0) \in K_R$, *and* $r \in (0, R/8)$. *Then, for every* $\lambda \geq 0$, *if* u *is a weak solution of* (3.13) *satisfying*

$$\fint_{K_{16\kappa r}(z_0)} |\nabla u|^2 dz \leq 1, \quad \textit{and} \quad [[\lambda u]]_{\mathrm{BMO}(K_R, R)} \leq M,$$

then there is $h \in L^\infty(\Gamma_{4\kappa r}(t_0), L^2(\Omega_{4\kappa r}(x_0))) \cap L^2(\Gamma_{4\kappa r}(t_0), W^{1,2}(\Omega_{4\kappa r}(x_0)))$ *such that the following estimate holds*

$$\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla h|^2 dz \leq \epsilon^2, \quad \|\nabla h\|_{L^\infty(K_{2\kappa r}(z_0))} \leq C(\Lambda, n). \tag{3.34}$$

Proof. Let

$$\delta = \min\Big\{\delta_2\Big(\frac{1}{2}\Big[\frac{1}{2}\Big]^{\frac{n+2}{2}}\epsilon, \Lambda, M, n, \alpha\Big), \delta'(\Lambda, n, \epsilon/[2(1 + 2^{(n+2)/2})])\Big\},$$

where δ_2 is defined in Lemma 3.6, and δ' is defined in Lemma 3.7. Moreover, let $\kappa = \kappa(\Lambda, M, n, \alpha_0, \frac{1}{2}\Big[\frac{1}{2}\Big]^{\frac{n+2}{2}}\epsilon)$ be the number defined in Lemma 3.6. Then, by applying Lemma 3.6 with r replaced by $8r$, we can find $w \in L^\infty(\Gamma_{8\kappa r}(t_0), L^2(\Omega_{8\kappa r}(x_0))) \cap L^2(\Gamma_{8\kappa r}(t_0), W^{1,2}(\Omega_{8\kappa r}(x_0)))$ satisfying

$$\left(\fint_{K_{8\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\right)^{1/2} \leq \frac{1}{2}\Big[\frac{1}{2}\Big]^{\frac{n+2}{2}}\epsilon \quad \text{and} \quad \left(\fint_{K_{8\kappa r}(z_0)} |\nabla w|^2 dz\right)^{1/2} \leq 1 + 2^{\frac{n+2}{2}}. \tag{3.35}$$

Then, we can apply Lemma 3.7 with $\mathbf{A}_0(x, t, \xi) = \mathbf{A}(x, t, \lambda \bar{u}_{K_{8\kappa r}(z_0)}, \xi), \rho = 2\kappa r$, and with some suitable scaling, we can find a function $h \in L^\infty(\Gamma_{4\kappa r}(t_0), L^2(\Omega_{4\kappa r}(x_0))) \cap L^2(\Gamma_{4\kappa r}(t_0), W^{1,2}(\Omega_{4\kappa r}(x_0)))$ such that the following estimate holds

$$\left(\fint_{K_{4\kappa r}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \leq \epsilon/2, \quad \|\nabla h\|_{L^\infty(K_{2\kappa r}(z_0))} \leq C(\Lambda, n). \tag{3.36}$$

It then follows from (3.35), (3.36), and the triangle inequality that

$$\begin{aligned}\left(\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla h|^2 dz\right)^{1/2} &\leq \left(\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\right)^{1/2} + \left(\fint_{K_{4\kappa r}(z_0)} |\nabla w - \nabla h|^2 dz\right)^{1/2} \\ &\leq \big[2\big]^{\frac{n+2}{2}} \left(\fint_{K_{8\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\right)^{1/2} + \epsilon/2 \leq \epsilon.\end{aligned}$$

The proof is therefore complete. □

4. Level set estimates

This section gives the key level set estimates needed in the proofs of the main theorems, i.e. Theorem 1.1, and Theorem 1.2. We can assume $R = 1$ as Theorem 1.1, and Theorem 1.2 can be retrieved to general $R > 0$ by using the dilation (2.3), Remark 2.1, and the dilation property of Lorentz quasi-norms, Remark 2.2.

Let $\epsilon > 0$ be a sufficiently small number to be determined depending only on the given numbers n, Λ, p, q, and α_0 . Let $\delta = \delta(\epsilon, \Lambda, M, n, \alpha_0)$, $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon)$ be the numbers defined in Proposition 3.8. Note that since ϵ depends on n, Λ, p, q and α_0, the numbers κ and δ also only depend on these numbers. Assume that all assumptions in Theorem 1.2 are valid with this δ and $R = 1$. For each $\lambda \geq 0$, and if u is a weak solution of (1.15), recall that

$$\mathbf{G}(x, t) \approx [[u]]_{\mathrm{BMO}(K_1, 1)} \mathbf{b}(x, t), \quad (x, t) \in K_2.$$

We fix $\eta > 2$ such that $\eta < \min\{2 + \epsilon_0, p\}$, where $\epsilon_0 = \epsilon_0(\Lambda, n)$ validates both Lemma 2.10 and Lemma 2.11. Let us also denote

$$F(x, t) = |\mathbf{F}(x, t)| + |\mathbf{G}(x, t)| + \mathcal{G}(f)|f(x, t)|^{n_*}, \quad (x, t) \in K_2, \tag{4.1}$$

where n_* is defined in (3.2), and

$$\mathcal{G}(f) = \left(\int_{K_2} |f(x,t)|^{2n_*} dz\right)^{\frac{1-n_*}{2n_*}}. \tag{4.2}$$

As we will see in (4.12) and (4.13) below, the function $\mathcal{G}$ plays an essential role in our proof. Observe also that since $p \geq 2$

$$\mathcal{G}(f) \left\| |f|^{n_*} \right\|_{L^{p,q}(K_2)} = \|f\|_{L^{2n_*}(K_2)}^{1-n_*} \|f\|_{L^{n_* p, n_* q}(K_2)}^{n_*} \leq C(n) \|f\|_{L^{n_* p, n_* q}(K_2)}. \tag{4.3}$$

From now on, let $\tau_0 > 0$ be the number defined by

$$\tau_0 = \left(\fint_{K_2} |\nabla u|^2 dz\right)^{1/2} + \frac{1}{\delta}\left(\fint_{K_2} |F|^\eta dz\right)^{1/\eta} < \infty. \tag{4.4}$$

For fixed numbers $1 \leq \mu \leq 2$, and $\tau > 0$, we denote the upper-level set of ∇u in K_μ by

$$E_\mu(\tau) = \Big\{\text{Lebesgue point } (x,t) \in K_\mu \text{ of } \nabla u : |\nabla u(x,t)| > \tau\Big\}. \tag{4.5}$$

The following Proposition estimating the upper-level sets of ∇u is the main result of this section.

Proposition 4.1. *There exist $N_0 = N_0(\Lambda, n) > 1$ and $B_0 = B_0(n)$ such that*

$$|E_{s_1}(N_0\tau)| \leq \epsilon^2 \left[|E_{s_2}(\tau/4)| + \frac{1}{(\delta\tau)^\eta}\int_{\delta\tau/4}^\infty s^\eta |\{(x,t) \in K_2 : |F(x,t)| > s\}| \frac{ds}{s}\right],$$

for all $1 \leq s_1 < s_2 \leq 2$, for every $\tau > \hat{B}_0\tau_0$, where $\hat{B}_0 := B_0[(s_2 - s_1)\kappa]^{-\frac{n+2}{2}}$.

The rest of the section is to prove this proposition. We follow the approach developed in [1] and used in [2, 5, 6]. However, some nontrivial modifications are also required to treat the terms $f, \mathbf{b}$ and to obtain the sharp homogeneous estimates, see Remark 1.4. For each $\bar{z} \in \overline{K}_2$ and each $r > 0$, we define

$$CZ_r(\bar{z}) = \left(\fint_{K_r(\bar{z})} |\nabla u|^2 dz\right)^{1/2} + \frac{1}{\delta}\left(\fint_{K_r(\bar{z})} |F|^\eta dz\right)^{1/\eta}. \tag{4.6}$$

Several lemmas are needed to prove Proposition 4.4. Our first lemma is a stopping-time argument lemma.

Lemma 4.2. *There exists a constant $B_0 = B_0(n)$ such that for each $1 \leq s_1 < s_2 \leq 2$, $\tau > \hat{B}_0\tau_0$, and for $\bar{z} \in E_{s_1}(\tau)$, there is $r_{\bar{z}} < \frac{(s_2-s_1)\kappa}{40}$ such that*

$$CZ_{r_{\bar{z}}}(\bar{z}) = \tau, \quad \textit{and} \quad CZ_r(\bar{z}) < \tau, \quad \forall\, r \in (r_{\bar{z}}, 1).$$

Proof. The argument is quite standard, see [1, 2, 5, 6]. Observe that because $r < 1$, and $\eta > 2$, we have

$$CZ_r(\bar{z}) \leq C(n)\left[\left(\frac{1}{r}\right)^{(n+2)/2}\left(\fint_{K_2} |\nabla u|^2 dz\right)^{1/2} + \left(\frac{1}{r}\right)^{(n+2)/\eta}\frac{1}{\delta}\left(\fint_{K_2} |F|^\eta dz\right)^{1/\eta}\right] \leq \frac{C(n)\tau_0}{r^{(n+2)/2}}.$$

Therefore, if $r > \frac{(s_2-s_1)\kappa}{40}$, then for $B_0 = C(n)[40]^{(n+2)/2}$, we see that

$$\frac{C(n)\tau_0}{r^{(n+2)/2}} \leq C(n)\left(\frac{40}{(s_2-s_1)\kappa}\right)^{(n+2)/2}\tau_0 = B_0[(s_2-s_1)\kappa]^{-\frac{n+2}{2}}\tau_0 < \tau.$$

Then,

$$CZ_r(\bar{z}) < \tau, \quad \textit{when} \quad \frac{(s_2-s_1)\kappa}{40} \leq r \leq 1, \quad \text{and} \quad \tau > \hat{B}_0\tau_0.$$

On the other hand, when $\bar{z} \in E_{s_1}(\tau)$, by the Lebesgue's theorem, we see that if r is sufficiently small, then

$$CZ_r(\bar{z}) > \tau.$$

Due to the fact the $CZ_r(\bar{z})$ is absolutely continuous, we can find $r_{\bar{z}}$, which is the largest number in $(0, \frac{(s_2-s_1)\kappa}{40})$, such that $CZ_{r_{\bar{z}}}(\bar{z}) = \tau$. From this, the conclusion of the lemma follows. $\square$

Lemma 4.3. *For each $\tau > \hat{B}_0\tau_0$, and each $1 \le s_1 < s_2 \le 2$, there exists a countable, disjoint family $\{K_{r_i}(z_i)\}_{i\in\mathcal{I}}$ with $r_i < \frac{(s_2-s_1)\kappa}{40}$ and $z_i \in K_{s_1}$ such that the following holds*

(i) $E_{s_1}(\tau) \subset \cup_{i=1}^{\infty} K_{5r_i}(z_i)$

(ii) $CZ_{r_i}(z_i) = \tau$, *and* $CZ_{r_i}(r) < \tau$ *for all* $r \in (r_i, 1)$.

Moreover, for each $i \in \mathcal{I}$, the following estimate holds

$$|K_{r_i}(z_i)| \le C(\Lambda, p, n)\left[|K_{r_i}(z_i) \cap E_{s_2}(\tau/4)| + \frac{1}{(\tau\delta)^\eta}\int_{\tau\delta/4}^{\infty} s^\eta |\{(x,t) \in K_{r_i}(z_i) : |F(x,t)| > s\}|\frac{ds}{s}\right]. \tag{4.7}$$

Proof. The conclusions (i) and (ii) follow directly from Lemma 4.2, and Vitali's covering lemma. It remains now to prove (4.7). Observe that if

$$\frac{1}{\delta^\eta}\fint_{K_{r_i}(z_i)} |F(x,t)|^\eta dz \ge \frac{\tau^\eta}{2^\eta}, \tag{4.8}$$

then

$$\begin{aligned}
|K_{r_i}(z_i)| &\le \frac{2^\eta}{\tau^\eta\delta^\eta}\int_{K_{r_i}(z_i)} |F(x,t)|^\eta dz = \frac{2^\eta}{\tau^\eta\delta^\eta}\int_0^\infty s^\eta\left|\left\{(x,t) \in K_{r_i}(z_i) : |F(x,t)| > s\right\}\right|\frac{ds}{s} \\
&= \frac{2^\eta}{\tau^\eta\delta^\eta}\left[\int_0^{\delta\tau/4}\cdots + \int_{\delta\tau/4}^\infty \cdots\right] \\
&\le \frac{|K_{r_i}(z_i)}{2^\eta} + \frac{2^\eta}{\tau^\eta\delta^\eta}\int_{\delta\tau/4}^\infty s^\eta\left|\left\{(x,t) \in K_{r_i}(z_i) : |F(x,t)| > s\right\}\right|\frac{ds}{s}.
\end{aligned}$$

Hence, (4.7) follows. Otherwise, i.e. (4.8) is false, it follows from the fact that $CZ_{r_i}(z_i) = \tau$ that

$$\fint_{K_{r_i}(z_i)} |\nabla u|^2 dz \ge \frac{\tau^2}{2^2},$$

and therefore

$$K_{r_i}(z_i) \le \frac{2^2}{\tau^2}\int_{K_{r_i}(z_i)} |\nabla u|^2 dxdt.$$

Then, observe that since $r_i < \frac{(s_2-s_1)\kappa}{40}$, $K_{r_i}(z_i) \subset K_{s_2}$, and hence

$$\begin{aligned}
|K_{r_i}(z_i)| &\le \frac{2^2}{\tau^2}\int_{K_{r_i}(z_i)\setminus E_{s_2}(\tau/4)} |\nabla u|^2 dz + \frac{2^2}{\tau^2}\int_{K_{r_i}(z_i)\cap E_{s_2}(\tau/4)} |\nabla u|^2 dz \\
&\le \frac{|K_{r_i}(z_i)|}{4} + \frac{2^2}{\tau^2}\int_{K_{r_i}(z_i)\cap E_{s_2}(\tau/4)} |\nabla u|^2 dz.
\end{aligned}$$

Therefore,

$$|K_{r_i}(z_i)| \le \frac{16}{3\tau^2}\int_{K_{r_i}(z_i)\cap E_{s_2}(\tau/4)} |\nabla u|^2 dz.$$

This, and Hölder's inequality with some $\gamma_0 > 0$ yield that

$$|K_{r_i}(z_i)| \le \frac{6|K_{r_i}(z_i)|^{\frac{1}{1+\gamma_0}}}{\tau^2}\left(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\right)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}.$$

Hence,

$$|K_{r_i}(z_i)|^{1-\frac{1}{1+\gamma_0}} \le \frac{6}{\tau^2}\left(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\right)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}. \tag{4.9}$$

On the other hand, with γ_0 sufficiently small so that $2(1+\gamma_0) < \eta$, we can apply Lemma 2.10 and Lemma 2.11 to obtain

$$\begin{aligned}\left(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\right)^{\frac{1}{2(1+\gamma_0)}} &\leq C\left[\left(\fint_{K_{2r_i}(z_i)} |\nabla u|^2 dz\right)^{1/2} + \left(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{2(1+\gamma_0)} dz\right)^{\frac{1}{2(1+\gamma_0)}}\right. \\ &\quad \left. + \left(\fint_{K_{2r_i}(z_i)} |\mathbf{G}|^{2(1+\gamma_0)^2} dz\right)^{\frac{1}{2(1+\gamma_0)^2}} + \mathcal{G}(f)\left(\fint_{K_{2r_i}(z_i)} |f|^{2n_*(1+\gamma_0)}\right)^{\frac{1}{2(1+\gamma_0)}}\right]. \end{aligned} \tag{4.10}$$

Then, since $2(1+\gamma_0)^2 < \eta$, we can use Hölder's inequality to control the last three terms on the right hand of (4.10) as the following

$$\begin{aligned} &\left(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{2(1+\gamma_0)} dz\right)^{\frac{1}{2(1+\gamma_0)}} + \left(\fint_{K_{2r_i}(z_i)} |\mathbf{G}|^{2(1+\gamma_0)^2} dz\right)^{\frac{1}{2(1+\gamma_0)^2}} + (2r_i)\left(\fint_{K_{2r_i}(z_i)} |f|^{2n_*(1+\gamma_0)}\right)^{\frac{1}{2(1+\gamma_0)}} \\ &\leq \left(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{\eta} dz\right)^{\frac{1}{\eta}} + \left(\fint_{K_{r_i}(z_i)} |\mathbf{G}|^{\eta} dz\right)^{\frac{1}{\eta}} + \mathcal{G}(f)\left(\fint_{K_{2r_i}(z_i)} |f|^{n_*\eta}\right)^{\frac{1}{\eta}} \leq C\left(\fint_{K_{2r_i}(z_i)} |F|^{\eta} dz\right)^{\frac{1}{\eta}}. \end{aligned} \tag{4.11}$$

Collecting the estimates (4.9), (4.10), and (4.11), we conclude that

$$\begin{aligned} |K_{r_i}(z_i)|^{1-\frac{1}{1+\gamma_0}} &\leq \frac{6}{\tau^2}\left(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\right)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}} \\ &\leq \frac{C(\Lambda, n)}{\tau^2} CZ_{2r_i}(z_i)^2 |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}} \leq C(\Lambda, n)|K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}. \end{aligned}$$

This implies

$$|K_{r_i}(z_i)| \leq C(\Lambda, n)|K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|,$$

and (4.7) follows. The proof is therefore complete. □

Proof of Proposition 4.1. Fix s_1, s_2 and τ as in the statement of Proposition 4.1. For each $i \in \mathcal{I}$, observe that from (ii) of Lemma 4.3, $CZ_{40r_i}(z_i) < \tau$ and $CZ_{20\hat{r}_i}(z_i) < \tau$, where $\hat{r}_i = \kappa^{-1} r_i \in (0, 1/40)$. Therefore, we have

$$\left(\fint_{K_{40\kappa\hat{r}_i}(z_i)} |\nabla u|^2 dz\right)^{1/2} \leq \tau, \quad \left(\fint_{K_{20\hat{r}_i}(z_i)} |F|^{\eta} dz\right)^{1/\eta} \leq \delta\tau.$$

Moreover, since $K_{20\hat{r}_i}(z_i) \subset K_2$, it follows that there is some constant $C_0 = C_0(n) > 1$ such that

$$\begin{aligned} 20\hat{r}_i\left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{2n_*} dz\right)^{1/(2n_*)} &= 20\hat{r}_i |K_{20\hat{r}_i}|^{-\left(\frac{1}{2n_*} - \frac{1}{2}\right)}\left(\int_{K_2} |f|^{2n_*} dz\right)^{\frac{1}{2n_*} - \frac{1}{2}}\left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{2n_*} dz\right)^{1/2} \\ &\leq C_0(n)\mathcal{G}(f)\left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{2n_*} dz\right)^{1/2}. \end{aligned} \tag{4.12}$$

Now, with the C_0 defined in (4.12), we define $\tau' = 3C_0\tau$, $\hat{u} = u/\tau'$, and $\hat{\lambda} = \tau'\lambda$. We see that $\hat{u}$ is a weak solution of

$$\begin{cases} \hat{u}_t - \operatorname{div}[\hat{\mathbf{A}}(x,t,\hat{\lambda}\hat{u},\nabla\hat{u}) - \hat{u}\mathbf{b} - \hat{\mathbf{F}}] = \hat{f}, & \text{in } K_2 \\ \langle \hat{\mathbf{A}}(x,t,\hat{\lambda}\hat{u},\nabla\hat{u}) - \hat{u}\mathbf{b} - \hat{\mathbf{F}}, \vec{\nu}\rangle = 0, & \text{on } T_2, \end{cases}$$

where

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{\tau'}, \quad \hat{f} = \frac{f}{\tau'}, \quad \text{and} \quad \hat{\mathbf{A}}(x,t,s,\xi) = \frac{\mathbf{A}(x,t,s,\tau'\xi)}{\tau'}.$$

As from Remark 2.1, $\hat{\mathbf{A}}$ satisfies all conditions (1.2)–(1.4) and

$$[\hat{\mathbf{A}}]_{\mathrm{BMO}(K_1,1)} = [\mathbf{A}]_{\mathrm{BMO}(K_1,1)} \leq \delta,$$

and

$$[[\hat{\lambda}\hat{u}]]_{\mathrm{BMO}(K_1,1)} = [[\lambda u]]_{\mathrm{BMO}(K_1,1)} \le M, \quad \fint_{K_{40\kappa\hat{r}_i}(z_i)} |\nabla \hat{u}|^2 dz \le 1.$$

Also, with $\hat{\mathbf{G}} \approx [[\hat{u}]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}$, and some $\alpha = 2(1+\gamma_0) \in (2,\eta)$ it follows from (4.12) and the Hölder's inequality that

$$\begin{aligned}
&\left(\fint_{K_{20\hat{r}_i}(z_i)} |\hat{\mathbf{F}}|^2 dz\right)^{1/2} + \left(\fint_{K_{20\hat{r}_i}(z_i)} |\hat{\mathbf{G}}|^{\alpha}\Big] dz\right)^{1/\alpha} + 20\hat{r}_i \left(\fint_{K_{20\hat{r}_i}(z_i)} |\hat{f}|^{2n_*} dz\right)^{1/(2n_*)} \\
&= \frac{1}{\tau'}\left[\left(\fint_{K_{20\hat{r}_i}(z_i)} |\mathbf{F}|^2 dz\right)^{1/2} + \left(\fint_{K_{20\hat{r}_i}(z_i)} |\mathbf{G}|^{\alpha} dz\right)^{1/\alpha} + 20\hat{r}_i \left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{2n_*} dz\right)^{1/(2n_*)}\right] \\
&= \frac{C_0}{\tau'}\left[\left(\fint_{K_{40\hat{r}_i}(z_i)} |\mathbf{F}|^2 dz\right)^{1/2} + \left(\fint_{K_{20\hat{r}_i}(z_i)} |\mathbf{G}|^{\alpha} dz\right)^{1/\alpha} + \mathcal{G}(f) \left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{2n_*} dz\right)^{1/2}\right] \qquad (4.13)\\
&\le \frac{1}{3\tau}\left[\left(\fint_{K_{40\hat{r}_i}(z_i)} |\mathbf{F}|^{\eta} dz\right)^{1/\eta} + \left(\fint_{K_{20\hat{r}_i}(z_i)} |\mathbf{G}|^{\eta} dz\right)^{1/\eta} + \mathcal{G}(f) \left(\fint_{K_{20\hat{r}_i}(z_i)} |f|^{n_*\eta} dz\right)^{1/\eta}\right] \\
&\le \frac{1}{\tau}\left(\fint_{K_{20\hat{r}_i}} |F|^{\eta} dz\right)^{1/\eta} \le \delta.
\end{aligned}$$

Therefore, all assumptions in Proposition 3.8 are satisfied with $r = 5\hat{r}_i/2$. Hence, we can find a function $\hat{v}_i$ such that

$$\fint_{K_{10r_i}(z_i)} |\nabla \hat{u} - \nabla \hat{v}_i|^2 dz \le \epsilon^2, \quad \|\nabla \hat{v}_i\|_{L^\infty(K_{5r_i}(z_i))} \le C_0(\Lambda, n).$$

Then, by scaling back with $v_i = \tau' \hat{v}_i$, we obtain

$$\fint_{K_{10r_i}(z_i)} |\nabla u - \nabla v_i|^2 dz \le 9C_0^2 \tau^2 \epsilon^2, \quad \|\nabla v_i\|_{L^\infty(K_{5r_i}(z_i))} \le 3C_0(\Lambda, n)\tau. \qquad (4.14)$$

Now, let $N_0 = 6C_0(\Lambda, n)\sqrt{C_*(n)}$, where $C_*(n)$ is defined to be

$$C_*(n) \ge \frac{|K_{10r}(z_0)|}{|K_r(z_0)|}, \quad \forall r \in (0,1), \quad \forall z_0 \in \Omega \cap B_2.$$

Observe that from Lemma 4.3,

$$|E_{s_1}(N_0\tau)| \le \sum_{i\in\mathcal{I}} \Big|\Big\{(x,t) \in K_{5r_i}(z_i) : |\nabla u(x,t)| > N_0\tau\Big\}\Big|.$$

Therefore,

$$\begin{aligned}
|E_{s_1}(N_0\tau)| &\le \sum_{i\in\mathcal{I}} \Big|\Big\{(x,t) \in K_{5r_i}(z_i) : |\nabla u(x,t) - \nabla v_i(x,t)| > \frac{N_0\tau}{2}\Big\}\Big| \\
&\quad + \sum_{i\in\mathcal{I}} \Big|\Big\{(x,t) \in K_{5r_i}(z_i) : |\nabla v_i(x,t)| > \frac{N_0\tau}{2}\Big\}\Big| \\
&\le \sum_{i\in\mathcal{I}} \Big|\Big\{(x,t) \in K_{10r_i}(z_i) : |\nabla u(x,t) - \nabla v_i(x,t)| > \frac{N_0\tau}{2}\Big\}\Big| \\
&\le \Big(\frac{2}{N_0\tau}\Big)^2 \sum_{i\in\mathcal{I}} \int_{K_{10r_i}(z_i)} |\nabla u - \nabla u_i|^2 dz \\
&\le 9C_0^2\epsilon^2 \Big(\frac{2}{N_0}\Big)^2 \sum_{i\in\mathcal{I}} |K_{10r_i}(z_i)| \le 9C_0^2\epsilon^2 \Big(\frac{2}{N_0}\Big)^2 C_*(n) \sum_{i\in\mathcal{I}} |K_{r_i}(z_i)|
\end{aligned}$$

From this and our choice of N_0, it follows that

$$|E_{s_1}(N_0\tau)| \leq \epsilon^2 \sum_{i\in\mathcal{I}} |K_{r_i}(z_i)|,$$

and the conclusion of our proposition follows directly from (4.7) and the fact that $\{K_{r_i}(z_i)\}_{i\in\mathcal{I}}$ is a disjoint family. ◻

5. Proof of main Theorems

As already discussed, Theorem 1.3 follows immediately from Theorem 1.1, Theorem 1.2, and some standard energy estimate. The proof of Theorem 1.1 is however similar to that of Theorem 1.2 by using Proposition 3.4 instead of Proposition 3.8. We therefore skip its proof and focus on proving Theorem 1.2. Again, through the dilation (2.3), Remark 2.1, and Remark 2.2, we can assume without loss of generality that $R = 1$.

Proof of Theorem 1.2. With Proposition 4.1 in hand, the proof is now standard (see [1, 2, 6]). We however give it here for the sake of completeness. For each $k \in \mathbb{N}$, we define $(\nabla u)_k(x,t) = \max\Big\{|\nabla u(x,t)|, k\Big\}$. It should be noted that we do not know yet if ∇u is in $L^{p,q}(K_1)$. However, since $(\nabla u)_k$ is bounded, $(\nabla u)_k \in L^{p,q}(K_2)$ for all $p > 2$ and $0 < q \leq \infty$. For $\mu \in [1,2]$, we denote

$$E^k_\mu(\tau) = \Big\{(x,t) \in K_\mu : (\nabla u)_k(x,t) > \tau\Big\}.$$

By considering the cases $k < N_0\tau$ and $k \geq N_0\tau$, we can conclude from the Proposition 4.1 that

$$|E^k_{s_1}(\tau N_0)| \leq \epsilon^2 \left[|E^k_{s_2}(\tau/4) + \frac{1}{(\delta\tau)^\eta}\int_{\delta\tau/4}^\infty s^\eta |\{(x,t)\in K_2 : |F(x,t)| > s\}|\frac{ds}{s}\right], \tag{5.1}$$

for all $\tau > \hat{B}_0\tau_0 = B_0[(s_2 - s_2)\kappa]^{-(n+2)/2}\tau_0$. We now divide the proof into two cases depending on if $q = \infty$ or not.

Case I: We start with the easy case when $q = \infty$. In this case, it is trivial that

$$\begin{aligned} \|(\nabla u)_k\|_{L^{p,\infty}(K_{s_1})} &= \sup_{\tau>0} \tau\Big|\{(x,t)\in K_{s_1} : (\nabla u)_k > \tau\}\Big|^{1/p} \\ &\leq \left[\sup_{0<\tau<N_0\hat{B}_0\tau_0} \tau\Big|\{(x,t)\in K_{s_1} : (\nabla u)_k > \tau\}\Big|^{1/p} + \sup_{N_0\hat{B}_0\tau_0<\tau} \tau\Big|\{(x,t)\in K_{s_1} : (\nabla u)_k > \tau\}\Big|^{1/p}\right]. \end{aligned} \tag{5.2}$$

From (4.4), the first term on the right hand side of (5.2) is obviously controlled by

$$\begin{aligned} |K_2|^{1/p} N_0\hat{B}_0\tau_0 &\leq C[(s_2 - s_1)\kappa]^{-(n+2)/2}\left[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^\eta(K_2)}\right] \\ &\leq C[(s_2 - s_1)\kappa]^{-(n+2)/2}\left[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,\infty}(K_2)}\right]. \end{aligned}$$

On the other hand, with (5.1), the second term on the right hand side of (5.2) can be rewritten and then controlled as

$$\begin{aligned} &\sup_{N_0\hat{B}_0\tau_0<\tau} \tau\Big|\{(x,t)\in K_{s_1} : (\nabla u)_k > \tau\}\Big|^{1/p} = N_0 \sup_{\hat{B}_0\tau_0<\tau} \tau\Big|\{(x,t)\in K_{s_1} : (\nabla u)_k > N_0\tau\}\Big|^{1/p} \\ &\leq C\epsilon^{2/p}\sup_{\tau>\hat{B}_0\tau_0} \tau\left[\Big|\{(x,t)\in K_{s_2} : (\nabla u)_k > \tau/4\}\Big| + \frac{1}{(\delta\tau)^\eta}\int_{\delta\tau/4}^\infty s^\eta|\{(x,t)\in K_2 : |F(x,t)| > s\}|\frac{ds}{s}\right]^{1/p} \\ &\leq C\epsilon^{2/p}\left[\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-\eta/p}\sup_{\tau>\hat{B}_0\tau_0}\left(\tau^{p-\eta}\int_{\delta\tau/4}^\infty s^{\eta-p}s^p|\{(x,t)\in K_2 : |F(x,t)| > s\}|\frac{ds}{s}\right)^{1/p}\right] \\ &\leq C\epsilon^{2/p}\left[\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-\eta/p}\|F\|_{L^{p,\infty}(K_2)}\sup_{\tau>\hat{B}_0\tau_0}\left(\tau^{p-\eta}\int_{\delta\tau/4}^\infty s^{\eta-p-1}ds\right)^{1/p}\right] \\ &\leq C\left[\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-1}\|F\|_{L^{p,\infty}(K_2)}\right]. \end{aligned}$$

Hence, combining the previous two estimates, we see that for every $1 \leq s_1 < s_2 \leq 2$, there is a constant $C_1 = C_1(\Lambda, n) > 0$ such that

$$\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_1})} \leq C_1\epsilon^{2/p}\, \|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + C_1[(s_2-s_1)\kappa]^{-(n+2)/2}\left[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\, \|F\|_{L^{p,\infty}(K_2)}\right].$$

This and since ϵ is sufficiently small so that $C_1\epsilon^{2/p} \leq 1/2$, we can use the iteration Lemma 2.5 to imply that

$$\begin{aligned}\|(\nabla u)_k\|_{L^{p,\infty}(K_1)} &\leq C(\Lambda, n, p, \kappa)\left[\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,\infty}(K_2)}\right]\\ &\leq C\left[\|\nabla u\|_{L^2(K_2)} + \|\mathbf{F}\|_{L^{p,\infty}(K_2)} + \|\mathbf{G}\|_{L^{p,\infty}(K_2)} + \mathcal{G}(f)\, \left\||f|^{n_*}\right\|_{L^{p,\infty}(K_2)}\right].\end{aligned}$$

We note that the Lorentz quasi-norm is lower semi-continuous with respect to the a.e. convergence. Because of this, we can take $k \to \infty$, and use (4.3) to obtain the desired estimate (1.16).

Case II: We consider the case $0 < q < \infty$. In this case,

$$\begin{aligned}\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} &\leq C(N_0, p, q)\left(\int_0^\infty \left[s^p|\{(x,t) \in K_{s_1} : (\nabla u)_k(x,t) > N_0 s\}|\right]^{q/p}\frac{ds}{s}\right)^{1/q}\\ &\leq C\left[\left(\int_0^{\hat{B}_0\tau_0}\cdots\right)^{1/q} + \left(\int_{\hat{B}_0\tau_0}^\infty\cdots\right)^{1/q}\right] = I_1 + I_2.\end{aligned} \tag{5.3}$$

Using (4.4), the first term I_1 is easily controlled as followings

$$\begin{aligned}I_1 &\leq C|K_2|^{1/p}\hat{B}_0\tau_0 \leq C[(s_2-s_1)\kappa]^{-\frac{n+2}{2}}\left[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\, \|F\|_{L^\eta(K_2)}\right]\\ &\leq C[(s_2-s_1)\kappa]^{-\frac{n+2}{2}}\left[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\, \|F\|_{L^{p,q}(K_2)}\right],\end{aligned} \tag{5.4}$$

For the term I_2, we use (5.1) to control it as

$$\begin{aligned}I_2 &\leq C\epsilon^{2/p}\left(\int_{\hat{B}_0\tau_0}^\infty s^q\left|\{(x,t) \in K_{s_2} : (\nabla u)_k(x,t) > s/4\}\right|^{q/p}\frac{ds}{s}\right)^{1/q}\\ &\quad + C\epsilon^{2/p}\delta^{-\eta/p}\left(\int_{\hat{B}_0\tau_0}^\infty s^{(p-\eta)q/p}\left\{\int_{\delta s/4}^\infty \tau^\eta|\{(x,t) \in K_2 : F(x,t) > \tau\}|\frac{d\tau}{\tau}\right\}^{q/p}\frac{ds}{s}\right)^{1/q}\\ &\leq C\epsilon^{2/p}\, \|(\nabla u)_k\|_{L^{p,q}(K_{s_2})}\\ &\quad + C\delta^{-1}\left(\int_{\hat{B}_0\tau_0}^\infty (\delta s)^{(p-\eta)q/p}\left\{\int_{\delta s/4}^\infty \tau^\eta|\{(x,t) \in K_2 : F(x,t) > \tau\}|\frac{d\tau}{\tau}\right\}^{q/p}\frac{ds}{s}\right)^{1/q}\\ &= C\left[\epsilon^{2/p}\, \|(\nabla u)_k\|_{L^{p,q}(K_{s_2R})} + J\right],\end{aligned} \tag{5.5}$$

where

$$J = \delta^{-1}\left(\int_{\hat{B}_0\tau_0}^\infty (\delta s)^{(p-\eta)q/p}\left\{\int_{\delta s/4}^\infty \tau^\eta|\{(x,t) \in K_2 : F(x,t) > \tau\}|\frac{d\tau}{\tau}\right\}^{q/p}\frac{ds}{s}\right)^{1/q}. \tag{5.6}$$

To control J, we consider the cases $q > p$ and $q < p$. When $q > p$, we use the Hardy's inequality, Lemma 2.6 with

$$\kappa = \frac{q}{p} \geq 1, \quad r = \frac{(p-\eta)q}{p} > 0, \quad \text{and} \quad h(\tau) = \tau^{\eta-1}|\{(x,t) \in K_2 : F(x,t) > \tau\}|.$$

Observe that because $\eta < p$, $F \in L^\eta(K_2)$, and hence $h \in L^1((0,\infty))$. Therefore, Lemma 2.6 implies

$$\begin{aligned}J &\leq C\delta^{-1}\left[\int_0^\infty s^{(p-\eta)q/p}s^{\eta q/p}\left|\{(x,t) \in K_2 : F(x,t) > s\}\right|^{q/p}\frac{ds}{s}\right]^{1/q}\\ &= C\delta^{-1}\left[\int_0^\infty s^q\left|\{(x,t) \in K_2 : F(x,t) > s\}\right|^{q/p}\frac{ds}{s}\right]^{1/q}\\ &= C\delta^{-1}\, \|F\|_{L^{p,q}(K_2)}.\end{aligned}$$

This estimate, (5.3), (5.4), and (5.5) imply that

$$\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} \leq C_2 \left[\epsilon^{2/p} \|(\nabla u)_k\|_{L^{p,q}(K_{s_2})} + [(s_2 - s_1)\kappa]^{-\frac{n+2}{2}} \left(\|\nabla u\|_{L^2(K_2)} + \delta^{-1} \|F\|_{L^{p,q}(K_2)}\right)\right],$$

for some constant C_2 depending only on Λ, n, p, q. This, and by taking ϵ sufficiently small such that $C_2\epsilon^{2/p} \leq 1/2$, we can apply the iteration lemma, Lemma 2.5, to obtain

$$\|(\nabla u)_k\|_{L^{p,q}(K_1)} \leq C\left(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\right).$$

Then, as before, we can send $k \to \infty$ to infer that

$$\|\nabla u\|_{L^{p,q}(K_1)} \leq C\left(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\right)$$

This estimate and (4.3) imply the desired estimate (1.16).

It now only remains to consider the case $q \leq p$. In this case, by using Lemma 2.7 with

$$\kappa = \frac{p}{q} \geq 1, \quad r = \frac{\eta q}{p}, \quad \text{and} \quad h(\tau) = \left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}.$$

we see that

$$\begin{aligned}
&\left(\int_{\delta s/4}^{\infty} \tau^{\eta}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|\frac{d\tau}{\tau}\right)^{q/p} \\
&= \left(\int_{\delta s/4}^{\infty}\left[\tau^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\right]^{p/q} \frac{d\tau}{\tau}\right)^{q/p} \\
&\leq C\left[(s\delta)^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \delta s/4\right\}\right|^{q/p} + \int_{\delta s/4}^{\infty} \tau^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\frac{d\tau}{\tau}\right]
\end{aligned}$$

Pluging this estimate into the definition of J in (5.6), we infer that

$$\begin{aligned}
J &\leq C\delta^{-1}\left[\left(\int_0^{\infty}(s\delta)^{(p-\eta)q/p}(s\delta)^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \delta s/4\right\}\right|^{q/p}\frac{ds}{s}\right)^{1/q}\right. \\
&\quad \left.+ \left\{\int_0^{\infty}(s\delta)^{(p-\eta)q/p}\left(\int_{\delta s/4}^{\infty}\tau^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\frac{d\tau}{\tau}\right)\frac{ds}{s}\right\}^{1/q}\right] \\
&\leq C\delta^{-1}\left[\|F\|_{L^{p,q}(K_2)} + \left\{\int_0^{\infty}(s\delta)^{(p-\eta)q/p}\left(\int_{\delta s/4}^{\infty}\tau^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\frac{d\tau}{\tau}\right)\frac{ds}{s}\right\}^{1/q}\right] \\
&= C\delta^{-1}[\|F\|_{L^{p,q}(K_2)} + J'].
\end{aligned}$$

We control J' by using Fubini's theorem as follows

$$\begin{aligned}
J' &= \left(\int_0^{\infty}\tau^{\eta q/p}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\left(\int_0^{\tau/(4\delta)}(s\delta)^{(p-\eta)q/p}\frac{ds}{s}\right)\frac{d\tau}{\tau}\right)^{1/q} \\
&\leq C\left(\int_0^{\infty}\tau^{q}\left|\left\{(x,t) \in K_2 : F(x,t) > \tau\right\}\right|^{q/p}\frac{d\tau}{\tau}\right)^{1/q} = C\,\|F\|_{L^{p,q}(K_2)}.
\end{aligned}$$

Hence, we conclude in this case that $J \leq C\,\|F\|_{L^{p,q}(K_2)}$. From this estimate, (5.3), (5.4), and (5.5), we again conclude that

$$\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} \leq C_3\left[\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,q}(K_{s_2})} + [(s_2 - s_1)\kappa]^{-\frac{n+2}{2}}\left(|\,\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,q}(K_2)}\right)\right].$$

Argue as before, by choosing ϵ such that $C_3\epsilon^{2/p} \leq 1/2$ and then sending $k \to \infty$, we also obtain

$$\|\nabla u\|_{L^{p,q}(K_1)} \leq C\left(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\right).$$

This and (4.3) give (1.16). The proof is therefore complete once we chose $\epsilon < \min\left\{\frac{1}{2C_1}, \frac{1}{2C_2}, \frac{1}{2C_3}\right\}^{p/2}$, where C_1, C_2, C_3 are constants defined above and dependent only on $\Lambda, M, q, n, \alpha_0$. □

Appendix A. Proofs of Lemma 2.10 and Lemma 2.11

We only prove Lemma 2.11 since the other is simpler. We follow the approach used in [19, 44]. To this end, some notation is needed. We fix a cut-off function $\varphi \in C_0^\infty(B_2)$ with the following properties

$$\varphi(x) = 1, \quad \text{for} \quad x \in B_1.$$

For each $r > 0$, and each $x_0 \in \mathbb{R}^n$, we also define

$$\varphi_{x_0,2r}(x) = \varphi((x - x_0)/r).$$

As in [19], the following mean value of u will be used

$$u_{x_0,2r}(t) = \left(\int_{\Omega_{2r}(x_0)} \varphi^2_{x_0,2r}(x)dx\right)^{-1} \int_{\Omega_{2r}(x_0)} u(x,t)\varphi^2_{x_0,2r}(x)dx. \tag{A.1}$$

Without loss of generality, we can assume that $R = 1$. Hence, we consider the equation

$$\begin{cases} u_t - \operatorname{div}[\mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}] & = & f, & \text{in} & K_2, \\ \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu}\rangle & = & 0, & \text{on} & T_2. \end{cases} \tag{A.2}$$

We recall that if u is a solution of (A.2), we define

$$\mathbf{G}(x,t) = C_0^*(\gamma_0, n)[[u]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}(x,t), \quad (x,t) \in K_2,$$

for some constant $C_0^*(\gamma_0, n) \geq 1$ defined in (A.4) below.

Lemma A.1. *If u is a weak solution of* (A.2)*, then for every $t_1, t_2 \in (-4, 4)$ with $t_1 < t_2$, and for every $x_0 \in \Omega_2$, every $\rho \in (0,1)$*

$$\begin{aligned} |u_{x_0,2\rho}(t_2) - u_{x_0,2\rho}(t_1)| \leq C(\Lambda, n)\Bigg[&\frac{1}{\rho^{n+1}}\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} \Big(|\nabla u| + |\mathbf{F}|\Big)dz + \frac{1}{\rho^n}\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |f(x,t)|dz \\ &+ \frac{1}{\rho^n}\left(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |\mathbf{G}|^2 dz\right)^{1/2}\Bigg]. \end{aligned}$$

Proof. With Steklov's average as in [5, 13, 38], we can formally use $\varphi_{x_0,2\rho}$ as a test function for (A.2) to obtain

$$\begin{aligned} &\int_{\Omega_{2\rho}(x_0)} u(x,t_2)\varphi_{x_0,2\rho}(x)dx - \int_{\Omega_{2\rho}(x_0)} u(x,t_1)\varphi_{x_0,2\rho}(x)dx \\ &= -\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} \langle \mathbf{A}(x,t,u,\nabla u) - (u - \bar{u}_{K_2})\mathbf{b} - \mathbf{F}, \nabla\varphi_{x_0,2\rho}\rangle dx + \int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} f(x,t)\varphi_{x_0,2\rho}(x)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} |u_{x_0,2\rho}(t_2) - u_{x_0,2\rho}(t_2)| &\leq \frac{C(\Lambda,n)}{\rho^{n+1}}\Bigg[\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} \Big(|\nabla u| + |\mathbf{F}|\Big)dz + \rho\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |f(x,t)|dz \\ &\qquad + \left(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |\mathbf{b}|^2 dz\right)^{1/2}\left(\int_{K_2} |u - \bar{u}_{K_2}|^2 dz\right)^{1/2}\Bigg] \\ &\leq \frac{C(\Lambda,n)}{\rho^{n+1}}\Bigg[\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} \Big(|\nabla u| + |\mathbf{F}|\Big)dz + \rho\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |f(x,t)|dz \\ &\qquad + \left(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} |\mathbf{G}|^2 dz\right)^{1/2}\Bigg]. \end{aligned}$$

The proof is complete. □

Lemma A.2 (Caccioppoli's type estimate). *For each $z_0 = (x_0, t_0) \in K_1$, for each $\rho \in (0,1)$, if u is a weak solution of* (A.2)*, there holds*

$$
\begin{aligned}
&\rho^{-2} \sup_{\tau \in \Gamma_{2\rho}(t_0)} \int_{\Omega_\rho(x_0)} |u(x,t) - u_{x_0,2\rho}|^2 dx + \fint_{K_\rho(z_0)} |\nabla u|^2 dz \\
&\leq C(\Lambda, n)\left[\rho^{-2} \fint_{K_{2\rho}(z_0)} |u - u_{x_0,2\rho}|^2 dz + \left(\fint_{K_{2\rho}} |\mathbf{G}|^\alpha dz\right)^{2/\alpha} + \fint_{K_{2\rho}} |\mathbf{F}(x,t)|^2 dz + \rho^2 \left(\fint_{K_{2\rho}} |f(x,t)|^{2n_*} dz\right)^{1/n_*}\right].
\end{aligned}
$$

where $\alpha = 2(1+\gamma_0)$ with $\gamma_0 > 0$ is any fixed number, and $n_* = \frac{n+2}{n+4}$.

Proof. Let $\sigma \in C_0^\infty(\Gamma_{2\rho}(t_0))$ be a cut-off function satisfying $0 \leq \sigma \leq 1$ and

$$
\sigma(t) = 1, \quad \text{for all} \quad t \in \Gamma_\rho(t_0), \quad \text{and} \quad |\sigma'(t)| \leq \frac{100}{\rho^2}, \quad \forall\, t \in \Gamma_{2\rho}(t).
$$

By using Steklov's average as in [5, 13, 38], we can formally use $(u - u_{x_0,2\rho}(t))\sigma^2(t)\varphi^2_{x_0,2\rho}$ as a test function for the equation (A.2) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\Omega_{x_0,2\rho}(x_0)} \varphi^2_{2\rho} |u - u_{x_0,2\rho}(t)|^2 \sigma^2(t) dx + \partial_t u_{x_0,2\rho}(t)\sigma^2(t) \int_{\Omega_{2\rho}(x_0)} \varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho}) dx \\
&= -\int_{\Omega_{2\rho}(x_0)} \sigma^2(t) \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}, \nabla[(u - u_{x_0,2\rho})\varphi^2_{x_0,2\rho}]\rangle dx \\
&\quad + \int_{\Omega_{2\rho}(x_0)} f(x,t)\sigma^2(t)\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho}) dx + \int_{\Omega_{2\rho}(x_0)} \varphi^2_{2\rho} |u - u_{x_0,2\rho}(t)|^2 \sigma'(t)\sigma(t) dx.
\end{aligned}
$$

We observe that from Lemma A.1, $\partial_t u_{x_0,2\rho}(t)$ is integrable and it is defined a.e. $t \in \Gamma_{2\rho}(t_0)$. Moreover, it follows immediately from (A.1) that

$$
\int_{\Omega_{2\rho}(x_0)} \varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho}) dx = 0.
$$

From this and since (1.13) holds on Ω_2, it follows that for each $\tau \in \Gamma_{2\rho}(t_0)$,

$$
\begin{aligned}
&\frac{1}{2} \int_{\Omega_{x_0,2\rho}(x_0)} \varphi^2_{x_0,2\rho} |u(x,\tau) - u_{x_0,2\rho}(\tau)|^2 \sigma^2(\tau) dx + \int_{t_0 - 4\rho^2}^{\tau} \int_{\Omega_{2\rho}(x_0)} \langle \mathbf{A}(x,t,u,\nabla u), \nabla u\rangle \sigma^2(t)\varphi^2_{x_0,2\rho} dz \\
&= -2 \int_{t_0-4\rho^2}^{\tau} \int_{\Omega_{2\rho}(x_0)} \langle \mathbf{A}(x,t,u,\nabla u), \nabla\varphi_{x_0,2\rho}\rangle (u - u_{x_0,2\rho})\varphi_{x_0,2\rho}\sigma^2(t) dz \\
&\quad + \int_{t_0-4\rho^2}^{\tau} \int_{\Omega_{2\rho}(x_0)} \langle (u - \bar{u}_{K_{2\rho}(z_0)})\mathbf{b} + \mathbf{F}, \nabla[(u - u_{x_0,2\rho})\varphi^2_{x_0,2\rho}]\rangle dz \\
&\quad + \int_{t_0-4\rho^2}^{\tau} \int_{\Omega_{2\rho}(x_0)} f(x,t)\sigma^2(t)\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho}) dz + \int_{t_0-4\rho^2}^{\tau} \int_{\Omega_{2\rho}(x_0)} \varphi^2_{x_0,2\rho} |u - u_{x_0,2\rho}(t)|^2 \sigma'(t)\sigma(t) dz.
\end{aligned}
$$

This, and the conditions (1.2)-(1.3) imply that

$$
\begin{aligned}
&\sup_{\tau \in \Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} \varphi^2_{x_0,2\rho} |u(x,\tau) - u_{x_0,2\rho}(\tau)|^2 \sigma^2(\tau) dx + \int_{K_{2\rho}(z_0)} |\nabla(u - u_{x_0,2\rho})|^2 \varphi^2_{x_0,2\rho}\sigma^2(t) dz \\
&\leq C(\Lambda)\Bigg[\int_{K_{2\rho}(z_0)} |\nabla u| \varphi_{x_0,2\rho} |\nabla\varphi_{x_0,2\rho}| |u - u_{x_0,2\rho}| \sigma^2(t) dz \\
&\quad + \int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}||u - \bar{u}_{K_{2\rho}}| + |\mathbf{F}|\Big)\Big(|\nabla u|\varphi^2_{x_0,2\rho} + 2|u - u_{x_0,2\rho}||\nabla\varphi_{x_0,2\rho}|\varphi_{x_0,2\rho}\Big)\sigma^2(t) dz \\
&\quad + \int_{K_{2\rho}(z_0)} |f(x,t)|\sigma^2(t)\varphi^2_{x_0,2\rho} |u - u_{x_0,2\rho}| dz + \int_{K_{2\rho}(z_0)} \varphi^2_{x_0,2\rho} |u - u_{x_0,2\rho}(t)|^2 |\sigma'(t)||\sigma(t)| dz.
\end{aligned}
\tag{A.3}
$$

We now control the first two terms on the right hand side of (A.3). Let $\epsilon > 0$ sufficiently small, which will be determined. Use Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}
&\int_{K_{2\rho}(z_0)} |\nabla u|\varphi_{x_0,2\rho}|\nabla \varphi_{x_0,2\rho}||u-u_{x_0,2\rho}|\sigma^2(t)dz \\
&\leq \epsilon \int_{K_{2\rho}(z_0)} |\nabla u|^2\varphi_{x_0,2\rho}^2\sigma^2(t)dz + C(\epsilon)\int_{K_{2\rho}(z_0)} |\nabla\varphi_{x_0,2\rho}|^2|u-u_{x_0,2\rho}|^2\sigma^2(t)dz.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}||u-\bar{u}_{K_{2\rho}(z_0)}| + |\mathbf{F}|\Big)\Big(|\nabla u|\varphi_{x_0,2\rho}^2 + 2|u-u_{x_0,2\rho}||\nabla\varphi_{x_0,2\rho}|\varphi_{x_0,2\rho}\Big)\sigma^2(t)dz \\
&\leq \epsilon \int_{K_{2\rho}(z_0)} |\nabla u|^2\varphi_{x_0,2\rho}^2\sigma^2(t)dz + C(\epsilon)\left[\int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}|^2|u-\bar{u}_{K_{2\rho}(z_0)}|^2 + |\mathbf{F}|^2\Big)\varphi_{x_0,2\rho}^2\sigma^2(t)dz \right. \\
&\qquad \left. + \int_{K_{2\rho}(z_0)} |u-\bar{u}_{x_0,2\rho}|^2|\nabla\varphi_{x_0,2\rho}|^2\sigma^2(t)dz\right].
\end{aligned}$$

Apply Hölder's inequality again for the term involving $\mathbf{b}$, we see that

$$\begin{aligned}
&\fint_{K_{2\rho}(z_0)} |\mathbf{b}|^2|u-\bar{u}_{K_{2\rho}(z_0)}|^2\varphi_{x_0,2\rho}^2\sigma^2(t)dz \\
&\leq \left(\fint_{K_{2\rho}(z_0)} |\mathbf{b}|^\alpha dz\right)^{2/\alpha}\left(\fint_{K_{2\rho}(z_0)} |u-\bar{u}_{K_{2\rho}(z_0)}|^{\frac{2\alpha}{\alpha-2}}\right)^{(\alpha-2)/\alpha} \\
&\leq C_0^*(n,\gamma_0)[[u]]^2_{\mathrm{BMO}(K_1,1)}\left(\fint_{K_{2\rho}(z_0)} |\mathbf{b}|^\alpha dz\right)^{2/\alpha} = \left(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^\alpha dz\right)^{2/\alpha}.
\end{aligned} \tag{A.4}$$

Then, by writing $w = |u(x,t) - u_{x_0,2\rho}(t)||\varphi_{x_0,2\rho}(x)\sigma(t)$ and collecting all last estimates together with (A.3), and the choice that ϵ sufficiently small, we infer that

$$\begin{aligned}
&\rho^{-2}\sup_{\tau\in\Gamma_{2\rho}(t_0)}\int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2dz \\
&\leq C(\Lambda,n)\left[\fint_{K_{2\rho}(z_0)} |u-u_{x_0,2\rho}|^2\Big(\varphi_{x_0,2\rho}^2|\sigma'(t)|\sigma(t) + |\nabla\varphi_{x_0,2\rho}|^2\sigma^2(t)\Big) + \left(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^\alpha dz\right)^{2/\alpha}\right. \\
&\qquad \left. + \fint_{K_{2\rho}(z_0)} |\mathbf{F}|^2\varphi_{x_0,2\rho}^2\sigma^2(t)dz + \fint_{K_{2\rho}(z_0)} |f(x,t)||u-u_{x_0,2\rho}||\varphi_{x_0,2\rho}^2\sigma^2(t)dz\right].
\end{aligned} \tag{A.5}$$

Finally, it remains to control the last term on the right hand side of (A.5). This, however, can be done exactly the same as in (3.19) using Hölder's inequality, Sobolev imbedding [31, eqn (3.2), p. 74]), and Young's inequality with ϵ sufficiently small, we then obtained

$$\begin{aligned}
&\fint_{K_{2\rho}(z_0)} |f(x,t)||u-u_{x_0,2\rho}||\varphi_{x_0,2\rho}^2\sigma^2(t)dz \\
&\leq \epsilon\left[\rho^{-2}\sup_{t\in\Gamma_{2\rho}(t_0)}\int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2dxdt\right] + C(n,\epsilon)\rho^2\left(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*}dz\right)^{1/n_*}.
\end{aligned}$$

Then, with ϵ sufficiently small, it follows from (A.5) and the last estimate that

$$\begin{aligned}
&\rho^{-2} \sup_{\tau\in\Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2 dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2 dz \\
&\leq C(\Lambda,n) \Bigg[\fint_{K_{2\rho}(z_0)} |u-u_{x_0,2\rho}|^2 \Big(\varphi_{x_0,2\rho}^2 |\sigma'(t)|\sigma(t) + |\nabla \varphi_{x_0,2\rho}|^2 \sigma^2(t)\Big) + \left(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^{\alpha} dz\right)^{2/\alpha} \\
&\qquad + \fint_{K_{2\rho}(z_0)} |\mathbf{F}|^2 \varphi_{x_0,2\rho}^2 \sigma^2(t) dz + \rho^2 \left(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*} dz\right)^{1/n_*} \Bigg].
\end{aligned}$$

The proof of the lemma is now complete. □

Lemma A.3. *There is $\mu \in (1,2)$ which depends only on n such that the following holds. For every $\epsilon > 0, \alpha = 2(1+\gamma_0)$ with some $\gamma_0 > 0$, there exists $C_0 = C_0(\Lambda,n,\epsilon)$ such that the following holds. For every $z_0 = (x_0,t_0) \in K_1$, for each $\rho \in (0,1/4)$, if u is a weak solution of* (A.2)*, there holds*

$$\begin{aligned}
\fint_{K_\rho(z_0)} |\nabla u|^2 dz \leq \epsilon \fint_{K_{4\rho}} |\nabla u|^2 dz + C_0 \Bigg[\left(\fint_{K_{4\rho}(z_0)} |\nabla u|^{\mu} dz\right)^{2/\mu} + \left(\fint_{K_{4\rho}} |\mathbf{G}|^{\alpha} dz\right)^{2/\alpha} \\
+ \fint_{K_{4\rho}} |\mathbf{F}(x,t)|^2 dz + \mathcal{G}(f)^2 \left(\fint_{K_{4\rho}} |f(x,t)|^{2n_*} dz\right) \Bigg].
\end{aligned}$$

where $n_* = \frac{n+2}{n+4}$, and $\mathcal{G}(f)$ is defined in (4.2).

Proof. For simplicity in writing, let us denote

$$\mathcal{F}(2\rho) = \left(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^{\alpha} dz\right)^{2/\alpha} + \fint_{K_{2\rho}(z_0)} |\mathbf{F}(x,t)|^2 dz + \mathcal{G}(f)^2 \left(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*} dz\right).$$

By Poincaré's inequality, we see that

$$\rho^{-2} \fint_{K_{2\rho}(z_0)} |u-u_{x_0,2\rho}|^2 dz \leq C(n) \fint_{K_{2\rho}(z_0)} |\nabla u|^2 dz.$$

This estimate, Lemma A.2, and (4.12) imply that

$$\rho^{-2} \sup_{t\in\Gamma_\rho(t_0)} \fint_{\Omega_\rho(x_0)} |u-u_{x_0,2\rho}|^2 dx \leq C(\Lambda,n) \left[\fint_{K_{2\rho}(z_0)} |\nabla u|^2 dz + \mathcal{F}(2\rho)\right]. \tag{A.6}$$

We now denote $\hat{u} = u - u_{x_0,2\rho}$. Then, observe that

$$\begin{aligned}
\rho^{-2} \fint_{K_{2\rho}(z_0)} |\hat{u}|^2 dz &\leq \rho^{-2} \left[\sup_{t\in\Gamma_{2\rho}(t_0)} \left(\fint_{\Omega_{2\rho}(x_0)} |\hat{u}|^2 dx\right)^{1/2}\right] \left[\fint_{\Gamma_{2\rho}(t_0)} \left(\fint_{\Omega_{2\rho}(x_0)} |\hat{u}|^2 dx\right)^{1/2} dt\right] \\
&\leq C\rho^{-1} \left[\left(\fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz\right)^{1/2} + \mathcal{F}(4\rho)^{1/2}\right] \left[\fint_{\Gamma_{2\rho}(t_0)} \left(\fint_{\Omega_{2\rho}(x_0)} |\hat{u}|^2 dx\right)^{1/2} dt\right],
\end{aligned} \tag{A.7}$$

where we have used (A.6) in the last estimate with ρ replaced by 2ρ. We now control the last multiplier in the right hand side of (A.7). To this end, if $n > 2$, and we take $2^* = \frac{2n}{n-2}$, and when $n = 2$, we can take 2^* to be any number that is greater than 2. Then, let $\mu \in (1,2)$ such that $\frac{1}{2^*} + \frac{1}{\mu} = 1$ (observe that $\mu = \frac{2n}{n+2}$ if $n > 2$). From this, Hölder's inequality, Poincaré's inequality, and Sobolev-Poincaré's inequality, it follows that

$$\rho^{-1}\fint_{\Gamma_{2\rho}(t_0)}\left(\fint_{\Omega_{2\rho}(x_0)}|\hat{u}|^2dx\right)^{1/2}dt \le \rho^{-1}\fint_{\Gamma_{2\rho}(t_0)}\left[\left(\fint_{\Omega_{2\rho}(x_0)}|\hat{u}|^{\mu}dx\right)^{\frac{1}{2\mu}}\left(\fint_{\Omega_{2\rho}(x_0)}|\hat{u}|^{2^*}dx\right)^{\frac{1}{2\cdot 2^*}}\right]dt$$

$$\le C(n)\rho^{-1/2}\fint_{\Gamma_{2\rho}(t_0)}\left[\left(\fint_{\Omega_{2\rho}(x_0)}|\nabla u|^{\mu}dx\right)^{\frac{1}{2\mu}}\left(\fint_{\Omega_{2\rho}(x_0)}|\hat{u}|^{2^*}dx\right)^{\frac{1}{2\cdot 2^*}}\right]dt$$

$$\le C(n)\fint_{\Gamma_{2\rho}(t_0)}\left[\left(\fint_{\Omega_{2\rho}(x_0)}|\nabla u|^{\mu}dx\right)^{\frac{1}{2\mu}}\left(\fint_{\Omega_{2\rho}(x_0)}|\nabla u|^{2}dx\right)^{\frac{1}{4}}\right]dt.$$

We then use Hölder's inequality twice for the time integration in the last estimate to infer that

$$\rho^{-1}\fint_{\Gamma_{2\rho}(t_0)}\left(\fint_{\Omega_{2\rho}(x_0)}|\hat{u}|^2dx\right)^{1/2}dt \le C(n)\left[\left(\fint_{K_{2\rho}(z_0)}|\nabla u|^{\mu}dz\right)^{\frac{1}{2\mu}}\right]\left[\fint_{\Gamma_{2\rho}(t_0)}dt\left(\fint_{\Omega_{2\rho}(x_0)}|\nabla u|^2dx\right)^{\frac{1}{2}\frac{\mu}{2\mu-1}}\right]^{\frac{2\mu-1}{2\mu}}$$

$$\le C(n)\left(\fint_{K_{2\rho}(z_0)}|\nabla u|^{\mu}dz\right)^{\frac{1}{2\mu}}\left(\fint_{K_{2\rho}(z_0)}|\nabla u|^2dz\right)^{\frac{1}{4}}.$$

The last estimate, together with (A.7) imply that

$$\rho^{-2}\fint_{K_{2\rho}(z_0)}|\hat{u}|^2dz \le C(n)\left[\left(\fint_{K_{4\rho}(z_0)}|\nabla u|^2dz\right)^{1/2}+\mathcal{F}(4\rho)^{1/2}\right]\left(\fint_{K_{4\rho}(z_0)}|\nabla u|^{\mu}dz\right)^{\frac{1}{2\mu}}\left(\fint_{K_{4\rho}(z_0)}|\nabla u|^2dz\right)^{\frac{1}{4}}$$

$$\le \epsilon\fint_{K_{4\rho}(z_0)}|\nabla u|^2dz + C(n,\epsilon)\left[\left(\fint_{K_{4\rho}(z_0)}|\nabla u|^{\mu}dz\right)^{\frac{2}{\mu}}+\mathcal{F}(4\rho)\right],$$

From this estimate, and Lemma A.2, we see that

$$\fint_{K_{\rho}(z_0)}|\nabla u|^2dz \le \epsilon\fint_{K_{4\rho}(z_0)}|\nabla u|^2dz + C(\Lambda,n,\epsilon)\left[\left(\fint_{K_{4\rho}(z_0)}|\nabla u|^{\mu}dz\right)^{\frac{2}{\mu}}+\mathcal{F}(4\rho)\right].$$

Hence, Lemma A.3 is proved. □

Proof of Lemma 2.11. Lemma 2.11 follows from Lemma A.3 and the standard Gehring's type lemma (see [19, Proposition 1.3], [20, Proposition 5.1], or [23, Corollary 6.1, p. 204] for example). □

Acknowledgement. This research is supported by the Simons Foundation, grant # 354889. The author also would like to thanks anonymous referees for valuable comments which significantly improve the manuscript.

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