Stochastic heat equation with distributional drifts

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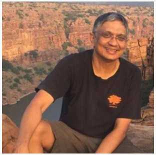
arXiv:2011.13498 - Joint work with

Siva Athreya



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The equation

$$(\partial_t - \partial^2_{xx})u(t,x) = b(u(t,x)) + \xi(t,x),\ (t,x) \in [0,1] \times D,\ D \subset \mathbb{R}$$

- b is a distribution in $\mathcal{B}^{\beta}_{q,\infty}(\mathbb{R})$
- u_0 is bounded measurable
- ξ is the space-time white noise

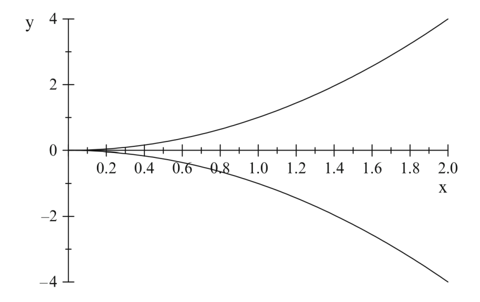
Examples of b:

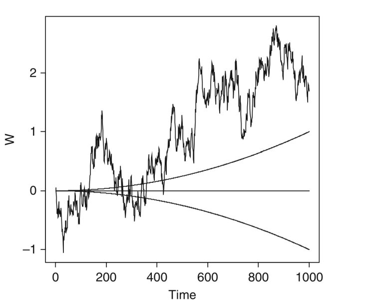
- $b = \delta$ (skew stochastic heat equation) or any measure
- $b(u) = \mathrm{pv}(1/u)$
- $b(u) = \mathrm{pv}(1/u^3)$

Regularization by noise

Figure: $dX = 2\text{sign}(x)\sqrt{X}dt$, $X_0 = 0$

Figure: $dX = 2\text{sign}(x)\sqrt{X}dt + dB_t$, $X_0 = 0$





[1]

[1]Pictures from Flandoli's Saint-Flour Lecture Notes in Mathematics 2015

Regularization by noise - Literatures

- ODE: $dX = b(X)dt$, b is bounded measurable, may have many / no solutions
  - When b is continuous bounded, there is at least one solution (Peano's theorem)
  - When b is Lipschitz, there exists a unique solution
- SDE: $dX = b(X)dt + dB$, B is a Brownian motion
  - When b is bounded measurable: there exists a unique solution (Veretenikov'80)
  - When $b = b(t,x) \in L^q([0,T]; L^p(\mathbb{R}^d))$, $\frac{d}{p} + \frac{2}{q} < 1$: there exists a unique solution (Krylov-Röckner'05)
- SPDE: $(\partial_t - \partial^2_{xx})u_t(x) = b(u_t(x)) + \xi_t(x)$
  - When b is in $L^q(\mathbb{R})$, $q > 1$: Gyöngy and Pardoux'93
  - When b is bounded: Butkovsky-Mytnik'19

Regularization by fBM

$dX = b(X)dt + dB^H$, B^H is a fractional Brownian motion, $H \in (0,1)$ is the Hurst parameter.

- Catellier-Gubinelli'16: When

$$b \in \mathcal{B}^{\beta}_{\infty,\infty} \quad \text{and} \quad \beta > 1 - \frac{1}{2H}$$

there exists a unique solution.

  - $H = 1/2$: $1 - \frac{1}{2H} = 0$, $b \in \mathcal{B}^{\beta}_{\infty,\infty}$, $\beta > 0$
  - $H = 1/4$: $1 - \frac{1}{2H} = -1$, $b \in \mathcal{B}^{\beta}_{\infty,\infty}$, $\beta > -1$
  - *the smaller the H, the more irregular b can be*

Solution concept: $(\partial_t - \partial_{xx}^2)u_t(x) = b(u_t(x)) + \xi_t(x)$, $(t,x) \in [0,1] \times D$

- $D = [0,1]$ under Neumann boundary condition or periodic condition
- $D = \mathbb{R}$

When b is a function, u is a solution if

$$u(t,x) = P_t u_0(x) + \int_0^t \int_D p_{t-s}(x,y) b(u(s,y)) dy du + V(t,x)$$

$$V(t,x) = \int_0^t p_{t-s}(x,y) \xi(dt,dy)$$

When b is a distribution, u is a solution if for any *suitable* smooth $(b_n) \to b$,

$$u(t,x) = P_t u_0(x) + \lim_n \int_0^t \int_D p_{t-s}(x,y) b_n(u(s,y)) dy du + V(t,x)$$

Results (Athreya-Butkovski-L.-Mytnik'20)

Well-posedness: $b \in \mathcal{B}^{\beta}_{p,\infty}(\mathbb{R})$

- when $\beta - \frac{1}{p} \geq -1$, $p \neq \infty$: there is unique solution, continuous w.r.t input (u_0, b)
- when $\beta - \frac{1}{p} > -3/2$: there exists a (probabilistic) weak solution

Some examples:

- δ or any measures belongs to $\mathcal{B}^0_{1,\infty}$
- $b(u) = |u|^{-\sigma}$, $\sigma \in (0,1)$, belongs to $\mathcal{B}^{1-\sigma}_{1,\infty}$
- $\mathrm{pv}(1/u)$, $\phi \mapsto \mathrm{pv}(\phi) = \int_0^\infty \frac{\phi(x)-\phi(-x)}{x} dx$ belongs to $\mathcal{B}^0_{1,\infty}$
- $L^p(\mathbb{R}) \hookrightarrow \mathcal{B}^0_{p,\infty}$, $p \geq 1$

Applications (Athreya-Butkovski-L.-Mytnik'20)

Scaling limit - integrable drifts

Suppose that f is *integrable* and u_λ solves

$$(\partial_t - \partial^2_{xx})u_\lambda(t,x) = \lambda^{-1} f(u_\lambda(t,x)) + \xi(t,x).$$

Then as $\lambda \to \infty$ the probability distribution of

$$\{\lambda^{-1/2} u_\lambda(\lambda t, \lambda^2 x) : (t,x) \in [0,1] \times D\}$$

converges to that of the skew stochastic heat equation

$$(\partial_t - \partial^2_{xx})u = \kappa_f \delta_0(u) + \xi, \quad \kappa_f = \int_{\mathbb{R}} f(x)dx$$

Applications (Athreya-Butkovski-L.-Mytnik'20)

Scaling limit - non-integrable drifts

Suppose that f is *locally integrable*, $xf(x)$ is uniformly bounded and

$$\lim_{a\to\infty}\int_{|x|<a} f(x)dx = 0 \quad \text{and} \quad \lim_{x\to\pm\infty} xf(x) = c_f$$

u_λ is as previously. Then as $\lambda \to \infty$ the probability distribution of

$$\{\lambda^{-1/2}u_\lambda(\lambda t, \lambda^2 x) : (t,x) \in [0,1]\times D\}$$

converges to that of the equation

$$(\partial_t - \partial^2_{xx})u = c_f \,\mathrm{pv}(\frac{1}{u}) + \xi, \quad \mathrm{pv}(\phi) = \int_0^\infty \frac{\phi(x)-\phi(-x)}{x}dx.$$

Heuristics of scaling limits

Scaling step: the random field

$$\{\lambda^{-1/2}u_\lambda(\lambda t,\lambda^2 x):(t,x)\in[0,1]\times D\}$$

has to same probability law as the solution to

$$(\partial_t-\partial^2_{xx})v=f_\lambda(v)+\xi,\quad f_\lambda(v)=\lambda f(\lambda v)$$

Limiting step:

- For integrable f: $\lim_{\lambda\to\infty}f_\lambda=\kappa_f\delta$ as Schwartz distributions
- For f locally integrable, $xf(x)$ is uniformly bounded and

$$\lim_{a\to\infty}\int_{|x|<a}f(x)dx=0\quad\text{and}\quad\lim_{x\to\pm\infty}xf(x)=c_f$$

we have $\lim_{\lambda\to\infty}f_\lambda=c_f\,\mathrm{pv}$ as Schwartz distributions

Common methods for regularization by noise

1. Zvonkin's transformation / Itô-Tanaka trick
   - Rely on Itô calculus: rewrite the drift in term of solutions of (Kolmogorov) PDE
   - Works for Brownian noise, some cases with space-time white noise
2. Non-linear Young formulation: (Davie'07, Catellier-Gubinelli'16)
   - Uses martingale inequalies, sewing techniques
   - Works for additive noise, both SDEs and SPDEs

Davie's estimates

B is Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded measurable
Davie'07: for a.e. sample path $B(\omega)$ the equation

$$dX(\omega) = b(X(\omega))dt + dB(\omega)$$

has a unique solution.
Method:

- Transform equation: $\psi = X - B$ then

$$\psi_t = x + \int_0^t b(B + \psi)dr$$

- Lipschitz estimate:

$$\| \int_s^t [b(B_r + x_1) - b(B_r + x_2)]dr \|_{L^m(\Omega)} \le C|t - s|^{1/2}|x_1 - x_2|$$

Method for SPDE

- Transform equation: $\psi = u - V$

$$\psi(t,x) = \lim_n \int_0^t \int_D p_{t-r}(x,y) b_n(V(r,y) + \psi(r,y)) dy dr$$

- Lipschitz estimate: when $b \in \mathcal{B}^\beta_{p,\infty}$, $\beta - \frac{1}{p} > -1$

$$\left\| \int_s^t \int_D p_{T-r}(x,y) \big[b(V(r,y) + \kappa_1(r,y)) - b(V(r,y) + \kappa_2(r,y)) \big] dr dy \right\|_{L_m} \le C \|b\|_{\mathcal{B}^\beta_p} \|\kappa_1 - \kappa_2\|_{C L_m} (t-s)^{1 + \frac{1}{4}(\beta - 1 - \frac{1}{p})}.$$

- Log-Lipschitz estimate when $b \in \mathcal{B}^\beta_{p,\infty}$, $\beta - \frac{1}{p} = -1$, $p \neq \infty$.
- Method to obtain above estimates: stochastic sewing lemma (L.'20)

Sewing method

General idea: Summing up local expansions to build/recover objects

Fundamental theorem of calculus:

$$\int_0^1 f'(r)dr = f(1) - f(0)$$

That is

$$\sum_{[s,t]\in\pi} f'(s)(t-s) \xrightarrow{|\pi|\downarrow 0} f(1) - f(0)$$

Link:

$$f'(s)(t-s) \approx f(t) - f(s)$$

so that

$$\sum_{[s,t]\in\pi} f'(s)(t-s) \approx \sum_{[s,t]\in\pi} f(t) - f(s) = f(1) - f(0)$$

Abstract Riemann sums

$$\sum_{[s,t]\in\pi} A_{s,t}$$

- $A_{s,t}$: local expansion around s
- When does the Riemann sums converge?

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Lyons'98 - Gubinelli'04 - Feyel&LaPradelle'06:

- $\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$, $s < u < t$
- *Sewing lemma:* If $|\delta A_{s,u,t}| \le C|t-s|^{1+\varepsilon}$ then:
  - Riemann sums converges to $\mathcal{A}$
  - the map $A \mapsto \mathcal{A}$ is continuous

Example: Young integrals

Problem

- $X \in C^\alpha$, $\alpha > 1/2$
- $f \in C^\infty$,
- Building object: integral $\int f(X)dX$

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Setup

- $A_{s,t} = f(X_s)(X_t - X_s)$
- Abstract Riemann sums: $\sum_{[s,t]\in\pi} f(X_s)(X_t - X_s)$

Sewing

- $\delta A_{s,u,t} = (f(X_u) - f(X_s))(X_t - X_u)$
- $|\delta A_{s,u,t}| \le |f|_1 |t-s|^{2\alpha}$
- since $2\alpha > 1$, Riemann sums converge

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- $X \in C^{\alpha}$, $\alpha \in (1/3, 1/2]$
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- $A_{s,t} = f(X_s)(X_t - X_s)$
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Sewing

- $\delta A_{s,u,t} = (f(X_u) - f(X_s))(X_t - X_u)$
- $|\delta A_{s,u,t}| \le |f|_1 |t-s|^{2\alpha}$
- since $2\alpha < 1$, sewing lemma can't be applied

Example: Rough integrals (cont.)

Problem

- $X \in C^\alpha$, $\alpha \in (1/3, 1/2]$
- $f \in C^\infty$,
- Building object: integral $\int f(X)dX$
- Assume: $\mathbb{X}_{s,t} =$ "$\int_s^t (X_r - X_s)dX_s$" is given

Example: Rough integrals (cont.)

Problem

- $X \in C^\alpha$, $\alpha \in (1/3, 1/2]$
- $f \in C^\infty$,
- Building object: integral $\int f(X)dX$
- Assume: $\mathbb{X}_{s,t} = \text{"} \int_s^t (X_r - X_s)dX_s \text{"}$ is given

Setup

- $A_{s,t} = f(X_s)(X_t - X_s) + Df(X_s)\mathbb{X}_{s,t}$,
- Abstract Riemann sums: $\sum_{[s,t]\in\pi} f(X_s)(X_t - X_s) + Df(X_s)\mathbb{X}_{s,t}$

Sewing

- $\delta A_{s,u,t} = -[f(X_u) - f(X_s) - Df(X_s)(X_u - X_s)](X_t - X_u) - (Df(X_u) - Df(X_s))\mathbb{X}_{u,t}$
- $|\delta A_{s,u,t}| \le |f|_2|t - s|^{3\alpha}$
- since $3\alpha > 1$, Riemann sums converges

Outreach

Further applications: Lyons'98, Hairer'14, Friz-Hairer'20

Optimality

Sewing: $|\delta A_{s,u,t}| \leq C|t-s|^{1+\varepsilon}$ implies convergence of Riemann sums

Example:

- B is a Brownian motion
- Building object: integral $\int BdB$

Fact: from Itô calculus, we know

$$\sum_{[s,t]\in\pi} B_s(B_t - B_s) \xrightarrow{|\pi|\downarrow 0} \int BdB \quad \text{in } L^2(\Omega).$$

However

$$|\delta A_{s,u,t}| = |(B_s - B_u)(B_t - B_u)| \approx |t-s|$$

Stochastic Sewing Lemma - L.'20

$\{A_{s,t}\}_{(s,t)\in[0,T]^2}$ in $L_m := L^m(\Omega), m \geq 2$. Suppose

- $A_{s,t}$ is $\mathcal{F}_t$-measurable
- $\|\delta A_{sut}\|_{L_m} \leq \Gamma_2|t-s|^{\frac{1}{2}+\varepsilon_2}$
- $\|\mathbb{E}^{\mathcal{F}_s}\delta A_{sut}\|_{L_m} \leq \Gamma_1|t-s|^{1+\varepsilon_1}$

Then abstract Riemann sums converge in L_m. [...]

Example: Itô integral

Problem

- B is Brownian motion
- $f \in C^\alpha$
- Building object: Itô integral $\int f(B)dB$

Example: Itô integral

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- B is Brownian motion
- $f \in C^\alpha$
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Setup

- $A_{s,t} = f(B_s)(B_t - B_s)$
- Abstract Riemann sums: $\sum_{[s,t]\in\pi} f(B_s)(B_t - B_s)$

SSL

- $\delta A_{sut} = -(f(B_u) - f(B_s))(B_t - B_u)$
- $\mathbb{E}^{\mathcal{F}_s}\delta A_{sut} = -\mathbb{E}^{\mathcal{F}_s}[f(B_u) - f(B_s)]\mathbb{E}^{\mathcal{F}_u}(B_t - B_u) = 0$
- $\|\mathbb{E}\delta A_{sut}\|_{L_m} \lesssim \|f\|_{C^\alpha}|t-s|^{\frac{1}{2}+\frac{\alpha}{2}}$

SSL yields Itô integral $\int f(B_s)dB_s$.

Davie estimate

B is Brownian motion, b is bounded measurable

$$\| \int_s^t [b(B_r + x_1) - b(B_r + x_2)] dr \|_{L^m(\Omega)} \le C|t-s|^{1/2}|x_1 - x_2|$$

- Define $f(u) = |x_1 - x_2|^{-1}[b(u + x_1) - b(u + x_2)]$
- $\|f\|_{\mathcal{B}^{-1}_{\infty,\infty}} \le \sup_x |b(x)|$
- Davie estimate is an estimate for $\| \int_s^t f(B_r) dr \|_{L^m(\Omega)}$

Example: distributive functionals of fBM

Suppose $f \in C^\nu = \mathcal{B}^\nu_{\infty,\infty}$, $\nu < 0$. B^H is a standard fBM.

Aim: define $\int_0^t f(B^H_r)dr$ in a robust way

- Whenever $\{f_n\} \subset C_b$ converges to f in C^ν,

$$\int_0^t f(B^H_r)dr = \lim_n \int_0^t f_n(B^H_r)dr$$

- Continuity of

$$(t, f, \psi) \mapsto \int_0^t f(B^H_r + \psi_r)dr$$

Here ψ is an adapted process which is more regular than B^H.

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Standard approaches:

1. Moment method
2. Zvonkin's transformation (Itô formula)

Approach using SSL: Naive guess $A_{s,t} = f(B^H_s)(t-s)$, «problematic».

Example: distributive functionals of fBM

Instead, consider $A_{s,t}[f] =$ "$\mathbb{E}^{\mathcal{F}_s} \int_s^t f(B_r^H) dr$". Note that

$$\mathbb{E}^{\mathcal{F}_s} \int_s^t f(B_r^H) dr = \mathbb{E}^{\mathcal{F}_s} \int_s^t f\left((B_r^H - \mathbb{E}^{\mathcal{F}_s} B_r^H) + \mathbb{E}^{\mathcal{F}_s} B_r^H \right) dr$$
$$= \int_s^t P_{\sigma_H^2(s,r)} f\left(\mathbb{E}^{\mathcal{F}_s} B_r^H \right) dr =: A_{s,t}[f]$$

where P_σ is the heat semigroup and

$$\sigma_H^2(s,r) = \mathbb{E}\left(B_r^H - \mathbb{E}^{\mathcal{F}_s} B_r^H \right)^2 .$$

Observations: (a) obviously, $\mathbb{E}^{\mathcal{F}_s} \delta A_{sut}[f] = 0$
(b) by *Schauder estimate* $\|P_\sigma f\|_\infty \lesssim \sigma^{\frac{\nu}{2}} \|f\|_{C^\nu}$
(c) *Local nondeterminism property* of fBM implies $\sigma_H^2(s,r) \gtrsim |r-s|^{2H}$.

Example: distributive functionals of fBM

So we have

$$\|A_{s,t}[f]\|_{L_m} \lesssim \|f\|_{C^\nu} \int_s^t \sigma_H^\nu(s,r)dr \lesssim \|f\|_{C^\nu} \int_s^t |r-s|^{\nu H} dr$$
$$\lesssim \|f\|_{C^\nu} |t-s|^{1+\nu H} .$$

If

$$1 + \nu H > \frac{1}{2} \quad \Leftrightarrow \quad \nu > -\frac{1}{2H}$$

then, by SSL, we can define

$$\text{“}\int_0^t f(B_r^H)dr\text{”} := \mathcal{A}_t[f] = \lim_{|\Delta t|\to 0} \sum_i \int_{t_i}^{t_{i+1}} P_{\sigma_H(s,r)} f(\mathbb{E}^{\mathcal{F}_s} B_r^H)dr$$

Furthermore, we have

$$\|\text{“}\int_s^t f(B_r^H)dr\text{”}\|_{L_m} \lesssim \kappa_m \|f\|_{C^\nu} |t-s|^{1+\nu H}$$

Example: distributive functionals of fBM, f is continuous

If f is continuous and bounded, then

$$\| \int_s^t f(B_r^H) dr \|_{L_m} \lesssim \|f\|_\infty |t-s|$$

By uniqueness part of SSL,

$$\mathcal{A}_t[f] = \int_0^t f(B_r^H) dr \quad \forall f \in C_b(\mathbb{R}^d)$$

- If f_n is a sequence of continuous functions converging to f in C^ν, then (SSL)- $\int_0^\cdot f(B_r^H) dr = \lim_n \int_0^\cdot f_n(B_r^H) dr$ in $C([0,T]; L_m)$

Thank you

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