

# GRADIENT ESTIMATES FOR WEAK SOLUTIONS OF LINEAR ELLIPTIC SYSTEMS WITH SINGULAR-DEGENERATE COEFFICIENTS

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**ABSTRACT.** This paper establishes Calderón-Zygmund type regularity estimates for solutions of the conormal derivative problem for a class of linear elliptic systems in divergence-form with singular, degenerate coefficients in bounded domains. In our class of equations, the principal terms are fourth order tensors of measurable functions that behave as some weight function in the Muckenhoupt class of  $A_2$ -weights. Regularity estimates for gradient of weak solutions in weighted Lebesgue spaces are established under some natural smallness conditions on the mean oscillation of coefficients. The results obtained recover known results when the coefficients are uniformly elliptic. These results can be considered as the Sobolev counterparts of the classical Hölder's regularity estimates established by B. Fabes, C. E. Kenig, and R. P. Serapioni.

## 1. INTRODUCTION

This work studies regularity estimates in weighted Sobolev spaces for weak solutions of a class of linear systems of elliptic equation with prescribed degenerate, singular coefficients in bounded domains with conormal boundary conditions. The system we are studying is given by

$$(1.1) \quad \begin{cases} \operatorname{div}[\mathcal{A}(x)\nabla u] &= \operatorname{div}[\mathbf{F}] & \text{in } \Omega, \\ \langle \mathcal{A}(x)\nabla u - \mathbf{F}, \mathbf{n} \rangle &= 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$  is an open bounded domain with  $C^1$  boundary,  $\mathbf{F} : \Omega \rightarrow \mathbb{R}^{n \times N}$  is a given vector field,  $u : \Omega \rightarrow \mathbb{R}^N$ , and  $n, N \in \mathbb{N}$ , and  $\mathbf{n}$  is the outward unit normal vector at points on the boundary of  $\Omega$ . The coefficient  $\mathcal{A} : \Omega \rightarrow \mathbb{R}^{nN \times nN}$  is a fourth order tensor  $(A_{ij}^{\alpha\beta})_{i,j=1}^N_{\alpha,\beta=1}^n$  of measurable functions that could be degenerate or singular as some weight function in some Muckenhoupt class of weight. Precisely, we assume that there exists a constant  $\Lambda > 0$  and a non-negative measurable function  $\mu$  on  $\mathbb{R}^n$

$$(1.2) \quad \Lambda \mu(x) |\xi|^2 \leq \langle \mathcal{A}(x)\xi, \xi \rangle = \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n A_{ij}^{\alpha\beta} \xi_\alpha^j \xi_\beta^i, \quad \forall \xi = (\xi_\alpha^i) \in \mathbb{R}^{nN}, \text{ a.e. } x \in \Omega, \text{ and}$$

$$(1.3) \quad |A_{ij}^{\alpha\beta}(x)| \leq \Lambda^{-1} \mu(x) \quad \text{a.e. } x \in \Omega, \text{ and all } \alpha, \beta = 1, \dots, N; i, j = 1, \dots, n.$$

Our goal is to establish the regularity of weak solutions of the degenerate system (1.1) in weighted Sobolev spaces. Regularity estimates for weak solutions of uniformly elliptic equations and systems of equations ( $\mu = 1$ ) for both Dirichlet and conormal derivative boundary value problems in Sobolev spaces is commonplace these days. One can find results in [1–3, 5, 12, 15–18, 23, 26] in which uniform elliptic coefficients are studied. Essentially one expects that the matrix  $\mathbf{F}$  and  $\nabla u$  have the same integrability property. However by now it is well known that the mere assumption on the uniform ellipticity of the tensor of coefficients  $\mathcal{A}$  is not sufficient for the gradient of the weak solution of (1.1) to have the same integrability as that of the data  $\mathbf{F}$ . This fact can be seen from the counterexample provided by N. G. Meyers in [19]. In the event that  $\mathcal{A}$  is uniformly elliptic and continuous, the  $L^p$ -norm of  $\nabla u$  can be controlled by the  $L^p$ -norm of the datum  $\mathbf{F}$  and this

is achieved via the Calderón-Zygmund theory of singular integrals and a perturbation technique, see [5, 12, 15–18] for this classical results for both elliptic and parabolic equations.

Our interest lies on equations with coefficients that may be degenerate or singular. The class of second order linear elliptic equations with degenerate coefficients with general case  $\mu \in A_2$  was investigated by B. Fabes, C. E. Kenig, and R. P. Serapioni for the first time in the pioneering paper [7] in 1982. In this classical paper [7], among other important results, existence, and uniqueness of weak solutions in weighted Sobolev space  $W_0^{1,2}(\Omega, \mu)$  were established; Harnack's inequality and Hölder's regularity of weak solutions were also obtained by adapting the Möser's iteration technique to the degenerate, non-uniformly elliptic equations (1.1). See also [22, 25] for some other earlier results on elliptic equations with measurable degenerate coefficients.

Recently, in [4], we have obtained estimates in Lebesgue spaces for gradient of solutions to zero Dirichlet boundary value problems for linear degenerate equations of type (1.1) with general  $\mu \in A_2$ . Two weighted Calderón-Zygmund type regularity estimates for quasilinear elliptic equations with prescribed singular-degenerate coefficients and non-homogeneous Dirichlet boundary conditions are also established in [24]. We aim to extend the above mentioned results in [4, 24] and establish the corresponding results for weak solutions of the conormal derivative problem for linear systems (1.1) by giving the right and optimal conditions on the coefficient tensor. As already indicated, the regularity estimate we obtain requires that coefficient matrix must not oscillate too much. To describe the condition precisely, we want to introduce a means of measuring oscillation. The following definition is a weighted version of functions of bounded mean oscillation that is compatible with degenerate and singular coefficients. This definition can be found in [9, 10, 20, 21].

**Definition 1.1.** *Given  $R_0 > 0$ , we say  $f : \Omega \rightarrow \mathbb{R}$  is function of bounded mean oscillation with weight  $\mu$  in  $\Omega$  if  $[f]_{\text{BMO}_{R_0}(\Omega, \mu)} < \infty$  where*

$$[f]_{\text{BMO}_{R_0}(\Omega, \mu)} = \sup_{\substack{x \in \Omega \\ 0 < \rho < R_0}} \frac{1}{\mu(B_\rho(x) \cap \Omega)} \int_{B_\rho(x) \cap \Omega} |f(y) - \langle f \rangle_{B_\rho(x) \cap \Omega}| dy,$$

and

$$\langle f \rangle_{B_\rho(x) \cap \Omega} = \frac{1}{|B_\rho(x) \cap \Omega|} \int_{B_\rho(x) \cap \Omega} f(y) dy.$$

Observe that the classical John-Nirenberg BMO in  $\Omega$  corresponds to  $\mu = 1$  and  $R_0 = \text{diam}(\Omega)$ . The class of Muckenhoupt  $A_p$ - weights will be recalled in Definition 2.2. Our first main result of this paper is the following theorem.

**Theorem 1.2.** *Let  $\Lambda > 0$ ,  $M_0 \geq 1$ , and  $p \geq 2$  be given. There exists a sufficiently small constant  $\delta = \delta(\Lambda, M_0, n, p) > 0$  such that the following statement holds. Suppose also that  $\mu \in A_2$  with  $[\mu]_{A_2} \leq M_0$ ,  $\Omega$  is a  $C^1$ -domain, (1.2)-(1.3) hold on  $\Omega$ , and  $[\mathcal{A}]_{\text{BMO}_{R_0}(\Omega, \mu)} \leq \delta$  for some  $R_0 > 0$ . Then every weak solution  $u \in W^{1,2}(\Omega, \mu, \mathbb{R}^N)$  of (1.1) corresponding to  $|\mathbf{F}|/\mu \in L^p(\Omega, \mu, \mathbb{R}^N)$  satisfies the estimate*

$$\|\nabla u\|_{L^p(\Omega, \mu)} \leq C \left\| \frac{\mathbf{F}}{\mu} \right\|_{L^p(\Omega, \mu)},$$

where  $C$  is a constant depending only on  $n, \Lambda, p, M_0, R_0$  and  $\Omega$ .

Local regularity estimates for weak solutions of (1.1) are not only of great interest by themselves, but also important in many applications since they only require local information on the data. This paper also establishes interior regularity estimates, and local boundary estimates. To state the local results we use the notation  $B_r$  to denote for a ball of radius  $r$  centered at the origin,  $B_r^+$  its upper part and  $T_r$  the flat part of the boundary of  $B_r^+$ . The next result presents the interior local gradient estimate.

**Theorem 1.3.** *Let  $\Lambda > 0$ ,  $M_0 \geq 1$ , and  $p \geq 2$  be given. There exists a sufficiently small constant  $\delta = \delta(\Lambda, M_0, n, p) > 0$  such that the following statement holds. Suppose that  $\mu \in A_2$  with  $[\mu]_{A_2} \leq M_0$ , and (1.2)-(1.3) hold for a given tensor matrix  $\mathcal{A}$  on  $B_2$ . Moreover, assume that  $[\mathcal{A}]_{\text{BMO}_1(B_1, \mu)} \leq \delta$ . Then, for every  $\mathbf{F}$  such that  $\mathbf{F}/\mu \in L^p(B_2, \mu, \mathbb{R}^N)$ , if  $u \in W^{1,2}(B_2, \mu, \mathbb{R}^N)$  is a weak solution to*

$$(1.4) \quad \operatorname{div}[\mathcal{A}\nabla u] = \operatorname{div}(\mathbf{F}) \quad \text{in } B_2,$$

*then  $\nabla u \in L^p(B_1, \mu)$  and*

$$\|\nabla u\|_{L^p(B_1, \mu)} \leq C \left( (\mu(B_1))^{\frac{1}{p}-\frac{1}{2}} \|\nabla u\|_{L^2(B_2, \mu)} + \|\mathbf{F}/\mu\|_{L^p(B_2, \mu)} \right).$$

*where  $C$  is a constant depending only on  $\Lambda, p, n, M_0$ .*

Our last result is a local boundary regularity one.

**Theorem 1.4.** *Let  $\Lambda > 0$ ,  $M_0 \geq 1$ , and  $p \geq 2$  be given. There exists a sufficiently small constant  $\delta = \delta(\Lambda, M_0, n, p) > 0$  such that the following statement holds. Suppose that  $\mu \in A_2$  with  $[\mu]_{A_2} \leq M_0$ , and (1.2)-(1.3) hold for the given tensor matrix  $\mathcal{A}$  on  $B_2^+$ . Moreover, assume that  $[\mathcal{A}]_{\text{BMO}_1(B_1^+, \mu)} \leq \delta$ . Then, for every  $\mathbf{F}$  such that  $\mathbf{F}/\mu \in L^p(B_2^+, \mu)$ , if  $u \in W^{1,2}(B_2^+, \mu, \mathbb{R}^N)$  is a weak solution to*

$$(1.5) \quad \begin{cases} \operatorname{div}[\mathcal{A}\nabla u] = \operatorname{div}(\mathbf{F}) & \text{in } B_2^+, \\ \langle \mathcal{A}\nabla u - \mathbf{F}, \mathbf{n} \rangle = 0 & \text{on } T_2, \end{cases}$$

*then  $\nabla u \in L^p(B_1^+, \mu)$  and*

$$\|\nabla u\|_{L^p(B_1^+, \mu)} \leq C \left( (\mu(B_1))^{\frac{1}{p}-\frac{1}{2}} \|\nabla u\|_{L^2(B_2^+, \mu)} + \|\mathbf{F}/\mu\|_{L^p(B_2^+, \mu)} \right),$$

*where  $C$  is a constant depending only on  $\Lambda, p, n, M_0$ .*

We now comment how we prove the main result. It is well-known that Theorem 1.2 can be obtained from Theorem 1.3 and Theorem 1.4 via standard arguments using partition of unity, and flattening the boundary. We therefore skip the proof of Theorem 1.2. Also, the proof of Theorem 1.3 can be done along the same line of arguments as in [4, Theorem 2.5] after making the necessary adjustment for systems. As such, we will not focus much on it but rather we state and use estimates that we may need in the proof of Theorem 1.4. To prove the latter, we will implement the approximation method of Caffarelli and Peral in [3]. The main idea in the approach is to locally consider the equation (1.5) as the perturbation of an equation for which the regularity of its solution is well understood. Key ingredients include Vitali's covering lemma, and the weak and strong  $(p, p)$  estimates of the weighted Hardy-Littlewood maximal operators. To be able to compare the solutions of the perturbed and un-perturbed equations, we prefer to use compactness argument as has been used in [1], but on weighted spaces. Such argument has been used in [4], similar regularity results as Theorem 1.4 are proved for zero-Dirichlet boundary value problem of equations. Essential properties of  $A_2$  weights such as reverse Hölder's inequality and doubling property are properly utilized in dealing with technical issues arising from the degeneracy and singularity of the coefficients.

We remark that obtaining estimates as in Theorems 1.2-1.4 for large values of  $p$  is not always possible even for smooth but degenerate coefficient matrix  $\mathcal{A}$  and  $\mu \in A_2$ . We refer [4] for a counterexample. In light of this and compared to [6], this paper provides the correct minimal conditions on the coefficients so that the linear map  $\frac{\mathbf{F}}{\mu} \mapsto \nabla u$  remain continuous on the smaller space  $L^p(\Omega, \mu)$ ,  $p \geq 2$ .

As in the Dirichlet boundary value problem studied in [4], to establish the weighted  $L^p$ -regularity estimates, we follow the approximation method of Caffarelli and Peral in [3] where we view (1.1) locally as a perturbation of an elliptic homogeneous equation with constant coefficients. The key to the success of this approach to equations with degenerate coefficients is the novel way of measuring mean oscillation of coefficients which is found to be compatible with the degeneracy of the

coefficients. The smallness of the measure, which is just a smallness of the mean oscillation with weights introduced in [20, 21] to study Hilbert transforms and characterize the dual of weighted Hardy space. This condition is optimal in the sense that it coincided with the well known result in [2] when  $\mu = 1$ . The counterexample given in [4] demonstrates the necessity of the smallness condition  $[\mathcal{A}]_{\text{BMO}}$  in Theorems 1.2-1.4.

The paper is organized as follows. In Section 2, we introduce notations, definitions, and provides some elementary estimates needed for the paper. Section 3 provides the approximation estimates. Level set estimates, and the proofs of the main theorems, Theorem 1.2-1.4 are given in Section 4, which is the last section of the paper.

## 2. NOTATIONS, DEFINITIONS, AND PRELIMINARY ESTIMATES

**2.1. Definitions and existence of weak solutions.** We start with some definitions, and notations.

**Definition 2.1.** Let  $\sigma$  be a non-negative measure on  $\mathbb{R}^n$  and a non-empty open set  $U \subset \mathbb{R}^n$ , we write

$$d\sigma = \sigma dx, \quad \sigma(U) = \int_U \sigma(x) dx.$$

For a locally integrable Lebesgue-measurable function  $f$  on  $\mathbb{R}^n$ , we always denote the average of  $f$  in  $U$  with respect of the measure  $d\sigma$  as

$$\bar{f}_{\sigma, U} = \int_U f(x) d\sigma = \frac{1}{\sigma(U)} \int_U f(x) \sigma dx.$$

For Lebesgue measure  $dx$ , we write

$$\bar{f}_U = \bar{f}_{dx, U}, \quad \text{and} \quad |U| = \int_U dx.$$

**Definition 2.2.** Let  $p \in (1, \infty)$  and  $\mu \in L^1_{\text{loc}}(\mathbb{R}^n)$  be non-negative. The weight function  $\mu$  is said to be of Muckenhoupt class  $A_p$  if

$$[\mu]_{A_p} := \sup_{B \subset \mathbb{R}^n} \left( \int_B \mu(y) dy \right) \left( \int_B \mu(y)^{-\frac{1}{p-1}} dy \right)^{p-1} < \infty,$$

where the supremum is taken over all balls  $B \subset \mathbb{R}^n$ .

It turns out that any  $\mu \in A_p$  defines a measure  $d\mu = \mu(x)dx$ . Moreover, we can define the corresponding Lebesgue and Sobolev spaces with respect to the measure.

**Definition 2.3.** Let  $\mu \in A_p$  with  $1 < p < \infty$ , let  $1 \leq q < \infty$  and  $\Omega \subset \mathbb{R}^n$  be open, bounded. A locally integrable function  $f$  define on  $\Omega$  is said to belong to the weighted Lebesgue space  $L^q(\Omega, \mu)$  if

$$\|f\|_{L^q(\Omega, \mu)} = \left( \int_{\Omega} |f(x)|^q \mu(x) dx \right)^{1/q} < \infty.$$

Let  $k \in \mathbb{N}$ . A locally integrable function  $f$  defined on  $\Omega$  is said to belong to the weighted Sobolev space  $W^{k,q}(\Omega, \mu)$  if all of its distributional derivatives  $D^\alpha f$  are in  $L^q(\Omega, \mu)$  for  $\alpha \in (\mathbb{N} \cup \{0\})^n$  with  $|\alpha| \leq k$ . The space  $W^{k,q}(\Omega, \mu)$  is equipped the the norm

$$\|f\|_{W^{k,q}(\Omega, \mu)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^q(\Omega, \mu)}^q \right)^{1/q}.$$

We also denote  $W_0^{1,q}(\Omega, \mu)$  be the closure of  $C_0^\infty(\Omega)$  in  $W^{1,q}(\Omega, \mu)$ .

Observe that both  $L^q(\Omega, \mu)$ , and  $W^{1,q}(\Omega, \mu)$  are Banach spaces with their natural norm. When  $q = 2$ ,  $W^{1,2}(\Omega, \mu)$  is a Hilbert space.

We now recall what we mean by weak solution of (1.1).

**Definition 2.4.** Assume that (1.2), (1.3) hold and  $|\mathbf{F}|/\mu \in L^2(\Omega, \mu)$ . A function  $u \in W^{1,2}(\Omega, \mu, \mathbb{R}^N)$  is a weak solution of (1.1) if

$$(2.1) \quad \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla \varphi \rangle dx = \int_{\Omega} \langle \mathbf{F}, \nabla \varphi \rangle dx, \quad \forall \varphi \in W^{1,2}(\Omega, \mu, \mathbb{R}^N).$$

Existence of a weak solution can be shown following the standard Hilbert Space method via the application of the theorem of Lax-Milgram. For Dirichlet boundary value problems, this has been done in [7]. Similar argument can be applied for the conormal derivative problem as well.

**Lemma 2.5.** Suppose  $\mathcal{A}$  satisfies (1.2) and (1.3). Then, for each  $\mathbf{F}$  with  $|\mathbf{F}|/\mu \in L^2(\Omega, \mu)$ , there exists a weak solution  $u \in W_0^{1,2}(\Omega, \mu, \mathbb{R}^N)$  to (1.1). Moreover,

$$(2.2) \quad \int_{\Omega} |\nabla u|^2 d\mu \leq C(\Lambda) \int_{\Omega} \left| \frac{\mathbf{F}}{\mu} \right|^2 d\mu.$$

The solution  $u$  is unique up to a constant vector.

**2.2. Weights and weighted norm inequalities.** In this section we review and collect some results needed in the paper. The first lemma is a standard result in measure theory.

**Lemma 2.6.** Assume that  $g \geq 0$  is a measurable function in a bounded subset  $U \subset \mathbb{R}^n$ . Let  $\theta > 0$  and  $K > 1$  be given constants. If  $\mu$  is a weight in  $\mathbb{R}^n$ , then for any  $1 \leq p < \infty$

$$g \in L_{\mu}^p(U) \Leftrightarrow S := \sum_{j \geq 1} K^{qj} \mu(\{x \in U : g(x) > \theta K^j\}) < \infty.$$

Moreover, there exists a constant  $C > 0$  such that

$$C^{-1}S \leq \|g\|_{L_{\mu}^p(U)}^p \leq C(\mu(U) + S),$$

where  $C$  depends only on  $\theta, K$  and  $p$ .

For a given locally integrable function  $f$  we define the weighted maximal function as

$$\mathcal{M}^{\mu} f(x) = \sup_{\rho > 0} \int_{B_{\rho}(x)} |f| d\mu = \sup_{\rho > 0} \frac{1}{\mu(B_{\rho}(x))} \int_{B_{\rho}(x)} |f| \mu(x) dx$$

For functions  $f$  that are defined on a bounded domain, we define

$$\mathcal{M}_{\Omega}^{\mu} f(x) = \mathcal{M}^{\mu}(f \chi_{\Omega})(x)$$

For  $M_0 > 0$  given, assume that  $\mu \in A_2$  such that  $[\mu]_{A_2} \leq M_0$ . The following two continuity results are well known for the maximal function.

- (Strong  $p - p$ ) For any  $1 < p < \infty$ , there exists a constant  $C = C(p, n, M_0)$  such that for any weight  $\mu$  with a strong doubling property we have

$$\|\mathcal{M}^{\mu}\|_{L_{\mu}^p \rightarrow L_{\mu}^p} \leq C.$$

- (Weak  $1 - 1$ ) When  $p = 1$ , there exists a constant  $C$  that depends on  $C(p, n, M_0)$  such that for any weight  $\mu$  with a strong doubling property and  $\lambda > 0$

$$\mu(x \in \mathbb{R}^n : \mathcal{M}^{\mu}(f) > \lambda) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| dx.$$

The proof of these estimates can be found in ([14, Proof of Lemma 7.1.10]). In this paper we will be using mostly  $A_2$  weights. From the definition, it is immediate that  $\mu \in A_2$ , then so is  $\mu^{-1}$  with

$$[\mu]_{A_2} = [\mu^{-1}]_{A_2}.$$

The following lemma is what is called reverse Hölder's inequality that holds for  $A_p$  weights.

**Lemma 2.7** (Theorem 9.2.2, [14], Remark 9.2.3 [14]). *For any  $M_0 > 0$ , there exist positive constants  $C = C(n, M_0)$  and  $\gamma = \gamma(n, M_0)$  such that for all  $\mu \in A_2$  satisfying  $[\mu]_{A_2} \leq M_0$  the reverse Hölder condition holds:*

$$\left( \frac{1}{|B|} \int_B \mu^{(1+\gamma)}(x) dx \right)^{\frac{1}{1+\gamma}} \leq \frac{C}{|B|} \int_B \mu(x) dx, \quad \text{and}$$

for every ball  $B \subset \mathbb{R}^n$ . The inequality holds true as well if  $\mu$  is replaced by  $\mu^{-1}$ .

As a consequence of Lemma 2.7 we have the following two important inequalities which will be used frequently in this paper.

**Lemma 2.8.** [4, Lemma 3.4] *For any  $M_0 > 0$ , let  $\beta = \frac{\gamma}{2+\gamma} > 0$  where  $\gamma$  is a constant as given in Lemma 2.7. Then for any  $\mu \in A_2$  satisfying  $[\mu]_{A_2} \leq M_0$ , for any ball  $B \subset \mathbb{R}^n$ , we have that*

- if  $u \in L^2(B, \mu)$ , then  $u \in L^{1+\beta}(B)$  and

$$\left( \int_B |u|^{1+\beta} dx \right)^{\frac{1}{1+\beta}} \leq C(n, M_0) \left( \int_B |u|^2 d\mu \right)^{1/2},$$

- if  $u \in L^q(B)$  with  $q \geq 1$ , then

$$\left( \int_B |u|^\tau d\mu \right)^{1/\tau} \leq C(n, M_0) \left( \int_B |u|^q dx \right)^{1/q}, \quad \text{with } \tau = \frac{q\gamma}{1+\gamma}.$$

Next, we recall the weighted Sobolev-Poincaré inequality whose prove can be found in [7, Theorem 1.5, Theorem 1.6]

**Lemma 2.9.** *Let  $M_0 > 0$  and assume that  $\mu \in A_2$  and  $[\mu]_{A_2} \leq M_0$ . Then, there exists a constant  $C = C(n, M_0)$  and  $\alpha = \alpha(n, M_0) > 0$  such that for every ball  $B_R \subset \mathbb{R}^n$ , and every  $u \in W^{1,2}(B_R, \mu)$ ,  $1 \leq \kappa \leq \frac{n}{n-1} + \alpha$ , the following estimate holds*

$$(2.3) \quad \left( \frac{1}{\mu(B_R)} \int_{B_R} |u - A|^{2\kappa} \mu(x) dx \right)^{\frac{1}{2\kappa}} \leq CR \left( \frac{1}{\mu(B_R)} \int_{B_R} |\nabla u|^2 \mu(x) dx \right)^{1/2},$$

where either

$$A = \frac{1}{\mu(B_R)} \int_{B_R} u(x) d\mu(x), \quad \text{or} \quad A = \frac{1}{|B_R|} \int_{B_R} u(x) dx.$$

The same result also holds if we replace the ball  $B_R$  with the half ball  $B_R^+$ .

**2.3. Boundary Lipschitz regularity estimates.** Consider the homogeneous system of equations with constant coefficients

$$(2.4) \quad \begin{cases} -\operatorname{div}[\mathcal{A}_0 \nabla v] &= 0 & \text{in } B_4^+, \\ \langle \mathcal{A}_0 \nabla v, \mathbf{n} \rangle &= 0 & \text{on } T_4, \end{cases}$$

with an elliptic constant tensor  $\mathcal{A}_0$  satisfying the inequality  $\Lambda|\xi|^2 \leq \langle \mathcal{A}_0 \xi, \xi \rangle \leq \Lambda^{-1}|\xi|^2$ .

**Definition 2.10.**  $v \in W^{1,q}(B_4^+; \mathbb{R}^N)$  is a weak solution to (2.4) in  $B_4^+$ , for some  $1 < q < \infty$ , if

$$\int_{B_4^+} \langle \mathcal{A}_0 \nabla v, \nabla \varphi \rangle dx = 0, \quad \forall \varphi \in C_0^\infty(B_4; \mathbb{R}^N).$$

We remark the class of test functions used in Definition 2.10 can be enlarged to  $W^{1,q'}(B_4^+, \mathbb{R}^N)$  whose trace vanish on the round part of  $\partial B_4^+$  where  $\frac{1}{q} + \frac{1}{q'} = 1$ . We recall the standard Lipschitz regularity estimate for weak solutions  $v$  of a system with uniformly elliptic constant coefficients. This result can be found in [11], see also [8, Theorem 4.1].

**Lemma 2.11.** *Let  $\mathcal{A}_0$  be a constant tensor satisfying conditions (1.2) and (1.3) with  $\mu = 1$ . Then there exists a constant  $C = C(n, \Lambda)$  such that if  $v \in W^{1,q}(B_4^+, \mathbb{R}^N)$  is a weak solution of (2.4) with  $q > 1$ , then*

$$\|\nabla v\|_{L^\infty(B_2^+)} \leq C \left( \int_{B_4^+} |\nabla v|^q dx \right)^{\frac{1}{q}}.$$

**2.4. Boundary weighted Caccioppoli's type estimates.** We now study the main equation of interest

$$(2.5) \quad \begin{cases} \operatorname{div}[\mathcal{A}(x)\nabla u] &= \operatorname{div}[F] & \text{in } B_4^+, \\ \langle \mathcal{A}(x)\nabla u - F, \mathbf{n} \rangle &= 0 & \text{on } T_4, \end{cases}$$

where

$$(2.6) \quad \Lambda \mu(x) |\xi|^2 \leq \langle \mathcal{A}(x)\xi, \xi \rangle \quad \text{for a.e. } x \in B_4^+, \quad \forall \xi \in \mathbb{R}^n \text{ and } |\mathcal{A}(x)| \leq \Lambda^{-1} \mu(x).$$

**Definition 2.12.**  $u \in W^{1,2}(B_4^+, \mu; \mathbb{R}^N)$  is a weak solution to (2.5) in  $B_4^+$  if

$$\int_{B_4^+} \langle \mathcal{A}\nabla u, \nabla \varphi \rangle dx = \int_{B_4^+} \langle F, \nabla \varphi \rangle dx, \quad \forall \varphi \in C_0^\infty(B_4; \mathbb{R}^N).$$

We study the system of equations (2.5) as a local perturbation of (2.4) corresponding to some constant tensor  $\mathcal{A}_0$  satisfying uniform ellipticity. In fact, if  $v \in W^{1,1+\beta}(B_4^+, \mathbb{R}^N)$  is a weak solution of (2.4) we have the following weighted Caccioppoli estimate of for  $u - v$  that is essential in the paper.

**Lemma 2.13.** *Suppose that  $M_0 > 0$  and  $c$  is a constant vector in  $\mathbb{R}^N$ . Let  $\mathcal{A}_0$  and  $v$  be as in Lemma 2.11, and let  $w = u - c - v$ . Assume that (2.6) holds on  $B_4^+$  and  $[\mu]_{A_2} \leq M_0$ . There exists a constant  $C = C(\Lambda, M_0, n)$  such that for all non-negative function  $\varphi \in C_0^\infty(B_4)$ ,*

$$\begin{aligned} \frac{1}{\mu(B_4)} \int_{B_4^+} |\nabla w|^2 \varphi^2 d\mu &\leq C(\Lambda, M_0, n) \left[ \frac{1 + \|\varphi \nabla v\|_{L^\infty(B_4^+)}^2}{\mu(B_4)} \int_{B_4^+} w^2 |\varphi|^2 d\mu + \frac{1}{\mu(B_4)} \int_{B_4^+} \left| \frac{F}{\mu} \right|^2 \varphi^2 d\mu \right. \\ &\quad \left. + \frac{\|\varphi \nabla v\|_{L^\infty(B_4^+)}^2}{\mu(B_4)} \int_{B_4^+} |\mathcal{A} - \mathcal{A}_0|^2 \mu^{-1} dx \right] \end{aligned}$$

*Proof.* It is clear that  $w$  is a weak solution of the system of equations

$$\begin{cases} \operatorname{div}[\mathcal{A}\nabla w] = \operatorname{div}[F - (\mathcal{A} - \mathcal{A}_0)\nabla v] & \text{in } B_4^+, \\ \langle \mathcal{A}\nabla w - F + (\mathcal{A} - \mathcal{A}_0)\nabla v, \mathbf{n} \rangle = 0 & \text{on } T_4. \end{cases}$$

For  $\varphi \in C_0^\infty(B_4)$ ,  $0 \leq \varphi \leq 1$  such that  $\varphi = 1$  on  $B_2$ . Since  $w\varphi^2 \in W^{1,2}(B_4^+, \mu; \mathbb{R}^N)$  and the trace of  $w\varphi^2$  on the round part of  $B_4^+$  is zero, by the remark stated above we can use  $w\varphi^2$  as a test function to the above equation. Now, noting that  $\nabla(w\varphi^2) = \varphi^2 \nabla w + w \otimes \nabla(\varphi^2)$ , we obtain

$$\begin{aligned} \int_{B_4^+} \langle \mathcal{A}\nabla w, \nabla w \rangle \varphi^2 dx &= - \int_{B_4^+} \langle \mathcal{A}\nabla w, w \otimes \nabla(\varphi^2) \rangle dx + \int_{B_4^+} \langle F, \nabla(w\varphi^2) \rangle dx \\ &\quad - \int_{B_4^+} \langle (\mathcal{A} - \mathcal{A}_0)\nabla v, \nabla(w\varphi^2) \rangle dx. \end{aligned}$$

We then have the following estimate

$$(2.7) \quad \left| \int_{B_4^+} \langle \mathcal{A} \nabla w, \nabla w \rangle \varphi^2 dx \right| \leq \left| \int_{B_4^+} \langle \mathcal{A} \nabla w, w \otimes \nabla(\varphi^2) \rangle dx \right| + \int_{B_4^+} |F| (|\nabla w| |\varphi|^2 + 2|\nabla \varphi| |\varphi| |w|) dx \\ + \int_{B_4^+} |(\mathcal{A} - \mathcal{A}_0)| (|\nabla v| |\nabla w| |\varphi|^2 + 2|\nabla v| |\nabla \varphi| |w| |\varphi|) dx.$$

Clearly, by (2.6),

$$\int_{B_4^+} \langle \mathcal{A} \nabla w, \nabla w \rangle \varphi^2 dx \geq \Lambda \int_{B_4^+} |\nabla w|^2 \varphi^2 d\mu.$$

For  $\epsilon > 0$ , using the boundedness of the coefficients assumed in (2.6) and Young's inequality, the first term on the right hand side can be estimated as

$$\left| \int_{B_4^+} \langle \mathcal{A} \nabla w, w \otimes \nabla(\varphi^2) \rangle dx \right| \leq 2\Lambda^{-1} \int_{B_4^+} \mu |\nabla w| |\varphi| |\nabla \varphi| |w| dx \\ \leq \epsilon \int_{B_4^+} |\nabla w|^2 |\varphi|^2 d\mu + C(\Lambda, \epsilon) \int_{B_4^+} |\nabla \varphi|^2 |w|^2 d\mu.$$

Similarly, we estimate the second term as

$$\int_{B_4^+} |F| (|\nabla w| |\varphi|^2 + 2|\nabla \varphi| |\varphi| |w|) dx \leq \int_{B_4^+} \frac{|F|}{\mu} (|\nabla w| |\varphi|^2 + 2|\nabla \varphi| |\varphi| |w|) \mu dx \\ \leq \epsilon \int_{B_4^+} |\nabla w|^2 \varphi^2 d\mu + C(\epsilon) \left[ \int_{B_4^+} \left| \frac{F}{\mu} \right|^2 \varphi^2 d\mu + \int_{B_4^+} w^2 |\nabla \varphi|^2 d\mu \right].$$

To estimate the last term, we apply Hölder's inequality and Young's inequality to obtain

$$\int_{B_4^+} |\mathcal{A} - \mathcal{A}_0| |\nabla v| [\varphi^2 |\nabla w| + 2|w| |\varphi| |\nabla \varphi|] dx \\ \leq \|\varphi \nabla v\|_{L^\infty(B_4^+)} \int_{B_4^+} |\mathcal{A} - \mathcal{A}_0| [\varphi |\nabla w| + 2|w| |\nabla \varphi|] \mu^{1/2} \frac{1}{\mu^{1/2}} dx \\ \leq \epsilon \int_{B_4^+} |\nabla w|^2 \varphi^2(x) d\mu + C(\epsilon) \|\varphi \nabla v\|_{L^\infty(B_4^+)}^2 \left[ \int_{B_4^+} |\mathcal{A} - \mathcal{A}_0|^2 \mu^{-1} dx + \int_{B_4^+} w^2 |\nabla \varphi|^2 d\mu \right].$$

Therefore, combining the above estimates and choosing  $\epsilon$  sufficiently small to absorb the term  $\int_{B_4^+} |\nabla w|^2 \varphi^2 d\mu$  on the right hand side yields the desired result.  $\square$

### 3. APPROXIMATION ESTIMATES

**3.1. Interior approximation estimates.** The following result is a modification of [4, Proposition 4.4] for systems, which will be needed in the paper.

**Proposition 3.1.** *Let  $\Lambda > 0$ ,  $M_0 > 0$  be fixed and  $\beta$  be as in Lemma 2.8. For every  $\epsilon > 0$  sufficiently small, there exists  $\delta > 0$  depending on only  $\epsilon, \Lambda, n, M_0$  such that the following statement holds true: If (1.2)-(1.3) hold on  $B_4$  for  $\mathcal{A}$ ,  $[\mu]_{A_2} \leq M_0$ , and*

$$\frac{1}{\mu(B_4)} \int_{B_4} |\mathcal{A} - \langle \mathcal{A} \rangle_{B_4}|^2 \mu^{-1} dx + \int_{B_4} \left| \frac{\mathbf{F}}{\mu} \right|^2 d\mu(x) \leq \delta^2,$$

*for every weak solution  $u \in W^{1,2}(B_4, \mu, \mathbb{R}^N)$  of*

$$\operatorname{div}[\mathcal{A}(x) \nabla u] = \operatorname{div}[\mathbf{F}] \quad \text{in } B_4$$



satisfying

$$\int_{B_4} |\nabla u|^2 d\mu \leq 1,$$

then, there exist a tensor of constant coefficients  $\mathcal{A}_0$  and a weak solution  $v \in W^{1,1+\beta}(B_4, \mathbb{R}^N)$  of

$$\operatorname{div}[\mathcal{A}_0 \nabla v] = 0 \quad \text{in } B_4$$

such that

$$|\langle \mathcal{A} \rangle_{B_4} - \mathcal{A}_0| \leq \frac{\epsilon \mu(B_4)}{|B_4|}, \quad \text{and} \quad \int_{B_2} |\nabla u - \nabla v|^2 d\mu \leq \epsilon.$$

Moreover, there is  $C = C(\Lambda, n, M_0)$  such that

$$(3.1) \quad \int_{B_3} |\nabla v|^2 dx \leq C.$$

*Proof.* The proof is the same as that of Proposition 3.2 below. We therefore skip it.  $\square$

**3.2. Boundary approximation estimates.** Our main result of the subsection is the following proposition which is an up to the boundary approximation similar to Proposition 3.1.

**Proposition 3.2.** *Let  $\Lambda > 0$ ,  $M_0 > 0$  be fixed and let  $\beta$  be as in Lemma 2.8. Suppose that  $[\mu]_{A_2} \leq M_0$  and  $\mathcal{A}$  satisfies (2.6). For every  $\epsilon > 0$ , there exists  $\delta = \delta(\epsilon, \Lambda, n, M_0)$  such that: if*

$$\frac{1}{\mu(B_4)} \int_{B_4^+} |\mathcal{A} - \langle \mathcal{A} \rangle_{B_4}|^2 \mu^{-1} dx + \left( \frac{1}{\mu(B_4)} \int_{B_4^+} \left| \frac{F}{\mu} \right|^2 d\mu(x) \right)^{1/2} \leq \delta,$$

and  $u \in W^{1,2}(B_4^+, \mu, \mathbb{R}^N)$  is a weak solution of (2.5) satisfying

$$(3.2) \quad \frac{1}{\mu(B_4)} \int_{B_4^+} |\nabla u|^2 d\mu \leq 1,$$

then there exist a constant matrix  $\mathcal{A}_0$  and a weak solution  $v \in W^{1,1+\beta}(B_4^+, \mathbb{R}^N)$  of (2.4) such that

$$(3.3) \quad \|\langle \mathcal{A} \rangle_{B_4} - \mathcal{A}_0\| \leq \frac{\epsilon \mu(B_4)}{|B_4|},$$

and

$$\int_{B_2^+} |\nabla u - \nabla v|^2 d\mu \leq \epsilon.$$

Moreover, there is  $C = C(\Lambda, n, M_0)$  such that

$$\int_{B_3^+} |\nabla v|^2 dx \leq C(\Lambda, n, M_0).$$

The proof of the lemma relies on the weighted Caccioppoli estimate, Lemma 2.13, and Lemma 3.1 below.

**Lemma 3.1.** *Let  $\Lambda > 0$ ,  $M_0 > 0$  be fixed and let  $\beta$  be as in Lemma 2.8. Suppose that  $[\mu]_{A_2} \leq M_0$  and  $\mathcal{A}$  satisfies (2.6). For every  $\epsilon > 0$  sufficiently small, there exists  $\delta > 0$  depending on only  $\epsilon, \Lambda, n$ , and  $M_0$  such that: if*

$$\frac{1}{\mu(B_4)} \int_{B_4^+} |\mathcal{A} - \langle \mathcal{A} \rangle_{B_4}|^2 \mu^{-1} dx + \left( \frac{1}{\mu(B_4)} \int_{B_4^+} \left| \frac{\mathbf{F}}{\mu} \right|^2 d\mu(x) \right)^{1/2} \leq \delta,$$

and  $u \in W^{1,2}(B_4^+, \mu, \mathbb{R}^N)$  is a weak solution of (2.5) satisfying

$$(3.4) \quad \frac{1}{\mu(B_4)} \int_{B_4^+} |\nabla u|^2 d\mu \leq 1,$$

then there exists a constant matrix  $\mathcal{A}_0$  and a weak solution  $v \in W^{1,1+\beta}(B_4^+, \mathbb{R}^N)$  of (2.4) such that

$$\|\langle \mathcal{A} \rangle_{B_4} - \mathcal{A}_0\| \leq \frac{\epsilon \mu(B_4)}{|B_4|},$$

and

$$\frac{1}{\mu(B_{7/2})} \int_{B_{7/2}^+} |\hat{u}_+ - v|^2 d\mu \leq \epsilon, \quad \hat{u}_+ = u - \bar{u}_{\mu, B_4^+}.$$

Moreover, there is  $C = C(\Lambda, n, M_0)$  such that

$$(3.5) \quad \oint_{B_3^+} |\nabla v|^2 dx \leq C(\Lambda, n, M_0).$$

*Proof.* We first notice that for each  $\lambda > 0$ , if we use the scaling  $\mathcal{A}_\lambda = \frac{1}{\lambda} \mathcal{A}$ ,  $\mu_\lambda = \mu/\lambda$  and  $\mathbf{F}_\lambda = \mathbf{F}/\lambda$ , then for a weak solution  $u$  of (2.5),  $u$  is also a weak solution of the system

$$\begin{cases} \operatorname{div}[\mathcal{A}_\lambda \nabla u] = \operatorname{div}[\mathbf{F}_\lambda] & \text{in } B_4^+, \\ \langle \mathcal{A}_\lambda \nabla u - \mathbf{F}, \mathbf{n} \rangle = 0 & \text{on } T_4. \end{cases}$$

Moreover,  $[\mu_\lambda]_{A_2} = [\mu]_{A_2}$ ,

$$\Lambda \mu_\lambda(x) |\xi|^2 \leq \langle \mathcal{A}_\lambda(x) \xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^{nN}, \quad \text{for a.e. } x \in B_4, \text{ and } |\mathcal{A}_\lambda(x)| \leq \Lambda^{-1} \mu_\lambda(x),$$

and Lemma 3.1 is invariant with respect to this scaling. Therefore, without loss of generality, we may assume that

$$(3.6) \quad \bar{\mu}_{B_4} = \frac{1}{|B_4|} \int_{B_4} \mu(x) dx = 1.$$

In this case, it follows from (2.6) and (3.6) that

$$(3.7) \quad \Lambda |\xi|^2 \leq \langle \langle \mathcal{A} \rangle_{B_4} \xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^{nN}, \quad \text{and } |\langle \mathcal{A} \rangle_{B_4}| \leq \Lambda^{-1}.$$

We will use a contradiction argument to prove the lemma. Suppose that the conclusion is not true. Then there exists  $\epsilon_0 > 0$  such that for each  $k \in \mathbb{N}$ , there are  $\mu_k \in A_2$ ,  $\mathcal{A}_k$  satisfying (2.6) with  $\mu_k$  and  $\mathcal{A}_k$  in place of  $\mu$  and  $\mathcal{A}$ , and  $\mathbf{F}_k$  and a weak solution  $u_k \in W^{1,2}(B_4^+, \mu_k, \mathbb{R}^N)$  to

$$(3.8) \quad \begin{cases} \operatorname{div}[\mathcal{A}_k \nabla u_k] &= \operatorname{div}[\mathbf{F}_k] & \text{in } B_4^+, \\ \langle \mathcal{A}_k \nabla u_k - \mathbf{F}_k, \mathbf{n} \rangle &= 0 & \text{on } T_4, \end{cases}$$

$$(3.9) \quad \begin{cases} \frac{1}{\mu_k(B_4)} \int_{B_4^+} |\mathcal{A}_k - \langle \mathcal{A}_k \rangle_{B_4}|^2 \mu_k^{-1} dx + \left( \frac{1}{\mu_k(B_4)} \int_{B_4^+} \left| \frac{\mathbf{F}_k}{\mu_k} \right|^2 d\mu_k \right)^{1/2} \leq \frac{1}{k^2} \\ [\mu_k]_{A_2} \leq M_0, \quad \bar{\mu}_{k, B_4} = \frac{1}{|B_4|} \int_{B_4} \mu_k(x) dx = 1, \end{cases}$$

and

$$(3.10) \quad \frac{1}{\mu_k(B_4)} \int_{B_4^+} |\nabla u_k|^2 d\mu_k \leq 1,$$

but for all constant matrix  $\mathcal{A}_0$  with  $\|\langle \mathcal{A}_k \rangle_{B_4} - \mathcal{A}_0\| \leq \epsilon_0$ , and for all weak solution  $v \in W^{1,1+\beta}(B_4^+, \mathbb{R}^N)$  of (2.4) it holds that

$$(3.11) \quad \frac{1}{\mu_k(B_{7/2})} \int_{B_{7/2}^+} |\hat{u}_{k+} - v|^2 d\mu_k \geq \epsilon_0.$$

Since  $\langle \mathcal{A}_k \rangle_{B_4}$  satisfies (3.7),  $\langle \mathcal{A}_k \rangle_{B_4}$  is a bounded sequence in  $\mathbb{R}^{nN \times nN}$ . Thus, there exists a subsequence denoting again by  $\langle \mathcal{A}_k \rangle_{B_4}$  and a constant tensor  $\bar{\mathcal{A}} \in \mathbb{R}^{nN \times nN}$  such that

$$(3.12) \quad \lim_{k \rightarrow \infty} \langle \mathcal{A}_k \rangle_{B_4} = \bar{\mathcal{A}}.$$

From (3.10) and weighted Sobolev-Poincaré inequality Lemma 2.9, [7, Theorem 1.5] (which still applicable for half-balls), we see that

$$\frac{1}{\mu_k(B_4)} \int_{B_4^+} |\hat{u}_{k+}|^2 d\mu_k \leq \frac{C(n, M_0)}{\mu_k(B_4)} \int_{B_4^+} |\nabla u_k|^2 d\mu_k \leq C(n, M_0).$$

Since  $\mu_k(B_4) = |B_4|$ , it implies that  $\|\hat{u}_{k+}\|_{W^{1,2}(B_4^+, \mu_k, \mathbb{R}^N)} \leq C(n, M_0)$ , for all  $k \in \mathbb{N}$ . This together with Lemma 2.8 implies that

$$\|\hat{u}_{k+}\|_{W^{1,1+\beta}(B_4^+, \mathbb{R}^N)} \leq C(n, M_0) \|\hat{u}_{k+}\|_{W^{1,2}(B_4^+, \mu_k, \mathbb{R}^N)} \leq C(n, M_0).$$

Therefore, using the compact imbedding  $W^{1,1+\beta}(B_4^+) \hookrightarrow L^{1+\beta}(B_4^+)$  and the diagonal argument, we find a subsequence of  $u_k$  denoted again by  $u_k$ , and  $u \in W^{1,1+\beta}(B_4^+, \mathbb{R}^N)$  such that

$$(3.13) \quad \begin{cases} \hat{u}_{k+} \rightarrow u \text{ strongly in } L^{1+\beta}(B_4^+, \mathbb{R}^N), & \nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^{1+\beta}(B_4^+, \mathbb{R}^N), \\ \hat{u}_{k+} \rightarrow u \text{ a.e. in } B_4^+. \end{cases} \quad \text{and}$$

As a consequence,

$$(3.14) \quad \|u\|_{W^{1,1+\beta}(B_4^+)} \leq C(n, M_0).$$

We claim that  $u$  is a weak solution of

$$(3.15) \quad \begin{cases} \operatorname{div}[\bar{\mathcal{A}} \nabla u] &= 0 & \text{in } B_4^+, \\ \langle \bar{\mathcal{A}} \nabla u, \mathbf{n} \rangle &= 0 & \text{on } T_4. \end{cases}$$

To do this, using  $\varphi \in C_0^\infty(B_4, \mathbb{R}^N)$  as a test function for equation (3.8), we have

$$(3.16) \quad \int_{B_4^+} \langle \mathcal{A}_k \nabla \hat{u}_{k+}, \nabla \varphi \rangle dx = \int_{B_4^+} \langle \mathbf{F}_k, \nabla \varphi \rangle dx.$$

By Hölder's inequality, we estimate the right hand side of (3.16) as

$$\begin{aligned} \left| \int_{B_4^+} \langle \mathbf{F}_k, \nabla \varphi \rangle dx \right| &\leq \left\{ \int_{B_4^+} \left| \frac{\mathbf{F}_k}{\mu_k} \right|^2 \mu_k dx \right\}^{1/2} \left\{ \int_{B_4^+} |\nabla \varphi|^2 \mu_k dx \right\}^{1/2} \\ &\leq \|\nabla \varphi\|_{L^\infty(B_4^+)} \left\{ \frac{1}{\mu_k(B_4^+)} \int_{B_4^+} \left| \frac{\mathbf{F}_k}{\mu_k} \right|^2 d\mu_k(x) \right\}^{1/2} \frac{\mu_k(B_4^+)}{|B_4^+|} \\ &\leq C \frac{\|\nabla \varphi\|_{L^\infty(B_4)}}{k}. \end{aligned}$$

Therefore, letting  $k \rightarrow \infty$  yields

$$(3.17) \quad \lim_{k \rightarrow \infty} \int_{B_4^+} \langle \mathbf{F}_k, \nabla \varphi \rangle dx = 0.$$

On the other hand, it follows from (3.9), (3.10), and Hölder's inequality that

$$\begin{aligned}
\left| \int_{B_4^+} \langle (\mathcal{A}_k - \langle \mathcal{A}_k \rangle_{B_4}) \nabla u_k, \nabla \varphi \rangle dx \right| &\leq \int_{B_4^+} |\mathcal{A}_k - \langle \mathcal{A}_k \rangle_{B_4}| |\nabla u_k| \mu_k^{1/2} |\nabla \varphi| \mu_k^{-1/2} dx \\
&\leq \|\nabla \varphi\|_{L^\infty(B_4^+)} \left\{ \int_{B_4^+} |\mathcal{A}_k - \langle \mathcal{A}_k \rangle_{B_4}|^2 \mu_k^{-1} dx \right\}^{1/2} \left\{ \frac{1}{|B_4^+|} \int_{B_4^+} |\nabla u_k|^2 d\mu_k \right\}^{1/2} \\
&\leq \frac{\|\nabla \varphi\|_{L^\infty(B_4^+)}}{\sqrt{k}} \left\{ \int_{B_4^+} |\nabla u_k|^2 d\mu_k \right\}^{1/2} \\
&\leq C \frac{\|\nabla \varphi\|_{L^\infty(B_4^+)}}{\sqrt{k}} \rightarrow 0, \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Thus,

$$0 = \lim_{k \rightarrow \infty} \int_{B_4^+} \langle (\mathcal{A}_k - \langle \mathcal{A}_k \rangle_{B_4}) \nabla u_k, \nabla \varphi \rangle dx = \lim_{k \rightarrow \infty} \left[ \int_{B_4^+} \langle \mathcal{A}_k \nabla u_k, \nabla \varphi \rangle dx - \int_{B_4^+} \langle \langle \mathcal{A}_k \rangle_{B_4} \nabla u_k, \nabla \varphi \rangle dx \right].$$

We also notice that  $\nabla u_k$  converges weakly to  $\nabla u$  in  $L^{1+\beta}(B_4, \mathbb{R}^N)$  from (3.13), and  $\langle \mathcal{A}_k \rangle_{B_4}$  converges strongly to constant tensor  $\bar{\mathcal{A}}$ , hence,

$$\lim_{k \rightarrow \infty} \int_{B_4^+} \langle \langle \mathcal{A}_k \rangle_{B_4} \nabla u_k, \nabla \varphi \rangle dx = \int_{B_4^+} \langle \bar{\mathcal{A}} \nabla u, \nabla \varphi \rangle dx.$$

Consequently,

$$(3.18) \quad \lim_{k \rightarrow \infty} \int_{B_4^+} \langle \mathcal{A}_k \nabla u_k, \nabla \varphi \rangle dx = \int_{B_4^+} \langle \bar{\mathcal{A}} \nabla u, \nabla \varphi \rangle dx.$$

Combining (3.17) and (3.18) yields

$$\int_{B_4^+} \langle \bar{\mathcal{A}} \nabla u, \nabla \varphi \rangle dx = 0, \quad \forall \varphi \in C_0^\infty(B_4, \mathbb{R}^N).$$

Now, since  $\bar{\mathcal{A}} = \lim_{k \rightarrow \infty} \langle \mathcal{A}_k \rangle_{B_4}$ , and by (3.7), we see that

$$\Lambda |\xi|^2 \leq \langle \bar{\mathcal{A}} \xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^{nN},$$

i.e., the constant tensor  $\bar{\mathcal{A}}$  is uniformly elliptic. Hence,  $u \in C^\infty(\bar{B}_{15/4}^+)$  by Lemma 2.11. Moreover, it follows from Lemma 2.11 and (3.14) that

$$(3.19) \quad \int_{B_{7/2}^+} |\nabla u|^2 d\mu_k \leq \|\nabla u\|_{L^\infty(B_{7/2})}^2 \leq C(n, \Lambda) \left( \int_{B_4^+} |\nabla u|^{1+\beta} dx \right)^{\frac{2}{1+\beta}} \leq C(n, M_0, \Lambda), \quad \forall k \in \mathbb{N}.$$

Using the above and following the argument used in [4, Lemma 4.3] we obtain that

$$(3.20) \quad \lim_{k \rightarrow \infty} \int_{B_{7/2}^+} |\hat{u}_k - u - c_k|^2 d\mu_k = 0, \quad \text{where } c_k = \int_{B_{7/2}^+} [\hat{u}_k - u] dx$$

However, note that since  $\langle \mathcal{A}_k \rangle_{B_4} \rightarrow \bar{\mathcal{A}}$  and  $\epsilon_0 > 0$ ,

$$\|\langle \mathcal{A}_k \rangle_{B_4} - \bar{\mathcal{A}}\| \leq \epsilon_0$$

for sufficiently large  $k$ . But, this contradicts to (3.11) if we take  $\mathcal{A}_0 = \bar{\mathcal{A}}$ ,  $v = u - c_k$  and  $k$  large enough.

We now turn to prove (3.5). Suppose that we have  $\beta$ ,  $\mathcal{A}_0$  and  $v \in W^{1,1+\beta}(B_4^+)$  satisfying the first part of the lemma. Then, by taking  $\epsilon$  sufficiently small, we see that

$$\frac{\Lambda}{2}|\xi|^2 \leq \langle \mathcal{A}_0 \xi, \xi \rangle \leq 2\Lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Hence, by the standard regularity theory for elliptic systems, Lemma 2.11,  $v$  is in  $C^\infty(B_{15/4}^+)$ . Moreover, from Lemma 2.8, we also have

$$\int_{B_{16/5}^+} |v|^2 dx \leq C(n, \Lambda) \left\{ \int_{B_{7/2}^+} |v|^{1+\beta} dx \right\}^{\frac{2}{1+\beta}}.$$

and as a consequence

$$\int_{B_{16/5}^+} |v|^2 dx \leq C(n, \Lambda, M_0) \int_{B_{7/2}^+} |v|^2 d\mu.$$

The preceding estimate together with the energy estimate for  $v$  implies that

$$\int_{B_3^+} |\nabla v|^2 dx \leq C(n, \Lambda, M_0) \int_{B_{16/5}^+} |v|^2 dx \leq C(n, \Lambda, M_0) \int_{B_{7/2}^+} |v|^2 d\mu.$$

Therefore,

$$\begin{aligned} \int_{B_3^+} |\nabla v|^2 dx &\leq C(n, \Lambda, M_0) \left[ \int_{B_{7/2}^+} |\hat{u} - v|^2 d\mu + \int_{B_{7/2}^+} |u - \bar{u}_{\mu, B_4}|^2 d\mu \right] \\ &\leq C(n, \Lambda, M_0) \left[ \epsilon + \frac{\mu(B_4^+)}{\mu(B_{7/2}^+)} \int_{B_4^+} |u - \bar{u}_{\mu, B_4}|^2 d\mu \right] \\ &\leq C(n, \Lambda, M_0) \left[ \epsilon + \frac{\mu(B_4)}{\mu(B_{7/2})} \int_{B_4^+} |\nabla u|^2 d\mu \right], \end{aligned}$$

where we have used the Poincaré's inequality, Lemma 2.9. Thus, by choosing  $\epsilon < 1$  and using (3.4), and the doubling property of  $[\mu]$ , we deduce that

$$\int_{B_3^+} |\nabla v|^2 dx \leq C(n, \Lambda, M_0) [1 + C] = C(n, \Lambda, M_0).$$

Therefore, it completes the proof of Lemma 3.1. □

#### 4. LEVEL SET ESTIMATES, AND PROOFS OF MAIN THEOREMS

**4.1. Level set estimates.** We begin with the following result.

**Lemma 4.1.** *Suppose that  $M_0 > 0$  and  $\mu \in A_2$  such that  $[\mu]_{A_2} \leq M_0$ . Then there exists a constant  $\varpi > 1$  so that for every  $\epsilon > 0$ , there is  $\delta = \delta(\Lambda, M_0, n, \epsilon) > 0$  sufficiently small such that if  $\mathcal{A}$  satisfies (2.6) on  $B_2^+$ ,*

$$[\mathcal{A}]_{\text{BMO}_1(B_1^+, \mu)} \leq \delta,$$

*$u \in W^{1,2}(B_2^+, \mu)$  is a weak solution to*

$$(4.1) \quad \begin{cases} \operatorname{div}[\mathcal{A}(x)\nabla u] &= \operatorname{div}[\mathbf{F}] & \text{in } B_2^+, \\ \langle \mathcal{A}(x)\nabla u - \mathbf{F}, \mathbf{n} \rangle &= 0 & \text{on } T_2, \end{cases}$$

*and if*

$$(4.2) \quad B_\rho^+(\bar{y}) \cap \{x \in B_2^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) \leq 1\} \cap \{x \in B_2^+ : \mathcal{M}^\mu\left(\left|\frac{\mathbf{F}}{\mu}\right|^2 \chi_{B_2^+}\right) \leq \delta^2\} \neq \emptyset,$$

for some  $\bar{y} = (y', 0) \in T_1$  and some  $\rho \in (0, 1/2)$ , then

$$\mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\} \cap B_\rho^+(\bar{y})) < \epsilon \mu(B_1).$$

*Proof.* The proof of this lemma is standard. We give it here for the sake of completeness. For a given  $\epsilon > 0$ , let  $\eta = \epsilon/C^*$  where  $C^*$  is a positive number to be determined depending only on  $M_0, n$ . Then, let  $\delta = \delta(\eta, \Lambda, M_0, n)$  be defined as in Proposition 3.2. We now prove the lemma with this choice of  $\delta$ . Observe that from the hypothesis (4.2), there exists  $x_0 \in B_\rho^+(\bar{y})$  such that for all  $l > 0$ ,

$$(4.3) \quad \int_{B_l(x_0)} \chi_{B_2^+} |\nabla u|^2 d\mu \leq 1, \quad \text{and} \quad \int_{B_l(x_0)} \chi_{B_2^+} \left| \frac{\mathbf{F}}{\mu} \right|^2 d\mu \leq \delta^2.$$

Since  $B_{4\rho}^+(\bar{y}) \subset B_{5\rho}(x_0) \cap B_2^+$ , it follows that

$$\frac{1}{\mu(B_{4\rho}(\bar{y}))} \int_{B_{4\rho}(\bar{y})} \chi_{B_2^+} |\nabla u|^2 d\mu \leq \frac{\mu(B_{5\rho}(x_0))}{\mu(B_{4\rho}(\bar{y}))} \int_{B_{5\rho}(x_0)} \chi_{B_2^+} |\nabla u|^2 d\mu \leq \frac{5^{2n} M_0}{4^{2n}}.$$

Similarly, we have that

$$\frac{1}{\mu(B_{4\rho}(\bar{y}))} \int_{B_{4\rho}(\bar{y})} \chi_{B_2^+} \left| \frac{\mathbf{F}}{\mu} \right|^2 d\mu \leq \delta^2 M_0 \left( \frac{5}{4} \right)^{2n}.$$

From these estimates,  $[\mathbb{A}]_{\text{BMO}_1(B_1^+, \mu)} \leq \delta$ , and after some appropriate scaling, dilation, and translation, we can apply Proposition 3.2 to see that there exists a constant matrix  $\mathcal{A}_0$  and a weak solution  $v$  to

$$\begin{cases} \operatorname{div}[\mathcal{A}_0 \nabla v] &= 0 & \text{in } B_{4\rho}^+(\bar{y}), \\ \langle \mathcal{A}_0 \nabla v, \mathbf{n} \rangle &= 0 & \text{on } T_{4\rho}(\bar{y}), \end{cases}$$

satisfying

$$(4.4) \quad \frac{1}{\mu(B_{2\rho}(\bar{y}))} \int_{B_{2\rho}(\bar{y})} |\nabla u - \nabla v|^2 d\mu < \eta M_0 (5/4)^{2n}, \quad \text{and} \quad \|\nabla v\|_{L^\infty(B_{3\rho}^+(\bar{y}))} \leq C_0,$$

for some positive constant  $C_0$  that depends only on  $n, \Lambda$  and  $M_0$ .

Now let  $\varpi > 0$  such that  $\varpi^2 = \max\{M_0 3^{2n}, 4C_0^2\}$ , where  $C_0$  is from (4.4). We will show that

$$\{x : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\} \cap B_\rho^+(\bar{y}) \subset \{x : \mathcal{M}^\mu(\chi_{B_{2\rho}^+(\bar{y})} |\nabla u - \nabla v|^2) > C_0^2\} \cap B_\rho^+(\bar{y}).$$

To prove the claim, we consider  $x \in B_\rho^+(\bar{y})$  such that

$$(4.5) \quad \mathcal{M}^\mu(\chi_{B_{2\rho}^+(\bar{y})} |\nabla u - \nabla v|^2)(x) \leq C_0^2,$$

and we only need to show that for any  $r > 0$

$$\int_{B_r(x)} \chi_{B_2^+} |\nabla u|^2 d\mu \leq \varpi^2.$$

Indeed, if  $r < \rho$ , then  $B_r(x) \cap B_2^+ \subset B_{2\rho}^+(\bar{y})$ , using (4.4) and (4.5), we see that

$$\begin{aligned} \int_{B_r(x)} \chi_{B_2^+} |\nabla u|^2 d\mu &\leq 2 \int_{B_r(x)} \chi_{B_{2\rho}^+(\bar{y})} |\nabla u - \nabla v|^2 d\mu + 2 \int_{B_r^+(x)} |\nabla v|^2 d\mu \\ &\leq 2 \mathcal{M}^\mu(\chi_{B_{2\rho}^+(\bar{y})} |\nabla u - \nabla v|^2)(x) + 2C_0^2 \leq 4C_0^2 \leq \varpi^2. \end{aligned}$$

Also, if  $r \geq \rho$ , then note that  $B_r(x) \subset B_{3r}(x_0)$  and by (4.3), we obtain that

$$\int_{B_r(x)} \chi_{B_2^+} |\nabla u|^2 d\mu(x) \leq \frac{\mu(B_{3r}(x_0))}{\mu(B_r(x))} \int_{B_{3r}(x_0)} \chi_{B_2^+} |\nabla u|^2 d\mu(x) \leq M_0 3^{2n} \leq \varpi^2,$$

and this proves the claim. From this claim, we can deduce that

$$\begin{aligned}
& \mu(B_\rho^+(\bar{y}) \cap \{x \in B_2^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\}) \\
& \leq \mu(\{x \in B_\rho^+(\bar{y}) : \mathcal{M}^\mu(\chi_{B_{2\rho}^+(\bar{y})} |\nabla u_k - \nabla v_k|^2) > C_0^2\}) \\
& \leq \frac{C(n, M_0)}{C_0^2} \mu(B_{2\rho}(\bar{y})) \frac{1}{\mu(B_{2\rho}(\bar{y}))} \int_{B_{2\rho}^+(\bar{y})} |\nabla u_k - \nabla v_k|^2 d\mu \\
& \leq C^* \eta \mu(B_1),
\end{aligned}$$

where we have used the weak (1, 1) estimates of  $\mathcal{M}^\mu$ , (4.4) and the doubling property of  $\mu$ . The proof is then complete once we choose  $\eta > 0$  such that  $C^* \eta = \epsilon$ .  $\square$

**Lemma 4.2.** *Suppose that  $M_0 > 0$  and  $\mu \in A_2$  such that  $[\mu]_{A_2} \leq M_0$ . Then there exists a constant  $\varpi > 1$  so that for every  $\epsilon > 0$ , there is  $\delta = \delta(\Lambda, M_0, n, \epsilon) > 0$  sufficiently small such that if  $\mathcal{A}$  satisfies (2.6) on  $B_2^+$ ,*

$$[\mathcal{A}]_{\text{BMO}_1(B_1^+, \mu)} \leq \delta,$$

$u \in W^{1,2}(B_2^+, \mu)$  is a weak solution to (4.1) and

$$(4.6) \quad \mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\} \cap B_\rho(y)) \geq \epsilon \mu(B_\rho(y)),$$

whenever  $y \in B_1^+$ , and  $\rho < \frac{1}{8}$ , then

$$(4.7) \quad B_\rho(y) \cap B_2^+ \subset \{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > 1\} \cup \{x \in B_1^+ : \mathcal{M}^\mu\left(\left|\frac{\mathbf{F}}{\mu}\right|^2 \chi_{B_2^+}\right) > \delta^2\}.$$

*Proof.* We begin by noting that in the case that  $y \in B_1^+$ , and  $B_{4\rho}(y) \cap \{x_n = 0\} = \emptyset$ , then we have  $B_{2\rho}(y) \subset B_2^+$ , and the situation is exactly as in [4, Proposition 4.7]. So we skip the proof of this case. In the event  $B_{2\rho}(y) \cap \{x_n = 0\} \neq \emptyset$ , we prove the lemma using a contradiction argument. Suppose that (4.6) holds but (4.7) fails. Then there exists  $x_0 \in B_\rho(y) \cap B_2^+$  such that

$$\mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2)(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}^\mu\left(\left|\frac{\mathbf{F}}{\mu}\right|^2 \chi_{B_2^+}\right)(x_0) \leq \delta^2.$$

Set  $\bar{y} = (y', 0) \in B_{2\rho}(y) \cap T_1$ . Then we have that  $B_{2\rho}(y) \cap B_2^+ \subset B_{4\rho}(\bar{y}) \cap B_2^+$ . As a consequence we have that

$$x_0 \in B_\rho(y) \cap B_2^+ \subset B_{4\rho}(\bar{y}) \cap B_2^+,$$

where we have used  $4\rho \in (0, 1)$ . Now all the hypotheses of Lemma 4.1 are satisfied with  $y$  replaced by  $\bar{y}$  and  $\rho$  replaced by  $4\rho$ . Applying using  $\frac{\epsilon}{M_0(6)^{2n}}$

$$\begin{aligned}
\mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\} \cap B_\rho^+(y)) & \leq \mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2\} \cap B_{4\rho}^+(\bar{y})) \\
& < \frac{\epsilon}{M_0(6)^{2n}} \mu(B_{4\rho}(\bar{y})) \\
& < \frac{\epsilon}{M_0(6)^{2n}} \mu(B_{6\rho}(y)) \leq \frac{\epsilon}{M_0(6)^{2n}} M_0(6)^{2n} \mu(B_\rho(y)) \\
& = \epsilon \mu(B_\rho(y)),
\end{aligned}$$

where we have used the inclusion  $B_{4\rho}(\bar{y}) \subset B_{6\rho}(y)$ . The last inequality obviously contradicts (4.6).  $\square$

**4.2. Proof of main theorems.** As we already discussed, Theorem 1.2 follows from Theorems 1.3-1.4, the partition of unity, the procedure of flattening the boundary, and the energy estimates. This is standard, and therefore we do not provide the proof of Theorem 1.2. Proof of Theorem 1.3 is similar, and much simpler than that of Theorem 1.4. Therefore, this proof is also skipped. It remains to provide the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Let us choose  $M$  large such that for  $u_M = u/M$

$$(4.8) \quad \mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2) > \varpi^2\}) \leq \epsilon \mu(B_1(y)), \quad \forall y \in B_1^+.$$

We can always choose such  $M$  because from the weak  $(1, 1)$  estimate for the maximal function  $\mathcal{M}^\mu$

$$\mu(\{x \in B_2^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2) > \varpi^2\}) \leq \frac{C(n, M_0)}{M^2 \varpi^2} \int_{B_2^+} |\nabla u|^2 d\mu.$$

Now take  $M$  according to the formula

$$\frac{C}{M^2 \varpi^2} \int_{B_2^+} |\nabla u|^2 d\mu = \epsilon \frac{\mu(B_2)}{M_0 \left(\frac{|B_2|}{|B_1|}\right)^2},$$

while keeping in mind that since  $B_1(y) \subset B_2$ , we have that  $B_1(y) \cap B_2 = B_1(y)$ . It follows from the above calculations that

$$(4.9) \quad M^2 \mu(B_1) \leq M^2 \mu(B_2) = M_0 (2)^{2n} \frac{C}{\epsilon \varpi^2} \|\nabla u\|_{L^2(B_2^+, \mu)}^2.$$

Now, let  $\varpi > 0$  sufficiently large defined as in Lemma 4.2. Let  $\epsilon > 0$  sufficiently small such that  $\epsilon_1 \varpi^p \leq 1/2$ , where  $\epsilon_1 = M_0 10^{2n} \epsilon$ . Then, with this epsilon, let  $\delta > 0$  defined as in Lemma 4.2. From Lemma 4.2, and the modified Vitali covering lemma in [15, 26], it follows that

$$\mu(B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^2) \leq \epsilon_1 \left[ \mu(\{B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > 1\}) + \mu(\{B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} \left|\frac{\mathbf{F}}{\mu}\right|^2) > \delta^2\}) \right].$$

See, for examples [4, Lemma 5.10] or [24, Proposition 4.9] for the details on the proof of this claim. Using this claim, and by induction, we then infer that for any  $k \in \mathbb{N}$ ,

$$(4.10) \quad \begin{aligned} \mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > \varpi^{2k}\}) &\leq \sum_{i=1}^k \epsilon_1^i \mu\left(\{x \in B_1^+ : \mathcal{M}^\mu\left(\left|\frac{\mathbf{F}}{\mu}\right|^2 \chi_{B_2^+}\right) > \delta^2 \varpi^{2(k-i)}\}\right) \\ &\quad + \epsilon_1^k \mu(\{x \in B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u|^2) > 1\}). \end{aligned}$$

Now consider the sum

$$S = \sum_{k=1}^{\infty} \varpi^{pk} \mu(\{B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2)(x) > \varpi^{2k}\}).$$

Observe that, when finite,  $S^{2/p}$  is comparable to the  $L^{p/2}$ - norm of  $\mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|)$  in  $B_1^+$ .

Applying (4.10) to the summand we have that

$$\begin{aligned} S &\leq \sum_{k=1}^{\infty} \varpi^{pk} \left[ \sum_{i=1}^k \epsilon_1^i \mu\left(\{x \in B_1^+ : \mathcal{M}^\mu\left(\left|\frac{\mathbf{F}_M}{\mu}\right|^2 \chi_{B_2^+}\right) > \delta^2 \varpi^{2(k-i)}\}\right) \right] \\ &\quad + \sum_{k=1}^{\infty} \varpi^{pk} \epsilon_1^k \mu(\{B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2)(x) > 1\}). \end{aligned}$$



Applying summation by parts we have that

$$\begin{aligned} S &\leq \sum_{j=1}^{\infty} (\varpi^p \epsilon_1)^j \left[ \sum_{k=j}^{\infty} \varpi^{p(k-j)} \mu \left( \left\{ x \in B_1^+ : \mathcal{M}^\mu \left( \left| \frac{\mathbf{F}_M}{\mu} \right|^2 \chi_{B_2^+} \right) > \delta^2 \varpi^{2(k-j)} \right\} \right) \right] \\ &\quad + \sum_{k=1}^{\infty} (\varpi^p \epsilon_1)^k \mu(\{B_1^+ : \mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2)(x) > 1\}) \\ &\leq C \left( \left\| \mathcal{M}^\mu \left( \chi_{B_2^+} \left| \frac{\mathbf{F}_M}{\mu} \right|^2 \right) \right\|_{L^{p/2}(B_1^+, \mu)}^{p/2} + \|\nabla u_M\|_{L^2(B_2^+, \mu)}^2 \right) \sum_{k=1}^{\infty} (\varpi^p \epsilon_1)^k, \end{aligned}$$

where we have applied the weak  $(1, 1)$  estimate of the maximal function. Now chose  $\epsilon$  small so that  $\varpi^p \epsilon_1 < 1$  to obtain that

$$S \leq C \left( \left\| \mathcal{M}^\mu \left( \chi_{B_2^+} \left| \frac{\mathbf{F}_M}{\mu} \right|^2 \right) \right\|_{L^{p/2}(B_1^+, \mu)}^{p/2} + \|\nabla u_M\|_{L^2(B_2^+, \mu)}^2 \right) \leq C \left( \left\| \frac{\mathbf{F}_M}{\mu} \right\|_{L^p(B_2^+, \mu)}^p + \|\nabla u_M\|_{L^2(B_2^+, \mu)}^2 \right),$$

where we have applied the strong  $(p, p)$  estimate for the maximal function operator  $\mathcal{M}^\mu$ .

Now applying Lemma 2.6 we have that

$$\|\nabla u_M\|_{L^p(B_1^+, \mu)}^p \leq C \|\mathcal{M}^\mu(\chi_{B_2^+} |\nabla u_M|^2)\|_{L^{p/2}(B_1^+, \mu)}^{p/2} \leq C(S + \mu(B_1)),$$

and therefore multiplying by  $M^p$  and applying (4.9) we obtain that

$$\|\nabla u\|_{L^p(B_1^+, \mu)}^p \leq C \left( \left\| \frac{\mathbf{F}}{\mu} \right\|_{L^p(B_2^+, \mu)}^p + (\mu(B_1))^{1-\frac{p}{2}} \|\nabla u\|_{L^2(B_2^+, \mu)}^2 \right).$$

This completes the proof of the theorem.  $\square$

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## REFERENCES

- [1] S.-S. Byun, L. Wang, *The conormal derivative problem for elliptic equations with BMO coefficients on Reifenberg flat domains*, Proceedings of London Mathematical Society (3), Volume 90, Issue 1, Pages 245–272, 2005.
- [2] S.-S. Byun, L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. 57 (2004), no. 10, 1283–1310.
- [3] L.A. Caffarelli and I. Peral, *On  $W^{1,p}$  estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. 51 (1998), no. 1, 1–21.
- [4] D. Cao, T. Mengesha and T. Phan, *Weighted  $W^{1,p}$  estimates for weak solutions of degenerate and singular equations*, to appear in the Indiana University Mathematics Journal.
- [5] L. C. Evans, *Partial Differential Equations*, AMS Graduate Studies in Mathematics, Vo 19, 2010.
- [6] G. Di Fazio, M. S. Fanciullo, M. S. and P. Zamboni,  *$L^p$  estimates for degenerate elliptic systems with VMO coefficients*, Algebra i Analiz 25 (2013), no. 6, 24–36; translation in St. Petersburg Math. J. 25 (2014), no. 6, 909–917.
- [7] E. B. Fabes, C. E. Kenig, R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations 7 (1982), no. 1, 77–116.
- [8] Q. Han, F. Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes in Mathematics, American Mathematical Society, Providence, RI.
- [9] J. Garcia-Cuerva, *Weighted  $H^p$ -spaces*, Thesis (Ph.D.) Washington University in St. Louis. 1975.
- [10] J. Garcia-Cuerva, *Weighted  $H^p$  spaces*, Dissertationes Math. (Rozprawy Mat.) 162 (1979).
- [11] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Edizioni Della Normale, 2012.

- [12] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
- [13] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Company, Tuck Link (2003).
- [14] L. Grafakos, *Modern Fourier Analysis*, Graduate texts in Math., Springer, 2009.
- [15] N. V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*. Graduate Studies in Mathematics, 96. American Mathematical Society, Providence, RI, 2008.
- [16] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Uralceva, *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society.
- [17] G.M. Lieberman, *Second Order Parabolic Differential Equations*. World Scientific Publishing Co, 2005.
- [18] A. Maugeri, D. K. Palagachev, L. G. Softova, *Elliptic and parabolic equations with discontinuous coefficients*. Mathematical Research, 109. Wiley-VCH Verlag Berlin GmbH, Berlin, 2000.
- [19] N.G. Meyers. *An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) 17 (1963) 189-206.
- [20] B. Muckenhoupt and R. L. Wheeden. *Weighted bounded mean oscillation and the Hilbert transform*. Studia Mathematica, T. LIV. 1976.
- [21] B. Muckenhoupt and R. L. Wheeden. *On the dual of weighted  $H^1$  of the half-space*. Studia Mathematica, T. LXIII. 1978.
- [22] M. K. V. Murthy, G. Stampacchia, *Boundary value problems for some degenerate-elliptic operators*. Ann. Mat. Pura Appl. (4) 80, 1968, 1-122.
- [23] I. Peral, F. Soria, *A note on  $W^{1,p}$  estimates for quasilinear parabolic equations*. Electron. J. Differ. Equ. Conf., 8 (2002), 121–131.
- [24] T. Phan, *Weighted Calderon-Zygmund estimates for weak solutions of quasi-linear degenerate elliptic equations*, submitted, arXiv:1702.08622.
- [25] E. Stredulinsky, *Weighted inequalities and applications to degenerate elliptic Partial Differential Eequations* Ph. D. Thesis, Indiana University, 1981.
- [26] L. Wang, *A geometric approach to the Calderón-Zygmund estimates*, Acta. Math. Sin. (Engl. Ser.), 19 (2003), 381-396.

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