Small solutions of nonlinear Schrödinger equations near first excited states

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Abstract

Consider a nonlinear Schrödinger equation in $\mathbb{R}^3$ whose linear part has three or more eigenvalues satisfying some resonance conditions. Solutions which are initially small in $H^1 \cap L^1(\mathbb{R}^3)$ and inside a neighborhood of the first excited state family are shown to converge to either a first excited state or a ground state at time infinity. An essential part of our analysis is on the linear and nonlinear estimates near nonlinear excited states, around which the linearized operators have eigenvalues with nonzero real parts and their corresponding eigenfunctions are not uniformly localized in space.

Keywords. nonlinear Schrödinger equation, first excited state.

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Contents

1 Introduction

Consider the nonlinear Schrödinger equation in $\mathbb{R}^3$,

$$i\partial_t \psi = H_0 \psi + \kappa |\psi|^2 \psi, \qquad \psi|_{t=0} = \psi_0, \tag{1.1}$$

where $H_0 = -\Delta + V$ is the linear Hamiltonian with a localized real potential V, $\kappa = \pm 1$, and $\psi(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the wave function. We often drop the x dependence and write $\psi(t)$. We assume $\psi_0 \in H^1$ is localized, say $\psi_0 \in L^1$, so that its dispersive component decays rapidly under the evolution. For any solution $\psi(t) \in H^1(\mathbb{R}^3)$ its L^2-norm and energy

$$\mathcal{E}[\psi] = \int \frac{1}{2}|\nabla \psi|^2 + \frac{1}{2} V |\psi|^2 + \frac{1}{4}\kappa |\psi|^4 \, dx \tag{1.2}$$

are constant in t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proven using these conserved quantities no matter what the sign of κ is.

We assume that H_0 has $K+1$ simple eigenvalues $e_0 < e_1 < \cdots < e_K (< 0)$ with normalized real eigenfunctions ϕ_k, $k = 0, 1, \ldots, K$, where $K \geq 2$. They are assumed to satisfy

$$e_0 < 2\, e_1 < 4\, e_2, \tag{1.3}$$

and some generic conditions to be specified later. Through bifurcation around zero along these eigenfunctions, one obtains $K+1$ families of *nonlinear bound states* $Q_{k,n} = n\phi_k + h$, $h = O(n^3)$ and[1] $(h, \phi_k) = 0$ for $k = 0, \ldots, K$, and $n > 0$ sufficiently small, which solve the equation

$$(-\Delta + V) Q + \kappa |Q|^2 Q = EQ, \tag{1.4}$$

for some $E = E_{k,n} = e_k + O(n^2)$, see Lemma 2.1. They are real and decay exponentially at spatial infinity. Each of them gives an exact solution $\psi(t,x) = Q(x)e^{-iEt}$ of (1.1). The family $Q_{0,n}$ are called the *nonlinear ground states* while $Q_{k,n}$, $k > 0$, are called the k-th *nonlinear excited states.*

Our goal is to understand the long-time dynamics of the solutions at the presence of nonlinear bound states. The first question is the stability problem of nonlinear ground states. It is well-known that nonlinear ground states are *orbitally stable* in the sense that the difference

$$\inf_{n,\theta} \left\| \psi(t) - Q_{0,n}\, e^{i\theta} \right\|_{H^1(\mathbb{R}^3)} \tag{1.5}$$

remains uniformly small for all time t if it is initially small. On the other hand, the difference is expected to approach zero locally since the majority of which is a dispersive wave that scatters to infinity. Hence one expects that it is *asymptotically stable* in the sense that

$$\left\| \psi(t) - Q_{0,n(t)}\, e^{i\theta(t)} \right\|_{L^2_{\text{loc}}} \to 0 \tag{1.6}$$

[1] The L^2 inner product $(\ ,\ )$ is $(f,g) = \int_{\mathbb{R}^3} \bar{f}\, g \, dx$. For a function $\phi \in L^2$, we denote by $\phi^\perp$ the L^2-subspace $\{g \in L^2 : (\phi, g) = 0\}$.

as $t \to \infty$, for a suitable choice of $n(t)$ and $\theta(t)$. Here $\|\cdot\|_{L^p_{\text{loc}}}$ denotes a local L^p norm,

$$\|\phi\|_{L^p_{\text{loc}}} = \|\phi\|_{L^p_{-r_0}} \tag{1.7}$$

for some fixed $r_0 > 10$, and for $r \in \mathbb{R}$

$$\|\phi\|_{L^p_r} = \|\langle x\rangle^r \phi(x)\|_{L^p(\mathbb{R}^3)}, \quad \langle x\rangle = (1+|x|^2)^{1/2}. \tag{1.8}$$

One is also interested in how fast (1.6) converges and whether $n(t)$ has a limit.

The second question is the asymptotic problem of the solution when $\psi(0)$ is small but not close to ground states. It is delicate since nonlinear excited states stay there forever but are expected to be unstable from physical intuition. Thus, a solution may stay near an excited state for an *extremely long time* but then moves on and approaches another excited state.

We now review the literature, assuming ψ_0 is small in $H^1 \cap L^1$.

If $-\Delta+V$ has only one bound state, i.e., with no excited states, the asymptotic stability of ground states is proved in [26, 27], with convergence rate $t^{-3/2}$. It is then shown in [22] that all solutions with small initial data, not necessarily near ground states, will locally converge to a ground state.

Suppose $-\Delta + V$ has two bound states. the asymptotic stability of ground states is proved in [31], with a slower convergence rate $t^{-1/2}$ due to the persistence of the excited state. The problem becomes more delicate when the initial data are away from ground states. It is proved in [33] that, near excited states, there is a finite co-dimensional manifold of initial data so that the corresponding solutions locally converge to excited states. Outside of a small wedge enclosing this manifold, all solutions exit the excited state neighborhood and relax to ground states [32]. It is further showed in [34] that for all small initial data in $H^1 \cap L^1$, there are exactly three types of asymptotic profiles: vacuum, excited states or ground states. The last problem is also considered in [29].

Suppose $-\Delta + V$ has three or more bound states. The asymptotic stability of ground states is proved in [30]. In fact, it is shown that all solutions with

$$\|\psi_0\|^{3-\varepsilon}_{H^1\cap L^1} \le |(\phi_0, \psi_0)| \ll 1, \quad 0 < \varepsilon \ll 1, \tag{1.9}$$

relax to ground states. It ensures that the solution is away from excited states but allows the ground state component to be much smaller than other components.

We also mention a few related results on the asymptotic stability of ground states of nonlinear Schrödinger equations with more general nonlinearities. For small solutions, one extension is to replace the resonance condition (1.3) by weaker conditions, e.g. those by [10] and by [8]. Another extension is to assume $\psi_0 \in H^1$ without assuming $\psi_0 \in L^1$. It is first proved in [12] for $K = 0$ and dimension $N = 3$ and then extended by [19, 20] for $K = 0$ and $N = 1, 2$. It is also extended by [8] for $K \ge 1$ with (1.3) replaced by weaker conditions used by [10]. A third extension is to allow subcritical nonlinearity $\pm|\psi|^{p-1}\psi$, $p < 1 + 4/N$, see e.g. [15]. A fourth extension is to assume $K = 1$ and e_1 has multiplicity, see [11, 13].

The stability of *large* solitary waves is considered for $K = 0, 1$, by [3, 4, 5] for $N = 1$ and by [6, 7] for $N = 3$.

See [18, 24, 13] and their references for construction of stable manifolds similar to that in [33].

In this paper, our goal is to continue the study of [30] under the same assumptions, with initial data ψ_0 now inside a neighborhood of the first excited state $Q_{1,n}$. This is the easiest interesting case not covered in [30]. Guided by the $K = 1$ case, one expects that the solution should either converge to a first excited state (with the ground state component always negligible), or leave the excited state neighborhood after some time (which may be extremely long, say greater than $e^{e^{-1/n}}$), and then relax to a ground state.

The new difficulty of the $K > 1$ case is the existence of higher excited state components. If the solution is to converge to a first excited state with the ground state component always negligible, one can think that the ground state component is absent and the first excited state as a new ground state. Thus, in the $K > 1$ case the convergence to a first excited state is expected to be in the rate $t^{-1/2}$, much slower than $t^{-3/2}$ in the $K = 1$ case.

When the difference is of order $t^{-3/2}$, one can use *centered orthogonal coordinates* as in [22, 34],

$$\psi(t) = Q_{1,n(t)} e^{i\theta(t)} + h(t), \quad h(t) = x_0(t)\phi_0 + \xi(t), \quad \xi \in \mathbf{E}_c(H_0). \tag{1.10}$$

The equations of $\dot{n}(t)$ and $\dot{\theta}(t)$ contain linear terms in h. When $x_0(t)$ is negligible, these linear terms are of order $t^{-3/2}$ and hence integrable in t, ensuring the convergence of the parameters. However, when $K > 1$, the difference is order $t^{-1/2}$ and one cannot show the convergence of the parameters if their equations contain linear terms. To remove linear terms, one is forced to use *linearized coordinates* around the first excited state, to be specified later in §3.2.

We now describe a few special properties of the linearized operator around an excited state. When the function ψ is close to a nonlinear bound state $Q = Q_{m,n}$ with corresponding frequency $E = E_{m,n}$, one writes $\psi = (Q(x)+h(t,x))e^{-iEt}$. The perturbation $h(t,x)$ satisfies

$$\partial_t h = \mathcal{L}h + \text{nonlinear terms}, \tag{1.11}$$

where the linearized operator $\mathcal{L}$ around Q is given by

$$\mathcal{L}h = -i\left\{ (H + \kappa Q^2)\, h + \kappa Q^2\, \overline{h} \right\}, \quad H = -\Delta + V - E + \kappa Q^2. \tag{1.12}$$

Note $HQ = 0$. Since $\mathcal{L}$ does not commute with i, it is not useful to consider its spectral properties. Instead one looks at its matrix version acting on $\left[\begin{smallmatrix} \operatorname{Re} h \\ \operatorname{Im} h \end{smallmatrix}\right]$:

$$\mathbf{L} = \begin{bmatrix} 0 & H \\ -H - 2\kappa Q^2 & 0 \end{bmatrix}. \tag{1.13}$$

The spectral property of $\mathbf{L}$ for $m > 0$ is studied in [33] and recalled in Proposition 2.4. It is a perturbation of $J(H_0 - e_m)$ with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which has eigenvalues $\pm i(e_k - e_m)$, $k = 0, \ldots, K$. When $m > 0$, $k < m$ and $e_k < 2e_m$, the eigenvalues $\pm i(e_k - e_m)$ are embedded in the continuous spectrum $\pm i\,[|e_m|, \infty)$. These embedded eigenvalues split into a quadruple of eigenvalues of $\mathbf{L}$, $\pm\lambda_k$ and $\pm\bar{\lambda}_k$, with $\operatorname{Im}\lambda_k = |e_k - e_m| + O(n^2)$ and $C^{-1}n^4 < \operatorname{Re}\lambda_k < Cn^4$ (assuming the generic condition (1.17)). The size of their corresponding eigenvectors are roughly[2]

$$O_{L^2_{100}}(1) + \frac{O(n^2)}{\langle x \rangle} 1_{|x| < n^{-4}}. \tag{1.14}$$

The second part is not localized; It is small in $L^\infty \cap L^3$, of order 1 in L^2, and of order $n^{6-12/p}$ in L^p for $p < 2$. In particular, the projection $P_c^{\mathbf{L}}$ onto the continuous spectral subspace $\mathbf{E}_c^{\mathbf{L}}$ of $\mathbf{L}$ is of order $n^{6-12/p} \gg 1$ in L^p for $p < 2$, giving an extra difficulty to the usual analysis.

[2]Denote $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^d$, $d \geq 1$. For $r \in \mathbb{R}$, denote by L^2_r the weighted L^2 spaces with $\|f\|_{L^2_r} = \|\langle x \rangle^r f(x)\|_{L^2}$.

To overcome this difficulty, we prove decay estimates of the form (see Lemma 2.11)

$$\left\| e^{t\mathbf{L}} P_c^\sharp \varphi \right\|_{L^p} \le C_p t^{-\frac{3}{2}+\frac{3}{p}} \langle t \rangle^{\frac{3}{2p}} \|\varphi\|_{L^{p'}}, \quad (t \ge 0) \tag{1.15}$$

for $3 \le p < 6$, with constant C_p independent of n. Here $P_c^\sharp$ is an extended projection: It is the sum of $P_c^{\mathbf{L}}$ and all projections onto eigenspaces whose corresponding eigenvalues have negative real parts. As shown in Remark (iii) after Lemma 2.11, these estimates with n-independent constant are false if $P_c^\sharp$ is replaced by $P_c^{\mathbf{L}}$. Also note that (1.15) is time-direction sensitive: it is true only for $t \ge 0$. The decay exponent above is not as good as the usual free Schrödinger evolution, but it is sufficient for us if we take $p < 6$ close to 6. A side benefit of extending P_c to $P_c^\sharp$ is that we no longer need to track the component $(P_c^\sharp - P_c)h$.

Our assumptions on the operator $H_0 = -\Delta + V$ are as follows:
Assumption A0. $H_0 = -\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has $K+1$ simple eigenvalues $e_0 < e_1 < \ldots < e_K < 0$, $K \ge 2$, with normalized real eigenfunctions $\phi_0, \ldots, \phi_K$.
Assumption A1. $V(x)$ is a real-valued function satisfying $|\nabla^\alpha V(x)| \lesssim \langle x \rangle^{-5-2r_1}$ for $|\alpha| \le 3$, for some $r_1 > 9/2$ to be given by Lemma 2.2. 0 is not an eigenvalue nor a resonance for H_0.
Assumption A2. Resonance condition. We assume that

$$e_0 < 2\, e_1 < 4\, e_2. \tag{1.16}$$

We further assume that, for some small $s_0 > 0$,

$$\gamma_0 \equiv \inf_{\substack{0 \le m \le 1, |s| < s_0 \\ m < k, l \le K}} \lim_{r \to 0+} \operatorname{Im} \left(\phi_m \phi_k^2 \,,\, \frac{1}{-\Delta + V + e_m - e_k - e_l - s - ri} P_c^{H_0} \phi_m \phi_k^2 \right) > 0. \tag{1.17}$$

Assumption A3. No-resonance condition (between eigenvalues). Let $j_{\max} = 3$. For all $j = 2, \ldots, j_{\max}$ and for all $k_1, \ldots, k_j, l_1, \ldots, l_j \in \{0, \ldots, K\}$, if $e_{k_1} + \cdots + e_{k_j} = e_{l_1} + \cdots + e_{l_j}$, then there is a permutation s of $\{1, \ldots, j\}$ such that $(l_1, \ldots, l_j) = (k_{s1}, \ldots, k_{sj})$.

Assumption A1 ensure several estimates for linear Schrödinger evolution such as decay estimates and the $W^{k,p}$ estimates for the wave operator $W_{H_0} = \lim_{t \to \infty} e^{itH_0} e^{it\Delta}$. They are certainly not optimal. The main assumption in A2 is the condition $e_{k-1} < 2e_k$. It ensures that $H_0 + e_m - e_k - e_l$ is not invertible in L^2 for $m < k, l$, and provides (for our cubic nonlinearity) the required resonance between eigenvalues through the continuous spectrum. Since the expression for γ_0 is quadratic, it is non-negative and $\gamma_0 > 0$ holds generically. Assumption A3 is a condition to avoid direct resonance between the eigenvalues. It is trivial if $K = 0, 1$. It holds true generically and is often seen in dynamical systems of ODE's. If we relax the assumption (1.16), we may need to increase $j_{\max}$.

Now we are ready to state our main theorem.

Theorem 1.1 *Assume Assumptions A0–A3 and fix $0 < \delta \le \frac{1}{10}$. There are constants $C_0, C_1 > 0$, and small $n_0 > 0$ such that the following hold. If $n = (\phi_1, \psi_0) \in (0, n_0)$ and $\|\psi_0 - n\phi_1\|_{H^1 \cap L^1} \le n^{1+\delta}$, then the solution $\psi(t)$ of (1.1) with $\psi(0) = \psi_0$ satisfies*

$$\limsup_{t \to \infty} \left\| \psi(t) - Q_{m,n_+} e^{i\theta(t)} \right\|_{L^2_{loc}} t^{1/2} \le C_0 / n \tag{1.18}$$

for $m = 0$ or $m = 1$, for some $n_+ \in (C_1^{-1} n, C_1 n)$ and some $\theta(t) \in C([0, \infty), \mathbb{R})$.

In fact we have more detailed estimates of the solution for all time, see Propositions 4.2, 5.1, 6.3, 6.7, and 7.2. In particular, if the initial data ψ_0 is placed in the neighborhood of an excited state $Q_{m,n}$ with $m \geq 2$, even if $K > 2$, Propositions 4.2, 5.1, 6.3, 6.7 show that the solution will either converge to Q_{m,n_+} for some n_+, or eventually exits the neighborhood, stays away from bound states for a time interval of order between $n^{-4}\log\frac{1}{n}$ and $n^{-4-2\delta}$, until it reaches the neighborhood of another bound state $Q_{m',n'}$, $m' < m$. If $m' = 0$, then Proposition 7.2 shows that $\psi(t)$ will converge to some Q_{0,n_+}. However, if $m' > 0$, our current analysis is not sufficient to control its evolution after this time.

We now sketch the structure of our proof and this paper.

In §2 we give the linear analysis, including the decay estimates (1.15).

In §3 we consider the decomposition of the solutions in different coordinates and the normal forms of their equations.

In §4 we start with the solution in a $n^{1+\delta}$-neighborhood of $Q_{1,n}$ and use linearized coordinates (3.17). We follow the evolution as long as the ground state component z_0 is negligible, characterized by $|z_0(t)| < n^{-3}(n^{-4-2\delta} + t)^{-1}$. If it is always negligible, we prove that the solution converges to an excited state with convergence rate $t^{-1/2}$.

In §5 we consider the case that $|z_0(t_c)| \geq n^{-3}(n^{-4-2\delta} + t_c)^{-1}$ in a first time $t_c \in [0,\infty)$, which may be 0 or extremely large, say $> e^{e^{-1/n}}$. After an initial layer, we show that $|z_0(t)|$ starts to grow exponentially with exponent Cn^4 until it reaches the size $2n^{1+\delta}$ at time t_o. The time it takes, $t_o - t_c$, is of order $n^{-4}\log\frac{2n^{1+\delta}}{|z_0(t_c)|}$. Along the way higher excited states may have size larger than $|z_0(t)|$ but can be controlled. This section is the most difficult part in the nonlinear analysis because it involves estimates not previously studied.

In §6 we study the dynamics after t_o when there are at least two components of size greater than $2n^{1+\delta}$, and change to *orthogonal coordinates*

$$\psi = x_0\phi_0 + \cdots + x_K\phi_K + \xi, \quad \xi \in \mathbf{E}_c(H_0). \tag{1.19}$$

Although $\xi(t_o)$ is already non-localized, we can prove "outgoing estimates" for $\xi(t_o)$, introduced in [32, 34], to capture the time-direction sensitive information of the dispersive waves. We show that, after a time of order between $n^{-4}\log\frac{1}{n}$ and $n^{-4-2\delta}$, the ground state component x_0 grows to order n while all other components become smaller than $n^{1+\delta}$. (This is called the *transition regime*.)

In §7 the ground state component becomes dominant and we change to linearized coordinates around it. Again we need to keep track of out-going estimates during the coordinate change. We show that the solutions will converge to ground states with convergence rate $t^{-1/2}$. The analysis is similar to §4 but easier because it has no unstable direction. (This is called the *stabilization regime*.)

Analysis similar to §6 and §7 is done in [30], (and in the two-eigenvalue case near ground states in [4, 31, 32, 7, 5]). However, with weaker decay estimates like (1.15), we need more refined analysis. For example, since the nonlinearity is of constant order n^3 in the transition regime, we need to make this time interval as short as possible by taking $\delta > 0$ small. We also take $p < 6$ close to 6 to minimize our loss in estimating the L^p-norm of the dispersive component during this interval.

New proof of linear decay estimates for ground states

We end this introduction by noting that, our linear analysis, Lemmas 2.11 and 2.13, in the case $m = 0$, provide a new proof of linear estimates for the linearized operators around ground states, which is used to prove the stability of ground states in 3D, see [6, 31, 30]. Proofs in these references either use the wave operator between $\mathcal{L}$ and $-i(H_0 - E)$, or use a similarity transform $\mathcal{L} = U(-iA)U^{-1}$ for some self-adjoint perturbation A of $H_0 - E$ and non-self-adjoint operator U. Our proof here use simple perturbation argument and requires less assumptions on the potential V. Moreover, this perturbation argument allows the operator V to be more general than a potential, as long as the decay and singular decay estimates for $-\Delta + V$ hold.

2 Linear analysis

In this section we will study various properties of the linearized operator around a fixed bound state, in particular an excited state. The starting point is the following lemma on the existence of nonlinear bound states and their basic properties, see [26, 12].

Lemma 2.1 (Nonlinear bound states) *Assume Assumptions A0–A1. There exists a small $n_1 > 0$ such that for each $k = 0, \dots, K$ and $n \in [0, n_1]$, there is a solution $Q_{k,n} \in H^2 \cap W^{1,1}$ of (1.4) with $E = E_{k,n} \in \mathbb{R}$ such that*

$$Q_{k,n} = n\phi_k + q(n), \quad (q, \phi_k) = 0. \tag{2.1}$$

The pair (q, E) is unique in the class $\|q\|_{H^2} + |E - e_k| \le n^2$. Moreover, $\|q\|_{H^2 \cap W^{1,1}} \lesssim n^3$, $\left\|\frac{\partial}{\partial n} q\right\|_{H^2 \cap W^{1,1}} + |E - e_k| \lesssim n^2$, and $|E - e_k - C_k n^2| \lesssim n^4$ where $C_k = \kappa \int \phi_k^4$. We also denote $\partial_E Q_{k,n} = \frac{\partial}{\partial E_k} Q_{k,n} = \frac{\partial}{\partial n} Q_{k,n} / \frac{\partial}{\partial n} E_{k,n} = \frac{1}{2Cn}\phi_k + O_{H^2 \cap W^{1,1}}(n)$, with $(Q_{k,n}, \partial_E Q_{k,n}) = \frac{1}{2C_k} + O(n)$.

In the following we fix $m \in \{0, \dots, K\}$ and $n \in [0, n_1]$. Let $Q = Q_{m,n}$, $\partial_E Q = \partial_E Q_{m,n}$ and $E = E_{m,n}$. The function Q satisfies $HQ = 0$ where

$$H = H_0 - E + \kappa Q^2. \tag{2.2}$$

The following lemma collects useful properties of H.

Lemma 2.2 *Assume Assumptions A0-A1 and let H be defined as in (2.2). The operator H has $K+1$ real eigenvalues $\tilde{e}_k = e_k - e_m + O(n^2)$ with normalized eigenfunctions $\tilde{\phi}_k = \phi_k + O(n^2)$. In particular, $\tilde{e}_m = 0$ and $\tilde{\phi}_m = CQ_m$. The projection to its continuous spectral subspace is $P_c^H f = f - \sum_k (\tilde{\phi}_k, f)\tilde{\phi}_k$. Furthermore, we have the following decay estimates*

$$\left\|e^{-itH} P_c^H \varphi\right\|_{L^q} \le C|t|^{-3/2+3/q} \|\varphi\|_{L^{q'}}, \quad (2 \le q \le \infty), \tag{2.3}$$

and singular decay estimates: for sufficiently large $r_1 > 9/2$, for $0 \le N \le 3$, for $\alpha_j \in \mathbb{C}$ with $\operatorname{Im} \alpha_j > 0$, $|\operatorname{Re} \alpha_j + e_m| \in [a_1, a_2] \subset (0, \infty)$, $j \le N$,

$$\left\|\langle x\rangle^{-r_1} e^{-itH} \Pi_{j=1}^N (H - \alpha_j)^{-1} P_c^H \varphi\right\|_{L^2} \le C \langle t\rangle^{-3/2} \left\|\langle x\rangle^{r_1} \varphi\right\|_{L^2}, \quad (t \ge 0). \tag{2.4}$$

Here the constant C is independent of n, φ and α_j.

Note that this lemma contains $H = H_0$ as a special case with $n = 0$. The proof of the first part is well-known by perturbation. Estimate (2.3) is by Journe-Soffer-Sogge [17]. Estimate (2.4) for $N = 0$ is by Jensen-Kato [16] and Rauch [23]. Estimate (2.4) for $\alpha_1 = \cdots = \alpha_N$, $N \ge 1$, was first proven by Soffer-Weinstein [28] for Klein-Gordon equations, then by Tsai-Yau [31] and Cuccagna [7] for (linearized) Schrödinger equations. The general case is similar and a proof based on Mourre estimate is sketched below for completeness. (See [7] for a different approach).

Denote the dilation operator $D = x \cdot p + p \cdot x$ with $p = -i\nabla$, and the commutators

$$\mathrm{ad}_D^0(H) = H, \quad \mathrm{ad}_D^{k+1}(H) = [\mathrm{ad}_D^k(H), D], \quad k \geq 0. \tag{2.5}$$

Fix $g_* \in C_c^\infty(\mathbb{R})$ with $g_* = 1$ on $[-1, 1]$ and $\operatorname{supp} g_* \subset (-2, 2)$. For each j, let $g_j(t) = g_*((t - \operatorname{Re} z_j)/\varepsilon)$. If $\varepsilon > 0$ is sufficiently small, $g_j(H)\mathrm{ad}_D^k(H)g_j(H)$ are bounded operators in L^2 for $k \leq 3$ and all j, and the Mourre estimate holds: For some $\theta > 0$,

$$g_j(H)[iH, D]g_j(H) \geq \theta g_j(H)^2, \quad \forall j. \tag{2.6}$$

See [9]. Thus the pair H, D satisfies the assumptions of the minimal velocity estimates in [14] and Theorem 2.4 of [25], and one has

$$\left\| \chi(D \leq \theta t/2) e^{-itH} g_j(H) \langle D \rangle^{-r_1} \right\|_{L^2 \to L^2} \leq C \langle t \rangle^{-r_1 + \varepsilon_1}, \tag{2.7}$$

where $0 < \varepsilon_1 \ll 1$ and $\chi(D \leq a)$ is the spectral projection of D associated to the interval $(-\infty, a]$. The same argument of [28] then gives (2.4).

Note that all ϕ_k, $\tilde{\phi}_k$, $Q_{k,n}$ and $R_{k,n}$ decay exponentially at infinity, see [2].

2.1 Linearized operator

A perturbation solution $\psi(x, t)$ of (1.1) of the exact solution $Q(x)e^{-iEt}$ can be written in the form

$$\psi(x, t) = [Q(x) + h(x, t)]e^{-iEt} \tag{2.8}$$

for some function h which is small in a suitable sense. Then, h satisfies

$$\partial_t h = \mathcal{L}h + \text{ nonlinear terms}, \tag{2.9}$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L}h = -i\{(H_0 - E + 2\kappa Q^2)h + \kappa Q^2 \bar{h}\}. \tag{2.10}$$

The operator $\mathcal{L}$ is linear over $\mathbb{R}$ but not over $\mathbb{C}$. As a result it is not useful to consider its spectral properties.

Consider the injection from scalar functions to vector functions

$$\jmath : L^2(\mathbb{R}^3, \mathbb{C}) \to L^2(\mathbb{R}^3, \mathbb{C}^2), \quad \jmath(\varphi) = [\varphi] := \begin{bmatrix} \operatorname{Re} \varphi \\ \operatorname{Im} \varphi \end{bmatrix}. \tag{2.11}$$

With respect to this injection, the operator $\mathcal{L}$ is naturally extended to a matrix operator acting on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ with the following form

$$\mathbf{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \quad \text{where} \begin{cases} L_- &= H = H_0 - E + \kappa Q^2, \\ L_+ &= H + 2\kappa Q^2 = H_0 - E + 3\kappa Q^2. \end{cases} \tag{2.12}$$

Note $\mathbf{L}$ is a perturbation of JH where $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. We will use $\mathcal{L} = \jmath^{-1}\mathbf{L}\jmath$ for computations involving $\mathcal{L}$.

The space $L^2(\mathbb{R}^3,\mathbb{C}^2)$ is endowed with the natural inner product

$$(f,g)=\int_{\mathbb{R}^3}(\bar{f}_1 g_1+\bar{f}_2 g_2)\,dx \tag{2.13}$$

for $f=\begin{bmatrix} f_1\\ f_2\end{bmatrix}$ and $g=\begin{bmatrix} g_1\\ g_2\end{bmatrix}$. We will use the Pauli matrices

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{2.14}$$

2.2 Invariant subspaces

In this subsection we study the spectral subspaces of $\mathbf{L}$. Since $\mathbf{L}$ is a perturbation of JH, we first give the following lemma for comparison. Recall $J=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $\tilde{\phi}_k$ are eigenfunctions of H with eigenvalues $\tilde{e}_k$ given in Lemma 2.2.

Lemma 2.3 (Invariant subspaces of JH) *Assume Assumptions A0–A2. The space $L^2(\mathbb{R}^3,\mathbb{C}^2)$ can be decomposed as the direct sum of JH-invariant subspaces*

$$L^2(\mathbb{R}^3,\mathbb{C}^2)=\mathbf{E}_0^{JH}\oplus\cdots\oplus\mathbf{E}_K^{JH}\oplus\mathbf{E}_c^{JH}. \tag{2.15}$$

For each $k\in\{0,\dots,K\}$, the space $\mathbf{E}_k^{JH}$ is spanned by 2 eigenvectors $\begin{bmatrix}1\\-i\end{bmatrix}\tilde{\phi}_k$ and $\begin{bmatrix}1\\i\end{bmatrix}\tilde{\phi}_k$ with eigenvalues $-i\tilde{e}_k$ and $i\tilde{e}_k$, respectively. Its corresponding orthogonal projection is $P_k^{JH}\begin{bmatrix}f_1\\f_2\end{bmatrix}=\begin{bmatrix}(\tilde{\phi}_k,f_1)\\(\tilde{\phi}_k,f_2)\end{bmatrix}\tilde{\phi}_k$. The subspace $\mathbf{E}_c^{JH}$ has projection $P_c^{JH}f=\begin{bmatrix}P_c^H f_1\\P_c^H f_2\end{bmatrix}$.

The proof is straightforward and skipped. We next give the corresponding statements for $\mathbf{L}$.

Proposition 2.4 (Invariant subspaces of $\mathbf{L}$) *Assume Assumptions A0–A2 and let $r_1>9/2$ be from Lemma 2.2. Fix $m\in\{0,\dots,K\}$ and $n\in(0,n_1]$. Let $Q=Q_{m,n}$, $\partial_E Q=\partial_E Q_{m,n}$ and $E=E_{m,n}$. The space $L^2(\mathbb{R}^3,\mathbb{C}^2)$ can be decomposed as the direct sum of $\mathbf{L}$-invariant subspaces*

$$L^2(\mathbb{R}^3,\mathbb{C}^2)=\mathbf{E}_0^{\mathbf{L}}\oplus\cdots\oplus\mathbf{E}_K^{\mathbf{L}}\oplus\mathbf{E}_c^{\mathbf{L}}. \tag{2.16}$$

If f and g belong to different subspaces, then

$$(\sigma_1 f,g)=0. \tag{2.17}$$

These subspaces and their corresponding projections satisfy the following.

(i) $\mathbf{E}_m^{\mathbf{L}}$ is the 0-eigenspace spanned by $\begin{bmatrix}0\\Q\end{bmatrix}$ and $\begin{bmatrix}\partial_E Q\\0\end{bmatrix}$, with $\mathbf{L}\begin{bmatrix}0\\Q\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ and $\mathbf{L}\begin{bmatrix}\partial_E Q\\0\end{bmatrix}=-\begin{bmatrix}0\\Q\end{bmatrix}$. Its projection is $P_m f=c_m(\sigma_1\begin{bmatrix}\partial_E Q\\0\end{bmatrix},f)\begin{bmatrix}0\\Q\end{bmatrix}+c_m(\sigma_1\begin{bmatrix}0\\Q\end{bmatrix},f)\begin{bmatrix}\partial_E Q\\0\end{bmatrix}$, $c_m=(Q,\partial_E Q)^{-1}$.

(ii) $\mathbf{E}_k^{\mathbf{L}}$ for $0\le k<m$, if such k exists, is spanned by 4 eigenvectors $\Phi_k=\begin{bmatrix}u_k\\-iv_k\end{bmatrix}$, $\bar{\Phi}_k$, $\sigma_3\Phi_k$ and $\sigma_3\bar{\Phi}_k$, with eigenvalues λ_k, $\bar{\lambda}_k$, $-\lambda_k$, and $-\bar{\lambda}_k$, respectively. Here $\lambda_k=-i(e_k-e_m)+O(n^2)$, $\frac{3}{4}\gamma_0 n^4\le\operatorname{Re}\lambda_k\le Cn^4$, ($\gamma_0$ is defined in (1.17)), u_k and v_k are complex-valued functions, $u_k=\bar{u}_k^+ +\bar{u}_k^-$ and $v_k=\bar{u}_k^+-\bar{u}_k^-$, with

$$u_k^+=\phi_k+O_{L^\infty_{3r_1}}(n^2),\quad u_k^-=(H-i\bar{\lambda}_k)^{-1}\phi_k^*+O_{L^\infty_{3r_1}}(n^2) \tag{2.18}$$

where $\phi_k^* = P_c^H \phi_k^* = O_{L^\infty_{3r_1}}(n^2)$*. Furthermore,* $(u_k, v_k) = 0$ *and* $(u_k, v_\ell) = (\bar u_k, v_\ell) = 0$ *for* $k \neq \ell$*. All* $(\bar u_k, v_k)$*,* $\|u_k^+\|_{L^2}$ *and* $\|u_k^-\|_{L^2}$ *are equal to* $1+O(n^2)$ *and* $\|u_k^-\|_{L^2_{loc}} \lesssim n^2$*. The projection to* $\mathbf{E}_k^{\mathbf{L}}$ *is* $P_k + P_k^\sharp$ *where*

$$\begin{aligned} P_k f &= c_k(\sigma_1 \bar\Phi_k, f)\Phi_k + \bar c_k(\sigma_1 \Phi_k, f)\bar\Phi_k, \\ P_k^\sharp f &= -c_k(\sigma_1\sigma_3 \bar\Phi_k, f)\sigma_3\Phi_k - \bar c_k(\sigma_1\sigma_3\Phi_k, f)\sigma_3\bar\Phi_k, \end{aligned} \tag{2.19}$$

and $c_k = (\sigma_1 \bar\Phi_k, \Phi_k)^{-1} = i/(\int 2u_k v_k) = i/2 + O(n^2)$*.*

(iii) $\mathbf{E}_k^{\mathbf{L}}$ *for* $m < k \le K$*, if such* k *exists, is spanned by 2 eigenvectors* $\Phi_k = \left[\begin{smallmatrix} u_k \\ -iv_k \end{smallmatrix}\right]$ *and* $\bar\Phi_k$ *with eigenvalues* λ_k *and* $\bar\lambda_k$*, respectively. Here* $\mathbb{R} \ni i\lambda_k = e_k - e_m + O(n^2)$*,* u_k *and* v_k *are real-valued, both equal to* $\phi_k + O_{L^\infty_{3r_1}}(n^2)$*, and normalized by* $(u_k, v_k) = 1$*. Its projection is* P_k*, also given by (2.19), with* $c_k = i/2$*.*

(iv) $\mathbf{E}_c^{\mathbf{L}} = \{g : (\sigma_1 f, g) = 0, \forall f \in \mathbf{E}_k, \forall k = 0, \dots, K\}$*. Its projection is* $P_c^{\mathbf{L}} f = f - \sum_{k=0}^K P_k f - \sum_{k<m} P_k^\sharp f$*.*

Note that λ_k is in the first quadrant and near the imaginary axis for $k < m$, and in the lower imaginary axis for $k > m$. They are all perturbations of $-i\tilde e_k$ of Lemma 2.3. When $k < m$, $-i\tilde e_k$ are inside the continuous spectrum $\pm i[|E_m|, \infty)$ and their resonance make the eigenvalues split.

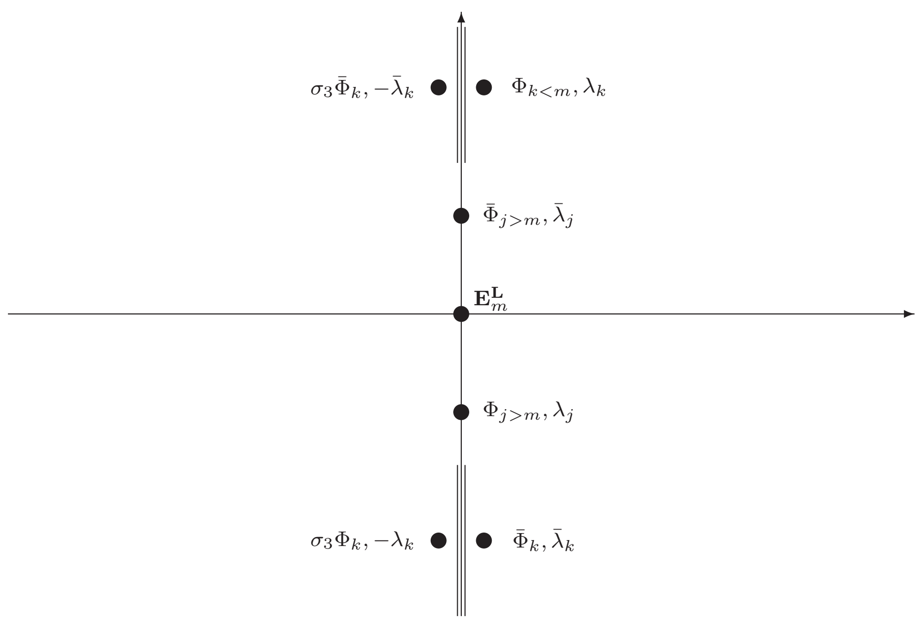

Figure 1: Spectrum of $\mathbf{L}$ around Q_m, $0 < m < K$.

Proof. The same proof of [33, Theorem 2.2] works in our many eigenvalue case. The only thing we need to check is the properties of u_k^+ and u_k^- when $k < m$. Fix $k < m$. Denote by Π the orthogonal projection from L^2 onto $\{\tilde\phi_k, Q_m\}^\perp$, and $B = 2\kappa Q_m^2$. We omit the subscript k below. By the defining equations $\mathcal{L}_m\Phi = \lambda\Phi$ and $\Phi = \left[\begin{smallmatrix} u \\ -iv \end{smallmatrix}\right]$, $\bar u$ satisfies

$$(H^2 + HB)\bar u = -\bar\lambda^2 \bar u. \tag{2.20}$$

By the same proof for the two-eigenvalue case in [33, section 2.1] (in which $\Pi = P_c^H$), $\bar{u}$ can be solved in the form

$$\bar{u} = \tilde{\phi} + h, \quad h = \Pi h = -(H^2 + \Pi H B \Pi + \bar{\lambda}^2)^{-1} \Pi H B \tilde{\phi}. \tag{2.21}$$

One can rewrite

$$h = (H^2 + \bar{\lambda}^2)^{-1}\Psi, \quad \Psi = \Pi\Psi = [1 + \Pi H B \Pi (H^2 + \bar{\lambda}^2)^{-1}]^{-1} \Pi H B \tilde{\phi}. \tag{2.22}$$

By resolvent estimates and a power series expansion as in [33], the function Ψ is localized and $\|\Psi\|_{L^2_{3r_1}} \le Cn^2$. Since $v = (i\lambda)^{-1}(H+B)u$, we have $u^\pm = \mp\frac{1}{2z}(H \mp z + B)\bar{u}$ with $z = i\bar{\lambda} = |e_k - e_m| + O(n^2)$. For u^+,

$$u^+ = -\frac{1}{2z}(H-z)\tilde{\phi} - \frac{1}{2z}(H+z)^{-1}\Psi - \frac{1}{2z}B\bar{u}. \tag{2.23}$$

The first term is equal to $(1+O(n^2))\tilde{\phi}$. Since $(H+z)^{-1}\Pi$ is order one, the remaining two terms are $O_{L^\infty_{3r_1}}(n^2)$, and so is $\phi - \tilde{\phi}$. This shows $u^+ = \phi + O_{L^\infty_{3r_1}}(n^2)$. For u^-,

$$u^- = \frac{1}{2z}(H+z)\tilde{\phi} + \frac{1}{2z}(H-z)^{-1}\Psi + \frac{1}{2z}B\bar{u}. \tag{2.24}$$

The first term is $O(n^2)\tilde{\phi}$. Since $(H-z)^{-1}(\Pi - P_c^H)\Psi$ are sum of eigenfunctions with $O(n^2)$ coefficients, we get (2.18) with $\phi_k^* = \frac{1}{2z}P_c^H\Psi = O_{L^\infty_{3r_1}}(n^2)$.

The orthogonality $(u,v) = 0$ is equivalent to $(\sigma_1\Phi, \Phi) = (\sigma_1\sigma_3\Phi, \Phi) = 0$, which follow from the general fact shown in [33, §2.6] that

$$(\sigma_1 f, g) = 0 \quad \text{if} \quad \mathbf{L}f = \lambda f,\ \mathbf{L}g = \mu g,\ \text{and}\ \bar{\lambda} \ne \mu. \tag{2.25}$$

It also follows from (2.25) that $(u_k, v_\ell) = (\bar{u}_k, v_\ell) = 0$ for $k \ne \ell$. That $\|u^+\|_{L^2} = 1 + O(n^2)$ and $\|u^-\|_{L^2_{loc}} \lesssim n^2$ follow from (2.18). Note

$$0 = (\bar{u}, \bar{v}) = (u^+ + u^-, u^+ - u^-) = (u^+, u^+) - (u^-, u^-) + (u^-, u^+) - (u^+, u^-). \tag{2.26}$$

Since the last two terms are $O(n^2)$, we get $\|u^-\|_{L^2} - \|u^+\|_{L^2} = O(n^2)$. Finally

$$(\bar{u}, v) = (u^+ + u^-, \bar{u}^+ - \bar{u}^-) = (u^+, \bar{u}^+) - (u^-, \bar{u}^-) + (u^-, \bar{u}^+) - (u^+, \bar{u}^-). \tag{2.27}$$

We have $(u^-, u^+) - (u^+, u^-) = O(n^2)$. By (2.18) we also have

$$(\bar{u}^-, u^-) = ((H-\bar{z})^{-1}\bar{\phi}_k^*, (H-z)^{-1}\phi_k^*) + O(n^4) = (\bar{\phi}_k^*, (H-z)^{-2}\phi_k^*) + O(n^4) = O(n^4) \tag{2.28}$$

by the singular decay estimate of Lemma 2.2 with $t = 0$. Thus $(\bar{u}_k, v_k) = 1 + O(n^2)$. Similarly, $(\bar{u}_k, v_\ell) = O(n^2)$ for $k \ne \ell$. □

In the following lemma we provide more properties of u_k^-.

Lemma 2.5 *Assume the same as in Proposition 2.4 and fix $k < m$. Let $r = r_1$. Then*

(i) $\|u_k^-\|_{L^p} \le C_p(n^2 + n^{6-\frac{12}{p}})$ *for* $1 \le p \le \infty$, *in particular* $\|u_k^-\|_{L^2_{-r}} \le Cn^2$.

(ii) $\|e^{-isH}P_c^H u_k^-\|_{L^2_{-r}} + \|e^{-isH_0}P_c^{H_0}u_k^-\|_{L^2_{-r}} \le Cn^2 \langle s\rangle^{-3/2}$ *for* $s \ge 0$.

(iii) $\|u_k^-\|_{H^1} \le C$.

Proof. Denote $z = i\bar\lambda_k$ and $\varphi = \phi_k^*$. For (i), it suffices to check $(H-z)^{-1}\varphi$, the main part of u_k^- in (2.18). Write $H - z = -\Delta + \nu^2 + V_1$ where $V_1 = V + \kappa Q_m^2$, $\nu^2 = E_m + z$ with $\operatorname{Im}\nu > 0$. Thus $\operatorname{Im}\nu \sim +n^4$. By resolvent expansion,

$$(H-z)^{-1}\varphi = (-\Delta - \nu^2)^{-1}\varphi + (-\Delta - \nu^2)^{-1}V_1(H-z)^{-1}\varphi. \tag{2.29}$$

Since the resolvent $(-\Delta - \nu^2)^{-1}$ has the convolution kernel $G(x) = (4\pi|x|)^{-1}\exp(i\nu|x|)$,

$$\left\|(-\Delta - \nu^2)^{-1}\varphi\right\|_{L^p} \lesssim \|G * \varphi\|_{L^p} \lesssim (\|G\|_{L^p(B_1^c)} + \|G\|_{L^2(B_1)}) \cdot \|\varphi\|_{L^1\cap L^2} \tag{2.30}$$

which is bounded by $(n^{4-12/p} + 1)\cdot n^2$. Since $\left\|V_1(H-z)^{-1}\varphi\right\|_{L^1\cap L^2} \lesssim \left\|(H-z)^{-1}\varphi\right\|_{L^2_{-r}} \lesssim n^2$, we have the same bound for the second term. The above show (i).

For (ii), we only need to consider $e^{-isH_0}P_c^{H_0}u_k^-$ since the other term follows from Lemma 2.2. By resolvent expansion $R = (H-z)^{-1} = R_0(1+\kappa Q_m^2 R)$ where $R_0 = (H_0 - E_m - z)^{-1}$,

$$P_c^{H_0}u^- = R_0\varphi' + O_{L^\infty_{3r_1}}(n^2), \quad \varphi' = P_c^{H_0}(1+\kappa Q_m^2 R)\varphi = O_{L^\infty_{3r_1}}(n^2). \tag{2.31}$$

Thus

$$e^{-isH_0}P_c^{H_0}u^- = e^{-isH_0}R_0\varphi' + O_{L^2_{-r}}(n^2\langle s\rangle^{-3/2}). \tag{2.32}$$

By the singular decay estimate for H_0, the first term is also of order $O_{L^2_{-r}}(n^2\langle s\rangle^{-3/2})$.

To prove (iii), it suffices to prove that $\|\nabla v\|_{L^2} = O(1)$ where $v = (H-z)^{-1}\varphi$. It can be shown by multiplying the equation $(H-z)v = \varphi$ by $\bar v$ and then integrating it on $\mathbb{R}^3$. $\square$

We will need the following lemmas for scalar functions.

Lemma 2.6 *Fix $0 \le k \le K$, $k \neq m$. Let $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$ be a scalar function.*
(i) $P_k[\varphi] = \operatorname{Re}\alpha\Phi_k$, $\jmath^{-1}P_k[\varphi] = \alpha\bar u^+ + \bar\alpha u^-$, where

$$\alpha = 2c_k(\sigma_1\bar\Phi_k, [\varphi]) = -2c_k i[(u_k^+, \varphi) - (u_k^-, \bar\varphi)]. \tag{2.33}$$

(ii) $P_k\varphi = 0$ iff $(\sigma_1\Phi_k, [\varphi]) = 0$ iff $(u_k^+, \varphi) = (u_k^-, \bar\varphi)$.
(iii) For $k < m$, $P_k^\sharp\varphi = 0$ iff $(\sigma_1\sigma_3\Phi_k, [\varphi]) = 0$ iff $(u_k^+, \bar\varphi) = (u_k^-, \varphi)$.

Proof. Write $[\varphi] = \left[\begin{smallmatrix}\varphi_1\\ \varphi_2\end{smallmatrix}\right]$. Since $[\varphi]$ is real, we have by (2.19) that $P_k[\varphi] = \operatorname{Re}\alpha\Phi_k$ with $\alpha = 2c_k(\sigma_1\bar\Phi_k, [\varphi])$. Omitting the subscript k, we have

$$(\sigma_1\bar\Phi_k, [\varphi]) = (i\bar v, \varphi_1) + (\bar u, \varphi_2) = (u^+ - u^-, -i\varphi_1) + (u^+ + u^-, \varphi_2) = -i(u^+, \varphi) + i(u^-, \bar\varphi),$$

which gives the formula for α. Thus

$$\jmath^{-1}P_k[\varphi] = \jmath^{-1}\operatorname{Re}\alpha\begin{bmatrix} u \\ -iv\end{bmatrix} = \frac{1}{2}\{(\alpha u + \bar\alpha\bar u) + i(-i\alpha v + i\bar\alpha\bar v)\} = \alpha\bar u^+ + \bar\alpha u^-. \tag{2.34}$$

The claim (ii) follows from (i). For (iii), since $\sigma_3\sigma_1\sigma_3 = -\sigma_1$, $(\sigma_1\sigma_3\Phi_k, [\varphi]) = 0$ is equivalent to $0 = (\sigma_1\Phi_k, \sigma_3[\varphi]) = (\sigma_1\Phi_k, [\bar\varphi])$ and hence to $(u_k^+, \bar\varphi) = (u_k^-, \varphi)$. $\square$

The following lemma will be used to treat the linear term in the η equation.

Lemma 2.7 *(i) For $k < m$,*

$$J\Phi_k = i\Phi_k - 2i\left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right]\bar u_k^+. \tag{2.35}$$

(ii) If $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and $P_k f = 0$, then $\|P_k J f\|_{L^2} \lesssim \|f\|_{L^2_{-3r_1}}$.

Proof. For (i), rewrite

$$\Phi_k = \left[\begin{smallmatrix} u_k \\ -iv_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \bar{u}_k^+ + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \bar{u}_k^-. \tag{2.36}$$

Applying J

$$J\Phi_k = -i \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \bar{u}_k^+ + i \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \bar{u}_k^-. \tag{2.37}$$

Canceling u_k^- we get (2.35).

For (ii), we have $(\sigma_1 \bar{\Phi}_k, f) = (\sigma_1 \Phi_k, f) = 0$. Using $J^* = -J$, $J\sigma_1 = -\sigma_1 J$, and (2.35),

$$(\sigma_1 \bar{\Phi}_k, Jf) = -(J\sigma_1 \bar{\Phi}_k, f) = (\sigma_1 J \bar{\Phi}_k, f) = (\sigma_1(-i\bar{\Phi}_k + 2i \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] u_k^+), f) = (2i \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] u_k^+, f). \tag{2.38}$$

Similarly $(\sigma_1 \Phi_k, Jf) = (2i \left[\begin{smallmatrix} i \\ -1 \end{smallmatrix}\right] \bar{u}_k^+, f)$. This shows (ii). □

Note, in deriving (2.35) if we cancel u_k^+ instead of u_k^-, we get

$$J\Phi_k = -i\Phi_k + 2i \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \bar{u}_k^-. \tag{2.39}$$

2.3 Decay estimate

In the following two subsections we prove decay estimates for $e^{t\mathbf{L}}$ with the constant independent of n. This independence is essential for our analysis of the nonlinear dynamics both inside a neighborhood of an excited and away from bound states. For example, it ensures that the time spent traveling between bound states is no longer than $O(n^{-4-2\delta})$.

An estimate of the form $\left\| e^{t\mathbf{L}} P_c^{\mathbf{L}} \varphi \right\|_{L^p} \le C \left\| \varphi \right\|_{L^{p'}} t^{-\sigma}$ for $5 < p < 6$, some $\sigma > 0$, and a constant C independent of n, would be ideal. It is however false, see Remark (iii) after Lemma 2.11. This is related to the fact that the projection $P_c^{\mathbf{L}}$ as an operator acting on L^1 is of order $O(n^{-6})$ due to the presence of u_k^-. We cannot avoid the projection $P_c^{\mathbf{L}}$: Suppose F is the total nonlinearity in the equation of the perturbation h. Our choice of parameters $a(t)$ and $\theta(t)$ makes $P_m F = 0$, but does not make $F \in \mathbf{E}_c$. To avoid the large constant problem, we extend the continuous spectral subspace $\mathbf{E}_c$ and absorb the range of $P_k^\sharp$, $k < m$, which have exponential decay, into $\mathbf{E}_c$. The range of P_k for $k < m$, which have exponential growth, is left out and will be taken care of using the evolution with correct time direction.

Define $\mathbf{E}_c^\sharp$ as the direct sum of $\mathbf{E}_c^{\mathbf{L}}$ and eigenspaces whose eigenvalues have *negative* real parts

$$\mathbf{E}_c^\sharp = \mathbf{E}_c^{\mathbf{L}} \oplus \operatorname{span}_{\mathbb{C}}\{\sigma_3 \Phi_k, \sigma_3 \bar{\Phi}_k : 0 \le k < m\}. \tag{2.40}$$

Its corresponding projection is denoted as

$$P_c^\sharp f = P_c^{\mathbf{L}} f + \sum_{k<m} P_k^\sharp(f) = f - P_d f, \quad P_d f = \sum_{k=0}^{K} P_k(f). \tag{2.41}$$

We extend the definition of $P_c^\sharp$ to scalar functions by $P_c^\sharp \varphi = \jmath^{-1} P_c^\sharp [\varphi]$, and similarly for P_d. If a scalar function φ satisfies $[\varphi] \in \mathbf{E}_c^\sharp$, then $(\sigma_1 \Phi_k, [\varphi]) = 0$ for all k.

The next lemma is on the uniform bound of H^1-norm of $e^{t\mathbf{L}} P_c^\sharp \varphi$ for $t \ge 0$.

Lemma 2.8 *For any scalar function $\varphi \in H^1$ we have*

$$\left\| e^{t\mathbf{L}} P_c^\sharp \varphi \right\|_{H^1} \le C \left\| \varphi \right\|_{H^1}, \quad (t \ge 0), \tag{2.42}$$

where the constant C is independent of n and $t \ge 0$.

Proof. From (2.41) and (2.19), we have

$$e^{t\mathbf{L}}P_c^\sharp\varphi = e^{t\mathbf{L}}P_c^{\mathbf{L}}\varphi - \sum_{k<m}\Big[\bar{c}_k(\sigma_1\sigma_3\Phi_k,\varphi)e^{-\bar{\lambda}_k t}\sigma_3\bar{\Phi}_k + c_k(\sigma_1\sigma_3\bar{\Phi}_k,\varphi)e^{-\lambda_k t}\sigma_3\Phi_k\Big]. \tag{2.43}$$

By Lemma 2.5, we have $\|\Phi_k\|_{H^1} = O(1)$ for all $k < m$. From this and $\operatorname{Re}\lambda_k > 0$ for all $k < m$, we can find a constant $C > 0$ independent of n such that

$$\left\|e^{t\mathbf{L}}P_c^\sharp\varphi\right\|_{H^1} \le \left\|e^{t\mathbf{L}}P_c^{\mathbf{L}}\varphi\right\|_{H^1} + C\left\|\varphi\right\|_{H^1}. \tag{2.44}$$

Moreover, by following the proof of [33, (2.6)], we see that there exists a constant C independent of n such that

$$\left\|e^{t\mathbf{L}}P_c^{\mathbf{L}}\varphi\right\|_{H^1} \le C\left\|P_c^{\mathbf{L}}\varphi\right\|_{H^1}. \tag{2.45}$$

Again, since $\|\Phi_k\|_{H^1} = O(1)$ for all k, we also have $\left\|P_c^{\mathbf{L}}\varphi\right\|_{H^1} \le C\left\|\varphi\right\|_{H^1}$ for some constant C which is independent of n. From this, (2.44), and (2.45), Lemma 2.8 follows. □

Lemma 2.9 *If a scalar function η satisfies $[\eta] \in \mathbf{E}_c^\sharp$, then*

$$\left\|\eta - P_c^H\eta\right\|_{L^\infty_{3r}} \lesssim n^2\left\|\eta\right\|_{L^2_{loc}} + \sum_{k<m}|(\bar{u}_k^-, P_c^H\eta)|. \tag{2.46}$$

Proof. Write $\eta' = P_c^H\eta$ and

$$\eta - \eta' = (1 - P_c^H)\eta = \textstyle\sum_k(\tilde{\phi}_k,\eta)\tilde{\phi}_k. \tag{2.47}$$

For $k \ge m$, $|(\tilde{\phi}_k,\eta)| \le O(n^2\left\|\eta\right\|_{L^2_{loc}})$. For $k < m$, by Lemma 2.6 (ii),

$$(\tilde{\phi}_k,\eta) + O(n^2\left\|\eta\right\|_{L^2_{loc}}) = (u_k^+,\eta) = (u_k^-,\bar{\eta}) = (u_k^-,\bar{\eta}') + (u_k^-,\bar{\eta}-\bar{\eta}'). \tag{2.48}$$

Since $\left\|u_k^-\right\|_{L^2_{loc}} \lesssim n^2$,

$$(u_k^-,\bar{\eta}-\bar{\eta}') = \textstyle\sum_{j=0}^K(u_k^-,(\tilde{\phi}_j,\eta)\tilde{\phi}_j) = O(n^2\left\|\eta\right\|_{L^2_{loc}}). \tag{2.49}$$

The above show the lemma. □

The following lemma provides decay estimates for $e^{-itH}u_j^-$.

Lemma 2.10 *Let H_* be the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Let V be a localized real potential so that H_*+V satisfies the decay and singular decay estimates* (2.3) *and* (2.4)*. Let $0 < n < n_0 \ll 1$, $a > 0$, and $z = a + n^4 i$. Let $\varphi(t) = n^2(H_* + V - z)^{-1}e^{-it(H_*+V)}P_c g$ with $\left\|g\right\|_{L^1} \le 1$ and $P_c = P_c^{H_*+V}$. Then for all $p \in (3,\infty]$, $m = \frac{1}{2} - \frac{3}{2p} \in [0, 1/2]$,*

$$\left\|\varphi(t)\right\|_{L^p} \lesssim t^{-m}(1+t)^{-m-\min(m,1/4)}, \quad \forall t > 0. \tag{2.50}$$

Above the p-dependent constant is uniform in $a \in [a_1, a_2] \subset (0,\infty)$ and independent of t and n.

Proof. The case $V = 0$ is postponed to Subsection 2.4. For general case $V \neq 0$, denote $R_0 = (H_* - z)^{-1}$, $R = (H_* + V - z)^{-1}$, $S_0(t) = e^{-itH_*}$ and $S(t) = e^{-it(H_*+V)}$. By resolvent expansion and Duhamel's formula,

$$\varphi(t) = n^2(R_0 + R_0VR_0 + R_0VRVR_0)\left(S_0(t) + \int_0^t S_0(t-s)VS(s)\,ds\right)P_c g.$$

By the estimate for $V=0$ case, $\left\|n^2 R_0 S_0(t) P_c g\right\|_{L^p} \lesssim \tilde{\alpha}_p(t) := t^{-m}(1+t)^{-m-\min(m,1/4)}$. By (2.30), $p>3$, and $(L^2_r; L^2_{-r})$-estimate of R,

$$\left\|n^2 R_0(V+VRV)R_0 S_0(t) P_c g\right\|_{L^p} \lesssim \left\|n^2 |V|^{1/2} R_0 S_0(t) P_c g\right\|_{L^2} \lesssim \tilde{\alpha}_p(t).$$

Thus, also by (2.3) with $q=\infty$, and $\|V\|_{L^\infty\to L^1} \lesssim 1$,

$$\|\varphi(t)\|_{L^p} \lesssim \tilde{\alpha}_p(t) + \int_0^t \tilde{\alpha}_p(t-s)\,\langle s\rangle^{-3/2}\,ds \lesssim \tilde{\alpha}_p(t). \tag{2.51}$$

$\square$

The following is the main result of this subsection.

Lemma 2.11 (Decay estimate) *For any scalar function* $\varphi \in L^{9/8}\cap L^{3/2}$,

$$\left\|e^{t\mathbf{L}} P_c^\sharp[\varphi]\right\|_{L^\infty+L^2} \le C\alpha_\infty(t)\,\|\varphi\|_{L^{9/8}\cap L^{3/2}}\,, \quad (t\ge 0). \tag{2.52}$$

For $3<p<6$ *and any scalar function* $\varphi \in L^{p'}$,

$$\left\|e^{t\mathbf{L}} P_c^\sharp[\varphi]\right\|_{L^p} \le C_p\alpha_p(t)\,\|\varphi\|_{L^{p'}}\,, \quad (t\ge 0). \tag{2.53}$$

Above the constants are independent of n *and* φ, *and*

$$\alpha_\infty(t) := t^{-1/2}\langle t\rangle^{-2/3}\,, \quad \alpha_p(t) := t^{-\frac{3}{2}+\frac{3}{p}}\langle t\rangle^{\frac{3}{2p}}\,. \tag{2.54}$$

Remark. (i) For (2.52) we could have chosen $\varphi \in L^q\cap L^{3/2}$, $\frac{12}{11}\le q<\frac{6}{5}$. Then $\alpha_\infty(t) = t^{-1/2}\langle t\rangle^{-s}$, with $s=3/q-2\in(1/2,3/4]$ by the same proof. The exponent $q=\frac{12}{11}$ gives the optimal decay rate that Lemma 2.10 provides for $e^{-itH}P_c u_j^-$. However, when we estimate $\left\|\eta^3\right\|_{L^q} \lesssim \|\eta\|_{L^2}^{3-3\theta}\|\eta\|_{L^p}^{3\theta}$, we prefer a larger q. For convenience we choose $q=9/8$.

(ii) Suppose we keep $q=\frac{12}{11}$ with $\alpha_\infty(t)=t^{-1/2}\langle t\rangle^{-3/4}$, and estimate $\left\|\eta^3\right\|_{L^{12/11}} \lesssim \|\eta\|_{L^2}^{3-3\theta}\|\eta\|_{L^p}^{3\theta} \lesssim \alpha_\infty(t)$, we need $\frac{11}{2}<p<6$.

(iii) These estimates are false if $P_c^\sharp$ is replaced by P_c. Suppose the contrary, then they would be also true if $P_c^\sharp$ is replaced by $P_d^\sharp = P_c^\sharp - P_c$. Consider the case $m=1$ and $\varphi=\phi_0$ the e_0-eigenfunction of $-\Delta+V$. Then

$$\left\|e^{t\mathbf{L}} P_d^\sharp[\varphi]\right\|_{L^p} \sim e^{-cn^4 t}, \quad \|\varphi\|_{L^{p'}} \sim 1. \tag{2.55}$$

However the former is not bounded by Ct^{-k} for all $t>0$, for any $k>0$ and C independent of n.

Proof. Denote $\eta(t) = e^{t\mathbf{L}}P_c^\sharp[\varphi]$ and $\eta' = P_c^{JH}\eta$. Lemma 2.9 implies

$$\|\eta\|_X \lesssim \left\|\eta'\right\|_X + \textstyle\sum_{k<m}|(\bar{u}_k^-,\eta')|, \quad X=L^\infty+L^2. \tag{2.56}$$

Denote $\mathbf{L} = JH+W_1$ with $W_1 = \begin{bmatrix} 0 & 0 \\ -2\kappa Q_m^2 & 0\end{bmatrix}$. By Duhamel's formula,

$$\eta'(t) = e^{tJH}P_c^{JH}P_c^\sharp[\varphi] + \int_0^t P_c^{JH} e^{(t-s)JH} W_1\eta(s)\,ds. \tag{2.57}$$

By Lemma 2.6 (i),

$$\jmath^{-1}P_c^\sharp[\varphi] = \varphi - \jmath^{-1}\operatorname{Re}\textstyle\sum_{j=0}^K z_j\Phi_j = \varphi - \sum_{j=0}^K (z_j\bar u_j^+ + \bar z_j u_j^-) \tag{2.58}$$

where $z_j \in \mathbb{C}$ are bounded by $\|\varphi\|_{L^q}$ for any $q \le 2$. Using (2.18) for $j < m$ in particular $u_j^- = (H - i\bar\lambda_j)^{-1}\phi_j^* + O_{L^\infty_{3r_1}}(n^2)$, $\phi_j^* = O_{L^\infty_{3r_1}}(n^2)$, $\operatorname{Im} i\bar\lambda_j \sim n^4$, and by Lemma 2.2 and Lemma 2.10 (with $p = \infty$),

$$\left\|\eta'(t)\right\|_X \lesssim \alpha(t)\left\|\varphi\right\|_Y + \int_0^t \langle t-s\rangle^{-3/2} n^2 \left\|\eta(s)\right\|_X ds, \tag{2.59}$$

where $\alpha(t) = t^{-1/2}\langle t\rangle^{-2/3}$ and $Y = L^{9/8} \cap L^{3/2}$. By the same reasons,

$$|(\bar u_k^-, \eta')| = (\bar\phi_k^*, (H - i\bar\lambda_k)^{-1}\eta') + O(n^2\left\|\eta'\right\|_X), \tag{2.60}$$

and

$$|(\bar\phi_k^*, (H - i\bar\lambda_k)^{-1}\eta')| \lesssim n^2 \left\|(H - i\bar\lambda_k)^{-1}\eta'\right\|_{L^2_{loc}} \lesssim n^2 \cdot \text{ RHS of (2.59)}. \tag{2.61}$$

Summing the estimates, we get $\|\eta(t)\|_X \lesssim$ RHS of (2.59), which implies (2.52).

The estimate (2.53) is proved similarly with $X = L^p$, $Y = L^{p'}$ and $\alpha(t) = \alpha_p(t) \sim \max(\tilde\alpha_p(t), t^{-3(\frac12 - \frac1p)})$. ☐

2.4 Decay estimate for free evolution with resonant data

In this subsection we prove Lemma 2.10 for $H_* = -\Delta$, i.e. decay estimate for $\varphi(t) = n^2(H_* - z)^{-1}e^{-tH_*}g$ where $z = a + n^4 i$, $a \sim 1$, and $g \in L^1$. The operator $(H_* - z)^{-1}e^{-itH_*}$ has symbol $(\xi^2 - z)^{-1}e^{-it\xi^2}$ and thus its Green's function G is radial and, for $r = |x|$,

$$\begin{aligned} G(r,t) &= (2\pi)^{-3}\int_0^\infty (p^2 - z)^{-1}e^{-itp^2}\int_{|\omega|=1} e^{ipr\omega_1} dS(\omega)\, p^2 dp \\ &= (2\pi)^{-3}\int_0^\infty (p^2 - z)^{-1}e^{-itp^2} 4\pi \frac{\sin(rp)}{rp}\, p^2 dp \\ &= \frac{1}{4\pi^2 ir}\int_{\mathbb{R}} (p^2 - z)^{-1}e^{-itp^2}e^{irp}\, pdp. \end{aligned}$$

It is well known that $G(r,0) = \frac{1}{4\pi r}e^{i\sqrt{z}r}$. We are not aware of an explicit formula for $G(r,t)$. Because for $3 < p \le \infty$ we have

$$\|\varphi(t)\|_{L^p} = \left\|n^2 G(t) * g\right\|_{L^p} \lesssim n^2 \|G(t)\|_{L^p}\|g\|_{L^1} \lesssim n^2 \|G(t)\|_{L^{3,\infty}}^{3/p}\|G(t)\|_{L^\infty}^{1-3/p}\|g\|_{L^1}, \tag{2.62}$$

estimate (2.50) follows from (2.64) of the following lemma.

Lemma 2.12 *Let H_* be the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Let $G(x,t)$ be the Green's function of the operator $(H_* - z)^{-1}e^{-itH_*}$ where z is the same as in Lemma 2.10. Then $G(x,t) = G(|x|,t)$ and*

$$|G(r,t)| \lesssim \begin{cases} \dfrac{r^{-1/2}}{n^4 r + r^{1/2} + (t-r)_+}, & r > 1, \dfrac{t}{100}, \\ t^{-3/2}, & 1 < r < \dfrac{t}{100}, \\ \min(t^{-1/2}(1+t)^{-1},\ r^{-1}), & r < 1. \end{cases} \tag{2.63}$$

In particular,

$$\|G(\cdot,t)\|_{L^\infty_x} \lesssim t^{-1/2}(1+t)^{-1/2}(1+n^4t^{1/2})^{-1}, \quad \|G(\cdot,t)\|_{L^{3,\infty}_x} \lesssim 1. \tag{2.64}$$

Proof. We may assume $a = 1/4$. The general case follows from change of variables and is uniform for $a \in [a_1, a_2]$. Introduce a regularizing factor $e^{-\delta p^2}$ and write $(p^2-z)^{-1}$ as a time integral (using $\operatorname{Re} z > 0$)

$$\begin{aligned} G(r,t) &= \lim_{\delta\to 0_+} \frac{1}{4\pi^2 r}\int_{\mathbb{R}}\int_0^\infty e^{-itp^2-\delta p^2-is(p^2-z)+irp}ds\, pdp \\ &= \lim_{\delta\to 0_+} \frac{1}{4\pi^2 r}\int_0^\infty e^{isz+\frac{ir^2}{4\alpha}}\int_{\mathbb{R}} e^{-i\alpha(p-\frac{r}{2\alpha})^2}\, pdp\, ds, \quad \alpha = s+t-i\delta. \end{aligned} \tag{2.65}$$

Using $\int_{\mathbb{R}} e^{-p^2}dp = \sqrt{\pi}$ and

$$\int_{\mathbb{R}} e^{-i\alpha(p-\beta)^2}pdp = \int_{\mathbb{R}} e^{-i\alpha(p-\beta)^2}\beta dp = \beta\int_{\mathbb{R}} e^{-i\alpha p^2}dp = \beta(i\alpha)^{-1/2}\sqrt{\pi}, \tag{2.66}$$

we get

$$\begin{aligned} G(r,t) &= \lim_{\delta\to 0_+}\frac{1}{4\pi^2 r}\int_0^\infty e^{isz+\frac{ir^2}{4\alpha}}\frac{r}{2\alpha}(i\alpha)^{-1/2}\sqrt{\pi}\, ds \\ &= \frac{1}{8\pi^{3/2}\sqrt{i}}\int_0^\infty e^{isz+\frac{ir^2}{4(s+t)}}(s+t)^{-3/2}\, ds \\ &= \frac{1}{8\pi^{3/2}\sqrt{i}}\int_t^\infty e^{i\Phi}s^{-3/2}\, ds, \end{aligned} \tag{2.67}$$

where the phase Φ is

$$\Phi(r,s) = sz - tz + \frac{r^2}{4s}, \quad \Phi_s = z - \frac{r^2}{4s^2}, \quad \Phi_{ss} = \frac{r^2}{2s^3}. \tag{2.68}$$

Note $z = \frac14 + n^4 i$, Φ_s vanishes at $s = r/(2\sqrt{z}) \sim r$, and $\operatorname{Re} i\Phi < 0$ for $s > t$.

First note

$$|G(r,t)| \lesssim \int_t^\infty s^{-3/2}\, ds = Ct^{-1/2}, \tag{2.69}$$

which is valid for all $r > 0$ and $t > 0$. We will use a stationary phase argument to get a better estimate. The main contribution should come from $I \equiv r(1-\mu, 1+\mu)$ where $0 < \mu \le \frac{1}{200}$ will be chosen. Comparing (2.69) and (2.70) below, it is clear we do not get a better estimate unless μ is small.

We first consider the case $r > 1$.

Suppose $t \in I$. The contribution from $s \in (t, r+\mu r)$ is bounded by

$$\left|\int_t^{r+\mu r} e^{i\Phi}s^{-3/2}ds\right| \lesssim \int_I r^{-3/2}ds \lesssim \mu r^{-1/2}. \tag{2.70}$$

The contribution from $(r+\mu r, \infty)$ is, with $t_1 = r+\mu r$,

$$\int_{t_1}^\infty e^{i\Phi}s^{-3/2}ds = \int_{t_1}^\infty \partial_s(e^{i\Phi})\frac{1}{i\Phi_s}s^{-3/2}ds = \frac{1}{i\Phi_s}e^{i\Phi}s^{-3/2}|_{s=t_1} + \int_{t_1}^\infty e^{i\Phi}Jds, \tag{2.71}$$

where

$$J = -\frac{\partial}{\partial s}(\frac{1}{i\Phi_s}s^{-3/2}) = \frac{\Phi_{ss}}{i(\Phi_s)^2}s^{-3/2} + \frac{3}{2i\Phi_s s^{5/2}}. \tag{2.72}$$

For $s \geq t_1$, we have $|\Phi_s| \sim n^4 + (s-r)/r$ and $|\Phi_{ss}| \lesssim s^{-1}$. Thus $|J| \lesssim (|\Phi_s|^{-1} + |\Phi_s|^{-2})s^{-5/2}$, and the boundary term is bounded by

$$|\frac{1}{i\Phi_s}e^{i\Phi}s^{-3/2}|_{s=t_1}| \lesssim \frac{1}{|\Phi_s(t_1)|}t_1^{-3/2} \lesssim \frac{r^{-3/2}}{n^4+\mu}. \tag{2.73}$$

Decompose $(t_1, \infty) = (t_1, 100r) \cup (100r, \infty)$. On $(t_1, 100r)$, we have

$$|\int_{t_1}^{100r} e^{i\Phi} J ds| \lesssim \int_{t_1}^{100r} \frac{r^{2-5/2}}{(n^4r + s - r)^2} ds \lesssim \frac{r^{-1/2}}{n^4 r + t_1 - r} = \frac{r^{-3/2}}{n^4+\mu}. \tag{2.74}$$

For $s > 100r$, we have $|\Phi_s| \gtrsim 1$ and

$$|\int_{100r}^{\infty} e^{i\Phi} J ds| \lesssim \int_{100r}^{\infty} s^{-5/2} ds \lesssim r^{-3/2}. \tag{2.75}$$

We now choose $\mu \leq \frac{1}{200}$ so that $\mu r^{-1/2} \sim \frac{r^{-3/2}}{n^4+\mu}$. If $r \geq 1$, we can choose $\mu = \frac{1}{200}r^{-1/2}(1 + n^8 r)^{-1/2}$ and get for $t/r \in (1-\mu, 1+\mu)$

$$|G(r,t)| \leq \frac{r^{-1/2}}{n^4 r + r^{1/2}}. \tag{2.76}$$

If $t \in (r+\mu r, 100r)$, we can take $t_1 = t$ in the above estimates and ignore the contribution from (2.70) to get the bound for $r > 1$

$$|G(r,t)| \lesssim \frac{r^{-1/2}}{n^4 r + |t-r|}. \tag{2.77}$$

If $t > 100r$, we can replace $100r$ by t in (2.75) and ignore the contribution from (2.70) and (2.74) to get (also true for $r < 1$),

$$|G(r,t)| \lesssim t^{-3/2}. \tag{2.78}$$

If $t \in (\frac{r}{100}, r - \mu r)$ and $r > 1$, the additional contribution from $s \in (t, r-\mu r)$ is estimated as in (2.71)–(2.74) with $t_1 = r - \mu r$ and $100r$ replaced by $r/100$, and bounded by (2.77), which is smaller than (2.76) for $r > 1$.

If $t \in (0, \frac{r}{100})$, we have $|\Phi_s| \sim r^2 s^{-2}$ and $|\Phi_{ss}| \sim r^2 s^{-3}$ for $s \in (t, \frac{r}{100})$. The additional contribution from $s \in (t, \frac{r}{100})$ is estimated as in (2.71)–(2.74) and bounded by

$$\left[r^{-2}s^{1/2}\right]_{s=t}^{r/100} + \int_{s=t}^{r/100} r^{-2}s^{-1/2} ds \leq r^{-3/2} \tag{2.79}$$

which is smaller than (2.76) for $r > 1$.

We now consider the case $r < 1$. Let $\alpha > 0$ be a small number to be chosen. The contribution from $s \geq \max(t, \alpha r)$ is bounded by

$$|\int_{\alpha r}^{\infty} e^{i\Phi} s^{-3/2} 1_{s>t}\, ds| \leq |\int_{\alpha r}^{\infty} s^{-3/2} ds| = C(\alpha r)^{-1/2}. \tag{2.80}$$

If $t < \alpha r$, we have $|\Phi_s|^{-1} \sim r^{-2}s^2$, $|\Phi_{ss}|/|\Phi_s| \lesssim s^{-1}$, and the contribution from $s < \alpha r$ is

$$\int_t^{\alpha r} e^{i\Phi} s^{-3/2}\, ds = \left[\frac{1}{i\Phi_s} e^{i\Phi} s^{-3/2} \right]_{s=t}^{s=\alpha r} + \int_t^{\alpha r} e^{i\Phi} J ds, \tag{2.81}$$

which is bounded by

$$r^{-2}(\alpha r)^{1/2}. \tag{2.82}$$

We want $(\alpha r)^{-1/2} \sim r^{-2}(\alpha r)^{1/2}$ and we can choose $\alpha = \frac{r}{100}$, which gives r^{-1} bound for $r < 1$.

In conclusion, we have proved (2.63) for all $r > 0$ and $t > 0$. $\square$

Remark. (i) Lemma 2.10 for the free case can be considered an estimate of $(f, n^2 G(t) g)$. If (2.63) cannot be improved, then Lemma 2.10 cannot be improved, even if assuming further that one of f, g is in L^2_r (but not both). To see it, let g be the characteristic function of the unit ball. Note $|I| \sim \mu r \gg 1$ for $r \gg 1$, thus $(n^2 G g)(r, t)$ has the optimal size at $r \sim t$. Since translation does not change the $L^1 \cap L^2$-norm of f, we can put the support of f at $r \sim t$, showing the optimality of Lemma 2.10.

(ii) Although the real part of the phase, $e^{-n^4(s-t)}$, is decaying, it does not seem to improve our estimate. In the case $t \sim r \sim n^{-8}$, we have $|I| \sim \mu r \sim n^{-4}$ and the estimate (2.70) does not improve because of the factor $e^{-n^4(s-t)}$, in view of the identity $\int_0^{n^{-4}} e^{-n^4 s} ds = C \int_0^{n^{-4}} ds$.

(iii) Since $|\operatorname{Im} \Phi_s| \sim |s - r|/r \lesssim \mu$ for $s \in I$, $e^{i\Phi}$ almost has no oscillation on I if $\mu^2 r \sim \mu \cdot |I| \ll 1$. Thus, if $\mu = \varepsilon r^{-1/2}$ with $0 < \varepsilon \ll 1$, then the upper bound in (2.70) is also a lower bound. In the case $t \sim r \gg \varepsilon^{-2} n^{-8}$, we have $\mu \ll n^4$ and $\mu r^{-1/2} \gg \frac{r^{-3/2}}{n^4} \sim \frac{r^{-3/2}}{n^4 + \mu}$. Thus (2.63) is optimal in this case.

2.5 Singular decay estimate

We will need to identify the main part of

$$\eta(t) = \int_0^t e^{(t-s)\mathcal{L}} P_c^\sharp e^{-i\alpha s} f(s) ds \tag{2.83}$$

where $\alpha \in \mathbb{C}$ with $\operatorname{Im} \alpha > 0$ and $f(s)$ is an L^2-valued function of s with $\dot{f}$ smaller than f in a suitable sense. We will rewrite it in matrix form in order to integrate by parts. Using

$$[\varphi] = \begin{bmatrix} \operatorname{Re} \varphi \\ \operatorname{Im} \varphi \end{bmatrix} = \operatorname{Re} \varphi \begin{bmatrix} 1 \\ -i \end{bmatrix}, \tag{2.84}$$

and denoting $R = (\mathbf{L} + i\alpha)^{-1}$, we have

$$\begin{aligned} \eta(t) &= \jmath^{-1} P_c^\sharp \int_0^t e^{(t-s)\mathbf{L}} \operatorname{Re} e^{-i\alpha s} f(s) \begin{bmatrix} 1 \\ -i \end{bmatrix} ds \\ &= \jmath^{-1} P_c^\sharp \operatorname{Re} \Bigg(- R e^{-i\alpha t} f(t) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &\qquad + e^{t\mathbf{L}} R f(0) \begin{bmatrix} 1 \\ -i \end{bmatrix} + \int_0^t e^{(t-s)\mathbf{L}} R e^{-i\alpha s} \dot{f}(s) \begin{bmatrix} 1 \\ -i \end{bmatrix} ds \Bigg). \end{aligned} \tag{2.85}$$

To estimate the last two terms, we need the following lemma.

Lemma 2.13 (Singular decay estimate) *There is a constant $C>0$ independent of $\alpha\in\mathbb{C}$ with $\operatorname{Im}\alpha>0$, $n\in[0,n_0]$, and vector function $\Psi\in L^2_r$, $r=3r_1$, so that*

$$\left\|\jmath^{-1}\operatorname{Re}e^{t\mathbf{L}}(\mathbf{L}+i\alpha)^{-1}P_c^\sharp\Psi\right\|_{L^2_{loc}}\le C\langle t\rangle^{-3/2}\|\Psi\|_{L^2_r},\quad (t\ge 0). \tag{2.86}$$

Proof. Denote by η the scalar function to be estimated, $\eta(t)=\jmath^{-1}\operatorname{Re}e^{t\mathbf{L}}RP_c^\sharp\Psi$, and $\eta'=P_c^H\eta$. Lemma 2.9 implies

$$\|\eta\|_{L^2_{loc}}\lesssim\|\eta'\|_{L^2_{loc}}+\textstyle\sum_{k<m}|(\bar u_k^-,\eta')|. \tag{2.87}$$

Denote $\mathbf{L}=JH+W_1$ with $W_1=\left[\begin{smallmatrix}0&0\\-2\kappa Q_m^2&0\end{smallmatrix}\right]$, $R=(\mathbf{L}+i\alpha)^{-1}$ and $R_0=(JH+i\alpha)^{-1}$. By Duhamel's formula and resolvent expansion,

$$\eta'(t)=P_c^H\jmath^{-1}\operatorname{Re}\left(e^{tJH}R_0(1+W_1R)P_c^\sharp\Psi+\int_0^t e^{(t-s)JH}W_1\eta(s)\,ds\right). \tag{2.88}$$

Denote the first term on the right side by $\eta_1'(t)$. Using $P_c^\sharp\Psi=\Psi-\sum_k P_k(\Psi)$,

$$\eta_1'(t)=\jmath^{-1}\operatorname{Re}e^{tJH}R_0P_c^{JH}\left(\Psi-\textstyle\sum_{k<m}P_k(\Psi)+\Psi_1\right), \tag{2.89}$$

where $\Psi_1=P_c^{JH}[-\sum_{k\ge m}P_k\Psi+W_1RP_c^\sharp\Psi]$ is localized with

$$\|\Psi_1\|_{L^2_r}\lesssim n^2\|\Psi\|_{L^2}+n^2\left\|RP_c^\sharp\Psi\right\|_{L^2_{loc}}\lesssim n^2\|\Psi\|_{L^2_r}. \tag{2.90}$$

Note that

$$e^{tJH}=\begin{bmatrix}\cos(tH)&\sin(tH)\\-\sin(tH)&\cos(tH)\end{bmatrix}=\sum_{\varepsilon=\pm1}e^{i\varepsilon tH}\tfrac12(I-i\varepsilon J), \tag{2.91}$$

$$(JH+i\alpha)^{-1}=(H^2-\alpha^2)^{-1}(-JH+i\alpha), \tag{2.92}$$

and

$$(I-i\varepsilon J)(-JH+i\alpha)=-\varepsilon i(H-\varepsilon\alpha)(I-\varepsilon iJ). \tag{2.93}$$

We conclude, for $R_0=(JH+i\alpha)^{-1}$,

$$e^{tJH}R_0=\sum_{\varepsilon=\pm1}e^{i\varepsilon tH}(H+\varepsilon\alpha)^{-1}\tfrac{-\varepsilon i}{2}(I-\varepsilon iJ). \tag{2.94}$$

By (2.94), (2.90), Lemma 2.2, and $\operatorname{Im}\alpha>0$,

$$\left\|\jmath^{-1}\operatorname{Re}e^{tJH}R_0P_c^{JH}(\Psi+\Psi_1)\right\|_{L^2_{loc}}\lesssim\langle t\rangle^{-3/2}\|\Psi\|_{L^2_r}. \tag{2.95}$$

For $k<m$, note

$$(I+iJ)\Phi_k=2\bar u_k^+\left[\begin{smallmatrix}1\\-i\end{smallmatrix}\right],\quad (I+iJ)\bar\Phi_k=2u_k^-\left[\begin{smallmatrix}1\\-i\end{smallmatrix}\right]. \tag{2.96}$$

Using (2.94) and writing $P_k\Psi=a\Phi_k+b\bar\Phi_k$, we have

$$\begin{aligned}\operatorname{Re}e^{tJH}R_0P_k\Psi&=\operatorname{Re}\textstyle\sum_{\varepsilon=\pm1}e^{i\varepsilon tH}(H+\varepsilon\alpha)^{-1}\frac{-\varepsilon i}{2}(I-\varepsilon iJ)P_k\Psi\\&=\operatorname{Re}e^{-itH}\left\{(H-\alpha)^{-1}i(a\bar u^++bu^-)+(H+\bar\alpha)^{-1}i(\bar b\bar u^++\bar u^-)\right\}\left[\begin{smallmatrix}1\\-i\end{smallmatrix}\right].\end{aligned}$$

By (2.84),

$$\jmath^{-1}\operatorname{Re} e^{tJH} R_0 P_k \Psi = e^{-itH}\left\{(H-\alpha)^{-1} i(a\bar{u}^+ + bu^-) + (H+\bar{\alpha})^{-1} i(\bar{b}\bar{u}^+ + \bar{a}u^-)\right\}. \quad (2.97)$$

Note $\operatorname{Im}(-\bar{\alpha}) = \operatorname{Im}\alpha > 0$. By Lemma 2.2 and (2.18),

$$\left\|P_c^H \jmath^{-1}\operatorname{Re} e^{tJH} R_0 P_k \Psi\right\|_{L^2_{loc}} \lesssim n^2 \langle t\rangle^{-3/2} \|\Psi\|_{L^2}. \quad (2.98)$$

Thus

$$\left\|\eta'(t)\right\|_{L^2_{loc}} \lesssim \langle t\rangle^{-3/2} \|\Psi\|_{L^2_r} + \int_0^t \langle t-s\rangle^{-3/2} n^2 \|\eta(s)\|_{L^2_{loc}} \, ds. \quad (2.99)$$

On the other hand, for $j < m$, by (2.18) again,

$$|(\bar{u}_j^-, \eta')| = (\bar{\phi}_j^*, (H - i\bar{\lambda}_j)^{-1}\eta') + O(n^2 \left\|\eta'\right\|_{L^2_{loc}}). \quad (2.100)$$

Note $\operatorname{Im} i\bar{\lambda}_j > 0$. By Lemma 2.2 and the previous decomposition of η',

$$|(\bar{\phi}_j^*, (H - i\bar{\lambda}_j)^{-1}\eta'(t))| \lesssim n^2 \left\|(H - i\bar{\lambda}_j)^{-1}\eta'\right\|_{L^2_{loc}} \lesssim n^2 \cdot \text{ RHS of (2.99)}. \quad (2.101)$$

By (2.87) and summing the estimates, we get $\|\eta(t)\|_{L^2_{loc}} \lesssim$ RHS of (2.99), which implies the lemma. □

2.6 Upper and lower spectral projections

In this subsection we prove various estimates for the spectral projections $\Pi_\pm$ which are defined in (2.104) and corresponds to $\pm\operatorname{Im} z \geq |E|$ in the spectrum of $\mathbf{L}$. In particular, Lemma 2.16 allows us to replace $P_c^\sharp$ by $P_\pm = P_c^\sharp \Pi_\pm$ in Lemmas 2.11 and 2.13.

Decompose $\mathbf{L} = JA + W_2 = JH + W_1$ where $A = -\Delta + |E|$, $W_2 = J(V + \kappa Q^2) + W_1$, and $W_1 = \begin{bmatrix} 0 & 0 \\ -2\kappa Q^2 & 0 \end{bmatrix}$. Let $R(z) = (\mathbf{L} - z)^{-1}$, $R_0(z) = (JA - z)^{-1}$ and $R_1(z) = (JH - z)^{-1}$ be their resolvents. Note $R_0(z)$ can be decomposed as

$$\begin{aligned} R_0(z) = (JA - z)^{-1} &= \begin{bmatrix} -z & A \\ -A & -z \end{bmatrix}^{-1} = (A^2 + z^2)^{-1} \begin{bmatrix} -z & -A \\ A & -z \end{bmatrix} \\ &= (A - iz)^{-1} M + (A + iz)^{-1}\bar{M}, \quad M = \frac{1}{2}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}. \end{aligned} \quad (2.102)$$

$R_1(z)$ has a similar formula with A replaced by H.

Let $\Gamma_{c\pm}$ be contours about the upper and lower continuous spectra $\Sigma_\pm = \pm[|E|i, +\infty i)$, respectively. For an eigenvalue λ of $\mathbf{L}$, let Γ_λ be a small circle centered at λ with radius $\sim n^4$. All contours are oriented clockwise and do not intersect. Let $P_* = \frac{1}{2\pi i}\int_{\Gamma_*} R(z)dz$, $* = c\pm, \lambda$, be their corresponding spectral projections. Note $P_{c\pm}$ are well defined in L^2 and L^p by the boundedness of wave operators between $\mathbf{L}P_c^{\mathbf{L}}$ and JHP_c^{JH} proved in [33], although the bounds depend on n. Decompose $P_c^\sharp$ as the sum of its upper and lower half plane components:

$$P_c^\sharp = P_+ + P_-, \quad P_\pm = P_{c\pm} + P_{L\pm}, \quad P_{L+} = \sum_{k<m} P_{-\bar{\lambda}_k}, \quad P_{L-} = \sum_{k<m} P_{-\lambda_k}. \quad (2.103)$$

Also denote

$$\Pi_\pm = P_\pm + P_{R\pm}, \quad P_{R+} = \sum_{k<m} P_{\lambda_k}, \quad P_{R-} = \sum_{k<m} P_{\bar\lambda_k}. \tag{2.104}$$

Note $P_\pm = P_c^\sharp \Pi_\pm$.

Let

$$\delta_0 = \frac{1}{4}\min\{|e_K|, |e_k - e_{k-1}| : 1 \le k \le K\}, \quad \tau_0 = \frac{1}{2}e_K - e_m. \tag{2.105}$$

Note $\operatorname{Im}\lambda_N < \tau_0 - \delta_0 < \tau_0 + \delta_0 < |E|$.

We collect a few estimates for $R_0(z)$ and $R(z)$.

Lemma 2.14 *Let $\sigma_d^0 = \{\pm i(e_k - e_m) : 0 \le k \le K\}$, $s > \frac{1}{2}$ and $1 \le p < \infty$. We have*

$$\begin{aligned}
&\|R_0(z)\|_{L^2_s \to L^2_{-s}} \le C\langle z\rangle^{-1/2}, \quad z \notin i\mathbb{R},\\
&\|R_1(z)\|_{L^2_s \to L^2_{-s}} + \|R(z)\|_{L^2_s \to L^2_{-s}} \le C\langle z\rangle^{-1/2}, \quad z \notin i\mathbb{R},\ \operatorname{dist}(z, \sigma_d^0) \ge \delta_0,\\
&\|R(z)\|_{L^2_s \to L^2_{-s}} \le Cn^{-4}, \quad 0 < |\operatorname{Re} z| < \frac{1}{4}\gamma_0 n^4,\ \operatorname{dist}(z, \sigma_d^0) < \delta_0,\\
&\|R_0(z)\|_{L^p \to L^p} + \|R_1(z)\|_{L^p \to L^p} + \|R(z)\|_{L^p \to L^p} \le C_p\langle z\rangle^{-1+\varepsilon_p}, \quad |\operatorname{Im} z| = \tau_0.
\end{aligned} \tag{2.106}$$

Above $\varepsilon_p = 0$ for $p > 1$ and $0 < \varepsilon_1 \ll 1$, and the constants are uniform in $n \in [0, n_0]$.

Proof. The first estimate is by the scalar case proved in [1, Remark 2 in Appendix A] and by (2.102). The second estimate for $R_1(z)$ is by the scalar case proved in [16, Theorem 9.2] and by (2.102) with A replaced by H. It is true for $R(z)$ using the resolvent series $R(z) = R_1(z)\sum_{k=0}^\infty [W_1 R_1(z)]^k$ and the fact W_1 is a small localized matrix potential. The third estimate is proved in [33, Lemma 2.5].

The last estimate for $R_0(z)$ is by the scalar case proved in [7, Lemma 7.4] and by (2.102). It is true for $R_1(z)$ because $\left\|(H-z)^{-1}\right\|_{L^p \to L^p} \lesssim \langle z\rangle^{-1+\varepsilon_p}$ for $|\operatorname{Im} z| = \tau_0$, which follows from

$$\begin{aligned}
(H-z)^{-1}f &= (H-z)^{-1}P_c f + (H-z)^{-1}\textstyle\sum_{k=0}^K (\tilde\phi_k, f)\tilde\phi_k\\
&= W^{-1}(A-z)^{-1}WP_c f + \textstyle\sum_{k=0}^K (\tilde e_k - z)^{-1}(\tilde\phi_k, f)\tilde\phi_k,
\end{aligned} \tag{2.107}$$

where W is the wave operator between H and A and $\tilde\phi_k$ are normalized eigenfunctions of H with eigenvalues $\tilde e_k$. Finally, the estimate for $R(z)$ follows from the resolvent series $R(z) = R_1(z)\sum_{k=0}^\infty [W_1 R_1(z)]^k$ again. □

Lemma 2.15 *Let $K_\pm = \Pi_\pm(J \mp i)$, initially defined from L^2_s to L^2_{-s}, $s > 1$. For any $1 \le p \le q < \infty$, there is a constant c so that $\|K_\pm u\|_p \le c\|u\|_q$ for any $u \in L^2_s \cap L^q$.*

This is clear for the reference self-adjoint operator JA, for which $K_\pm = 0$.

Proof. Recall R_0 is decomposed in (2.102), and $MJ = -iM$ and $\bar M J = i\bar M$. As z approaches $\Sigma_+ = [|E|i, +\infty i)$, the upper continuous spectrum of A, the resolvent $(A+iz)^{-1}$ is unbounded, and we write

$$R_0(z)J - iR_0(z) = -2iM(A - iz)^{-1}, \quad (z \sim \Sigma_+). \tag{2.108}$$

Note right side is bounded. Similarly, as z approaches $\Sigma_- = -\Sigma_+$, we write

$$R_0(z)J + iR_0(z) = 2i\bar M(A + iz)^{-1}, \quad (z \sim \Sigma_-). \tag{2.109}$$

We now prove the bound for K_+. The case of K_- is similar. Let $\Gamma = \Gamma_{c+} \cup \Gamma_p$ and $\Gamma_p = \cup_{k<m}(\Gamma_{\lambda_k} \cup \Gamma_{-\bar{\lambda}_k})$. By spectral projection formula and resolvent expansion,

$$\Pi_+ = \frac{1}{2\pi i}\int_\Gamma R(z)dz = \frac{1}{2\pi i}\int_\Gamma [1 + R_0(z)W_0 + R_0(z)W_0R(z)W_0]R_0(z)dz. \tag{2.110}$$

By (2.108),

$$\Pi_+(J-i) = \frac{-1}{\pi}\int_\Gamma [1+R_0(z)W_0+R_0(z)W_0R(z)W_0]M(A-iz)^{-1}dz = K_0+K_1+K_2. \tag{2.111}$$

The above sum is well-defined as operators from L^2_s to L^2_{-s}.

Note that K_0 is zero since $(A-iz)^{-1}$ is regular inside Γ and the rest of the integrand of K_0 does not depend on z.

For K_1, the integral over Γ_{c+} is bounded from L^q to L^p by Lemma 7.6 of Cuccagna [C2] using Coifman-Meyer multi-linear estimates. The integral over Γ_p is also bounded from L^q to L^p since

$$\begin{aligned} &\int_{\Gamma_p} \left\|R_0(z)W_0M(A-iz)^{-1}\right\|_{L^q\to L^p} |dz| \\ &\le \int_{\Gamma_p} \|R_0(z)\|_{L^p\to L^p} \left\|(A-iz)^{-1}\right\|_{L^q\to L^q} \lesssim \int_{\Gamma_p} n^{-4}\cdot 1 \lesssim 1. \end{aligned} \tag{2.112}$$

For K_2, the integrand is analytic in z and has enough decay in $B(L^2_s \to L^2_{-s})$ in $|z|$ by Lemma 2.14. Thus we can change the contour to $\Gamma_1 = \mathbb{R}+\tau_0 i$, By Lemma 2.14, $\|K_2\|_{L^q\to L^p}$ is bounded by

$$\int_{\Gamma_1} \|R_0(z)\|_{L^p\to L^p}\cdot\|R(z)\|_{L^q\to L^q}\cdot\|R_0(z)\|_{L^q\to L^q}\,|dz| \le C. \tag{2.113}$$

Summing the estimates we get the lemma. □

Lemma 2.16 *The projection operators $\Pi_\pm$ are bounded from L^2_s to L^2_{-s}, $s>1$, and from L^p to L^p for any $1\le p\le\infty$.*

Proof. From the definition of $K_\pm$ in Lemma 2.15, we have

$$K_+ = \Pi_+(J-i), \quad K_- = (1-\Pi_+-\Pi_0)(J+i), \tag{2.114}$$

where $\Pi_0 = \sum_{j\ge m} P_j$ is bounded in L^p. Thus

$$\Pi_+ = \frac{i}{2}[K_+ + K_- - (1-\Pi_0)(J+i)], \tag{2.115}$$

where shows Π_+ is bounded in L^p for $p<\infty$ by Lemma 2.15. Similarly Π_- and $\Pi^*_\pm$ are bounded in L^p for $p<\infty$. The boundedness of $\Pi_\pm$ in L^∞ follows from that of $\Pi^*_\pm$ in L^1 and duality. □

As a corollary, Lemmas 2.11 and 2.13 hold with $P_c^\sharp$ replaced by $P_\pm$ since $P_\pm = P_c^\sharp\Pi_\pm$.

2.7 Fermi Golden Rule

In this subsection we prove Corollary 2.20, which gives the key resonance coefficients in the normal form equations in Lemmas 3.7 and 3.8.

For any $k \neq m$, recall (2.36) that

$$\Phi_k = \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \bar{u}_k^+ + \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] \bar{u}_k^- . \tag{2.116}$$

From (2.18), we introduce Φ_k^+ and Φ_k^- which satisfy the equation $\Phi_k = \Phi_k^+ + \Phi_k^-$ where Φ_k^+ is localized and

$$\Phi_k^- = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] (H - \bar{\alpha}_k)^{-1} \bar{\phi}_k^*, \quad \Phi_k^+ = \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] \phi_k + O_{L_r^2}(n^2), \tag{2.117}$$

Note that $\phi_k^* = O_{L_r^2}(n^2)$ is defined in (2.18) and $\alpha_k = i\bar{\lambda}_k = |e_k - e_m| + O(n^2)$ with $\operatorname{Im} \alpha_k > 0$. Moreover, since $\Phi_k = \Phi_k^+ + \Phi_k^-$, from (2.19), we see that for all function $f \in L^2(\mathbb{R}^2, \mathbb{C}^2)$

$$\begin{aligned} P_k f &= c_k(\sigma_1 \bar{\Phi}_k, f)\Phi_k^+ + \bar{c}_k(\sigma_1 \Phi_k, f)\bar{\Phi}_k^+ + c_k(\sigma_1 \bar{\Phi}_k, f)\Phi_k^- + \bar{c}_k(\sigma_1 \Phi_k, f)\bar{\Phi}_k^-, \\ (P_k)^* f &= c_k(\bar{\Phi}_k, f)\sigma_1 \Phi_k^+ + \bar{c}_k(\Phi_k, f)\sigma_1 \bar{\Phi}_k^+ + c_k(\bar{\Phi}_k, f)\sigma_1 \Phi_k^- + \bar{c}_k(\Phi_k, f)\sigma_1 \bar{\Phi}_k^-. \end{aligned} \tag{2.118}$$

Since Φ_k^+ is localized and $\Phi_k^- = O_{L^2_{\mathrm{loc}}}(n^2)$, it follows from Lemma 2.3 and (2.118) that for all functions f such with $\|f\|_{L_r^2} = O(\delta)$

$$\begin{aligned} (P_k - P_k^{JH})f &= O(n^2\delta)\Phi_k + O(n^2\delta)\bar{\Phi}_k + O(\delta)\Phi_k^- + O(\delta)\bar{\Phi}_k^- + O_{L_r^2}(n^2\delta) \\ (P_k - P_k^{JH})^* f &= O(n^2\delta)\sigma_1\Phi_k + O(n^2\delta)\sigma_1\bar{\Phi}_k + O(\delta)\sigma_1\Phi_k^- + O(\delta)\sigma_1\bar{\Phi}_k^- + O_{L_r^2}(n^2\delta). \end{aligned} \tag{2.119}$$

Throughout this subsection, let ω and ϵ be two fixed numbers such that

$$\omega \pm \operatorname{Im} \lambda_k = O(1) \neq 0, \quad 0 < \epsilon \ll 1. \tag{2.120}$$

Let $\alpha = -i\omega + \epsilon$ and

$$R = (\mathbf{L} - \alpha)^{-1}, \quad R_0 = (JH - \alpha)^{-1}. \tag{2.121}$$

Note that we have

$$R = R_0 + R_0 W R_0 + R_0 W R W R_0, \tag{2.122}$$

where W is a localized potential which is of order $\|Q\|^2$.

Lemma 2.17 *For any $k \neq m$, there exist $C > 0$ independent of ϵ and n such that*

$$\left\|R_0 \Phi_k^-\right\|_{L^2_{\mathrm{loc}}}, \quad \left\|R_0 \bar{\Phi}_k^-\right\|_{L^2_{\mathrm{loc}}}, \quad \left\|(R_0)^* \sigma_1 \Phi_k^-\right\|_{L^2_{\mathrm{loc}}}, \quad \left\|(R_0)^* \sigma_1 \bar{\Phi}_k^-\right\|_{L^2_{\mathrm{loc}}} \leq C n^2. \tag{2.123}$$

Proof. We write

$$R_0 = (JH - \alpha)^{-1} = (H^2 + \alpha^2)^{-1} \begin{bmatrix} -\alpha & -H \\ H & -\alpha \end{bmatrix}. \tag{2.124}$$

Then, it follows that

$$\begin{aligned} R_0 \Phi_k^- &= \left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right] (H + i\alpha)^{-1} (H - \bar{\alpha}_k)^{-1} \bar{\phi}_k^* \\ R_0 \bar{\Phi}_k^- &= \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] (H - i\alpha)^{-1} (H - \alpha_k)^{-1} \phi_k^* \\ (R_0)^* \sigma_1 \Phi_k^- &= \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] (H + i\bar{\alpha})^{-1} (H - \bar{\alpha}_k)^{-1} \bar{\phi}_k^*, \\ (R_0)^* \sigma_1 \bar{\Phi}_k^- &= \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] (H - i\bar{\alpha})^{-1} (H - \alpha_k)^{-1} \phi_k^*. \end{aligned} \tag{2.125}$$

Since $\operatorname{Re} \alpha > 0$ and $\operatorname{Im}(\alpha_k) > 0$ and $\phi_k^* \in O_{L_r^2}(n^2)$, our claim follows. $\square$

Lemma 2.18 *There exists $C > 0$ such that for any function $f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $f, g = O_{L^2_r}(n)$:*

$$\begin{aligned} &|(f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp (P_c^\sharp - P_c^{JH}) g)| \le C n^4, \\ &|(f, (P_c^\sharp - P_c^{JH})(\mathbf{L}-\alpha)^{-1} P_c^\sharp g)| \le C n^4. \end{aligned} \tag{2.126}$$

Proof. Since the proofs of both estimates in (2.126) are similar, we shall only prove the first estimate. From (2.119), we have

$$(P_c^\sharp - P_c^{JH}) g = \sum_{k=0}^{K} \{O(n^3)\Phi_k + O(n^3)\bar{\Phi}_k + O(n)\Phi_k^- + O(n)\bar{\Phi}_k^-\} + O_{L^2_r}(n^3). \tag{2.127}$$

Since $\mathbf{L}\Phi_k = \lambda_k \Phi_k$ and $\lambda_k - \alpha$, $\bar{\lambda}_k - \alpha$ are all non-zero order one, we get

$$(f, P_c^\sharp R (P_c^\sharp - P_c^{JH}) g) = O(n^4) + (f, P_c^\sharp R[O(n)\Phi_k^- + O(n)\bar{\Phi}_k^-]). \tag{2.128}$$

By similarity, we only need to show that $|(f, P_c^\sharp R \Phi_k^-)| \le C n^3$. Let $\widetilde{g} = [W R_0 + W R W R_0]\Phi_k^-$. By Lemma 2.17, $\|\widetilde{g}\|_{L^2_r} \le C n^4$. Then, using (2.122), (2.118) and Lemma 2.17, we have

$$\begin{aligned} &|(f, P_c^\sharp R \Phi_k^-)| = |((P_c^\sharp)^* f, R_0 \Phi_k^- + R_0 \widetilde{g})| \\ &\le |((P_d)^* f, R_0 \Phi_k^-) + ((R_0)^* (P_d)^* f, \widetilde{g})| + C n^3 \\ &\le C \Big\{ n \sum_{j \ne m} |(\sigma_1 \Phi_j^-, R_0 \Phi_k^-)| + n \sum_{j\ne m}^{K} |(\sigma_1 \bar{\Phi}_j^-, R_0 \Phi_k^-)| + n^3 \Big\} \\ &\le C \Big\{ n \sum_{j \ne m}^{K} |(\sigma_1 \Phi_j^-, R_0 \Phi_k^-)| + n \sum_{j\ne m}^{K} |(\sigma_1 \bar{\Phi}_j^-, R_0 \Phi_k^-)| + n^3 \Big\} \end{aligned} \tag{2.129}$$

Note that from (2.117) and (2.125), we get

$$|(\sigma_1 \Phi_j^-, R_0 \Phi_k^-)| \le C n^4, \quad (\sigma_1 \bar{\Phi}_j^-, R_0 \Phi_k^-) = 0. \tag{2.130}$$

So, from (2.129), we obtain

$$|(f, P_c^\sharp R \Phi_k^-)| \le C n^3. \tag{2.131}$$

This completes the proof of Lemma 2.18. □

Corollary 2.19 *For any function $f, g \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ with $f, g = O_{L^2_r}(n)$, we have*

$$(f, P_c^\sharp (\mathbf{L}-\alpha)^{-1} P_c^\sharp g) = (f, P_c^{JH_0}(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0} g) + O(n^4). \tag{2.132}$$

Proof. Using (2.122) and Lemma 2.18, we have

$$(f, P_c^\sharp (\mathbf{L}-\alpha)^{-1} P_c^\sharp g) = (f, P_c^{JH}(JH - \alpha)^{-1} P_c^{JH} g) + O(n^4). \tag{2.133}$$

Now, since that $H - (H_0 - E) = \kappa Q^2 = O(n^2)$ and $P_c^{JH} - P_c^{JH_0} = O_{L^2_r}(n^2)$, we can use the same method as in Lemma 2.18 to obtain

$$(f, P_c^\sharp (\mathbf{L}-\alpha)^{-1} P_c^\sharp g) = (f, P_c^{JH_0}(J(H_0 - E) - \alpha)^{-1} P_c^{JH_0} g) + O(n^4). \tag{2.134}$$

This completes the proof of Corollary 2.19. □

Corollary 2.20 *Let $f, g \in L^2(\mathbb{R}^3, \mathbb{C})$ be localized real functions of order n and let $f_1 = \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right] f$ and $g_1 = \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] f$. We then have*

$$\begin{aligned}
&\langle \left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right] f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g\rangle = -2(f, P_c^{H_0}(H_0 - E - i\alpha)^{-1} P_c^{H_0} g) + O(n^4),\\
&\langle \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right] f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp \left[\begin{smallmatrix}-i\\ 1\end{smallmatrix}\right] g\rangle = -2(f, P_c^{H_0}(H_0 - E + i\alpha)^{-1} P_c^{H_0} g) + O(n^4),\\
&\langle \left[\begin{smallmatrix}1\\ i\end{smallmatrix}\right] f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g\rangle = O(n^4),\\
&\langle \left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right] f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp \left[\begin{smallmatrix}-i\\ 1\end{smallmatrix}\right] g\rangle = O(n^4).
\end{aligned} \tag{2.135}$$

Proof. By Corollary 2.19, we have

$$\langle \left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right] f, (\mathbf{L}-\alpha)^{-1} P_c^\sharp \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g\rangle = \langle \left[\begin{smallmatrix}1\\ -i\end{smallmatrix}\right] f, P_c^{JH_0}(J(H_0-E)-\alpha)^{-1} P_c^{JH_0} \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g\rangle + O(n^4). \tag{2.136}$$

On the other hand,

$$\begin{aligned}
(J(H_0-E)-\alpha)^{-1} P_c^{JH_0} \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g &= (H_0 - E + \alpha^2)^{-1} P_c^{H_0} \begin{bmatrix} -\alpha & -(H_0-E) \\ H_0 - E & -\alpha \end{bmatrix} \left[\begin{smallmatrix}i\\ 1\end{smallmatrix}\right] g \\
&= \left[\begin{smallmatrix}-1\\ i\end{smallmatrix}\right] \begin{bmatrix} (H_0 - E - i\alpha)^{-1} \\ (H_0 - E - i\alpha)^{-1} \end{bmatrix} P_c^{H_0} g.
\end{aligned}$$

So, the first identity of our corollary follows. Similarly, we can prove all of the last three identities of the corollary. □

3 Equations and main terms

In our analysis we use different coordinate systems. When the solution is away from bound states, we use the *orthogonal coordinates* (1.19), i.e., we decompose the solution as a sum of different spectral components with respect to $-\Delta + V$. When the solution is near a nonlinear bound state, we use the *linearized coordinates* (3.17), i.e., decomposition with respect to the corresponding linearized operator instead. In subsection 3.1 we recall the equations and normal forms in orthogonal coordinates from [30]. The rest of this section is devoted to analysis in linearized coordinates. We will not use the *centered orthogonal coordinates* (1.10).

3.1 Orthogonal coordinates

Let t_0 be a fixed initial time. For $t \ge t_0$ we may decompose the solution with respect to H_0 as

$$\psi(t) = \sum_{j=0}^{K} x_j(t)\phi_j + \xi, \quad \xi \in \mathbf{H}_c(H_0),\ \forall\, t \ge t_0. \tag{3.1}$$

Then for $t \ge t_0$, as in [30, Section 4] we have

$$\begin{aligned}
i\dot{x}_j &= e_j x_j + (\phi_j, G), \quad (j = 0, \dots, K),\\
i\partial_t \xi &= H_0 \xi + P_c^{H_0} G, \quad G := \kappa \psi^2 \bar{\psi}.
\end{aligned} \tag{3.2}$$

Let

$$G_3 := \kappa \left| \sum_{j=0}^{K} x_j \phi_j \right|^2 \left(\sum_{j=0}^{K} x_j \phi_j \right) = \kappa \sum_{l,m,j=0}^{K} x_l x_m \bar{x}_j \phi_l \phi_m \phi_j. \tag{3.3}$$

We then decompose ξ as (for details, see [30, Section 4])

$$\xi(t) = \xi^{(2)}(t) + \xi_1^{(3)}(t) + \xi_2^{(3)}(t) + \cdots + \xi_5^{(3)}(t),\ \forall\ t \geq t_0, \tag{3.4}$$

where

$$\begin{aligned} &\xi^{(2)}(t) := \sum_{l,m,j=0}^{K} x_l x_m \bar{x}_j(t) \xi_{lm}^j,\ \text{with} \\ &\xi_{lm}^j := -\kappa \lim_{r\to 0^+} [H_0 - e_l - e_m + e_j - ri]^{-1} P_c^{H_0} \phi_m \phi_l \phi_j, \end{aligned} \tag{3.5}$$

and, with $u_j(t) = e^{ie_j t} x_j(t)$ which have less oscillation than $x_j(t)$,

$$\begin{aligned} &\xi_1^{(3)}(t) := e^{-iH_0(t-t_0)} \xi(t_0), \qquad \xi_2^{(3)}(t) := -e^{-iH_0(t-t_0)} \xi^{(2)}(t_0), \\ &\xi_3^{(3)}(t) := -\int_{t_0}^t e^{-iH_0(t-s)} P_c^{H_0} \sum_{l,m,j=0}^{K} e^{i(-e_l-e_m+e_j)s} \frac{d}{ds}(u_l u_m \bar{u}_j) \xi_{lm}^j ds, \\ &\xi_4^{(3)}(t) := \int_{t_0}^t e^{-iH_0(t-s)} P_c^{H_0} i^{-1} (G - G_3 - \kappa \xi^2 \bar{\xi}) ds, \\ &\xi_5^{(3)}(t) := \int_{t_0}^t e^{-iH_0(t-s)} P_c^{H_0} i^{-1} (\kappa \xi^2 \bar{\xi}) ds. \end{aligned} \tag{3.6}$$

We recall the following two lemmas from [30]:

Lemma 3.1 (Lemma 4.1 [30]) *Let p, p' such that $4 \leq p < 6$, $(p)^{-1} + (p')^{-1} = 1$. Suppose that for a fixed time $t \geq t_0$ and for $0 < n \leq n_0 \ll 1$, we have*

$$\max_j |x_j(t)| \leq 2n, \quad \|\xi(t)\|_{L^2_{loc} \cap L^p} \leq 2n, \quad \|\xi(t)\|_{L^2} \ll 1. \tag{3.7}$$

Then for $u_j(t) = e^{ie_j t} x_j(t)$,

$$\|G\|_{L^1_{loc}} + \max_j |\dot{u}_j| \lesssim n^3 \quad \text{and} \quad \left\|G - G_3 - \kappa \xi^2 \bar{\xi}\right\|_{L^1 \cap L^{p'}} \lesssim n^2 \|\xi\|_{L^2_{loc}}. \tag{3.8}$$

Lemma 3.2 (Lemma 4.2 [30]) *Let p, u_j be as in Lemma 3.1. Suppose that for some $0 < n \leq n_0$ and for some $t \geq t_0$,*

$$\max_j |x_j(t)| \leq 2n, \quad \|\xi(t)\|_{L^2_{loc} \cap L^p} \leq 2n \quad \text{and} \quad \|\xi(t)\|_{L^2} \leq \alpha \ll 1. \tag{3.9}$$

Then, there are perturbations $\mu_j(t)$ of $u_j(t)$, $j \in I$, such that

$$\dot{\mu}_j(t) = \sum_{l=0}^{K} c_l^j |\mu_l|^2 \mu_j + \sum_{a,b=0}^{K} d_{ab}^j |\mu_a|^2 |\mu_b|^2 \mu_j + g_j, \tag{3.10}$$

and

$$\begin{aligned} &|u_j(t) - \mu_j(t)| \lesssim n^3, \\ &|g_j(t)| \lesssim n^7 + n^2 \left\|\xi^{(3)}\right\|_{L^2_{loc}} + n \|\xi\|_{L^2_{loc}}^2 + \|\xi\|_{L^2_{loc}}^{\frac{2(p-3)}{p-2}} \|\xi\|_{L^p}^{\frac{p}{p-2}}. \end{aligned} \tag{3.11}$$

Moreover, all of the coefficients c_l^j *and* d_{ab}^j *are of order one. The coefficients* c_l^j *are all purely imaginary and*

$$\mathrm{Re}\, d_{ab}^j = (2 - \delta_a^b)\gamma_{ab}^j - 2(2 - \delta_j^b)\gamma_{jb}^a, \tag{3.12}$$

with $\delta_a^b = 1$ *if* $a = b$ *and* $\delta_a^b = 0$ *if* $a \neq b$, *and*

$$\gamma_{ab}^l = \kappa^2 \,\mathrm{Im}\left(\phi_a\phi_b\phi_l, (H_0 - e_a - e_b + e_l - i0^+)^{-1} P_c^{H_0} \phi_a\phi_b\phi_l\right),\ \forall\, a, b, l \in I. \tag{3.13}$$

By the resonance condition Assumption (A2), the number $\gamma_{ab}^l \geq 0$ and it is positive if and only if $l < a, b$.

3.2 Linearized coordinates

When the solution ψ lies in a neighborhood of an excited state $Q = Q_{m,n}$, $m \in J$, it is natural to decompose $\psi - Q$ into invariant subspaces of the linearized operator around Q, see Lemma 2.4. The collection of these components is called the *linearized coordinates.*

Lemma 3.3 *There are small positive constants* n_0 *and* ε_3 *such that the following hold. Suppose* $\|\psi\|_{H^1} \leq n_0$ *satisfies* $\|\psi - (\psi, \phi_m)\phi_m\|_{L^2} \leq \varepsilon_3 |(\psi, \phi_m)|$.

(i) *For any* $0 < n < n_0$, *there exist unique* $a, \theta \in \mathbb{R}$ *such that*

$$\psi = [Q_{m,n} + a\partial_E Q_{m,n} + h]e^{i\theta}, \tag{3.14}$$

where $Q_{m,n}$ *and* $\partial_E Q_{m,n}$ *are given by Lemma 2.1,* $P_m h = 0$, *and* $|n^{-1}a| + \|h\|_{H^1} \leq \varepsilon_3 n$.

(ii) *There exist unique* $n(\psi) \in (0, n_0)$ *and* $\theta \in \mathbb{R}$ *such that* $a = 0$. *Moreover, if* ψ *is decomposed as in (i) with respect to another* n, *then*

$$n(\psi) = n + \frac{a}{2Cn} + O(n^3), \quad C = \kappa \int \phi_m^4. \tag{3.15}$$

(iii) *If* ψ *is decomposed as in (i) with respect to* n_1 *and* n_2 *with* $\|h_j\| \leq \rho \leq \varepsilon_3 n$, $|a_j| \leq C\rho^2$, *and* $|n_1 - n_2| \lesssim n^{-1}\rho^2$, *then*

$$C(n_1^2 - n_2^2) + a_1 - a_2 = O(\rho |n_1 - n_2|). \tag{3.16}$$

The proof of Lemma 3.3 is similar to those for [31, Lemmas 2.1–2.4].

By Lemma 3.3, when $\psi(t)$ is in a sufficiently small neighborhood of an excited state $Q = Q_{m,n}$, there is a unique choice of real $a(t)$ and $\theta(t)$ so that

$$\psi(t) = [Q + a(t)\partial_E Q + h(t)]e^{-iEt + i\theta(t)}, \quad P_m h(t) = 0. \tag{3.17}$$

Here $\partial_E Q = \partial_E Q_{m,n}$ and $E = E_{m,n}$. We can further decompose

$$h = \zeta + \eta, \quad \zeta = \sum_{k \neq m} \zeta_k, \quad [\eta] \in \mathbf{E}_c^\sharp, \tag{3.18}$$

where, for each $k \neq m$,

$$\zeta_k := \jmath^{-1}\,\mathrm{Re}(z_k \Phi_k) = z_k \bar{u}_k^+ + \bar{z}_k u_k^-, \quad u_k^\pm := \frac{1}{2}(\bar{u}_k \pm \bar{v}_k). \tag{3.19}$$

Substituting (3.17) into (1.1) and using $\mathcal{L}iQ = 0$ and $\mathcal{L}\partial_E Q = -iQ$, we get

$$\partial_t h - \mathcal{L}h = F_h \equiv i^{-1}(F + \dot\theta(Q + a\partial_E Q + h)) - aiQ - \dot a \partial_E Q, \tag{3.20}$$

where

$$F = \kappa Q(2|h_\sigma|^2 + h_\sigma^2) + \kappa |h_\sigma|^2 h_\sigma, \quad h_\sigma = a\partial_E Q + h. \tag{3.21}$$

We choose $\dot\theta$ and $\dot a$ so that $P_m F_h = 0$. Thus $F_h = (1 - P_m) i^{-1}(F + \dot\theta(a\partial_E Q + h))$ and

$$\begin{cases} \dot a = (c_m Q, \operatorname{Im}(F + \dot\theta h)), \\ \dot\theta = F_\theta \equiv -\left[a + (c_m \partial_E Q, \operatorname{Re} F)\right] \cdot \left[1 + (c_m \partial_E Q, \partial_E Q)a + (c_m \partial_E Q, \operatorname{Re} h)\right]^{-1}. \end{cases} \tag{3.22}$$

Taking $P_c^\sharp$ of (3.20), we get

$$\partial_t \eta - \mathcal{L}\eta = P_c^\sharp i^{-1}(F + \dot\theta(a\partial_E Q + h)). \tag{3.23}$$

Note $z_k = 2c_k(\sigma_1 \bar\Phi_k, [h])$. Taking $2c_k(\sigma_1\bar\Phi_k, [\cdot])$ of (3.20), $k \neq m$, we get

$$\dot z_k - \lambda_k z_k = Z_k := 2c_k(\sigma_1 \bar\Phi_k, [F_h]). \tag{3.24}$$

A direct computation using (2.36) shows[3]

$$Z_k = -2c_k \left\{ (u_k^+, F) + (u_k^-, \overline{F}) + \left[(u_k^+, h) + (u_k^-, \overline{h}) + (\bar u_k, \partial_E Q)a\right]\dot\theta \right\}. \tag{3.25}$$

Let $\omega_k := -\operatorname{Im}\lambda_k$ and let $p_k(t) = z_k(t)e^{i\omega_k t}$. We have

$$\dot p_k = (\operatorname{Re}\lambda_k)p_k + e^{i\omega_k t} Z_k. \tag{3.26}$$

Also, for any $k \neq m$, let $r_k := e^{-\lambda_k t} z_k$, we have,

$$\dot r_k = e^{-\lambda_k t} Z_k. \tag{3.27}$$

Note that $r_k = p_k$ for all $k > m$ and $r_k = e^{-\operatorname{Re}(\lambda_k)t} p_k$ for $k < m$. We shall use r_k in computing the normal form for the equation of a.

Definition 3.1 *Denote $I = \{0, 1, \cdots, K\}$, $I^* = \{0^*, 1^*, \cdots, K^*\}$. For all $m \in I$, let $I_{>m} = \{m+1, \cdots, K\}$, $I_{<m} = \{0, \cdots, m-1\}$, $I_m = I \setminus \{m\}$, $I_m^* = I^* \setminus \{m^*\}$ and $\Omega_m := I_m \cup I_m^*$. For $j \in I_m$, let*

$$\lambda_{j^*} = \bar\lambda_j, \quad \omega_{j^*} = -\omega_j, \quad z_{j^*} = \bar z_j, \quad r_{j^*} = \bar r_j, \quad p_{j^*} = \bar p_j, \quad u_{j^*}^\pm = \bar u_j^\pm, \quad \text{and} \quad v_{j^*}^\pm = \bar v_j^\pm. \tag{3.28}$$

It then follows that for all $j \in \Omega_m$, we have $z_j(t) = e^{-i\omega_j t} p_j(t)$ and $r_j = e^{-\lambda_j t} z_j$.

[3] Note $-2c_k \sim -i$ which is the coefficient of [30, page 242, line 5].

3.3 Decomposition of a

Recall $\dot{a} = (c_m Q, \mathrm{Im}(F + \dot{\theta} h))$. Let $F_1 := \kappa Q(2|\zeta|^2 + \zeta^2)$, $A^{(2)} := c_m(Q, \mathrm{Im}\, F_1)$ and $A^{(3)} := c_m(Q, \mathrm{Im}(F - F_1 + \dot{\theta} h))$. Then, we have $\dot{a} = A^{(2)} + A^{(3)}$. We shall impose the boundary condition of a at $t = T$, which is in fact the condition imposed on the choice of $E = E(T)$. Hence, we have

$$a(t) = a(T) + \int_T^t [A^{(2)}(s) + A^{(3)}(s)]ds. \tag{3.29}$$

Recall that

$$\zeta = \sum_{k \in I_m} \zeta_k, \quad \zeta_k = z_k \bar{u}_k^+ + \bar{z}_k u_k^-. \tag{3.30}$$

Therefore,

$$\mathrm{Im}\, \zeta_k \zeta_l = \mathrm{Im}[(z_k z_l)(\bar{u}_k^+ \bar{u}_l^+ - \bar{u}_k^- \bar{u}_l^-) + (z_k \bar{z}_l)(\bar{u}_k^+ u_l^- - \bar{u}_k^- u_l^+)]. \tag{3.31}$$

Let

$$a_{kl,1} := \kappa c_m(Q^2, \bar{u}_k^+ \bar{u}_l^+ - \bar{u}_k^- \bar{u}_l^-), \quad a_{kl,2} = \kappa c_m(Q^2, \bar{u}_k^+ u_l^- - \bar{u}_k^- u_l^+). \tag{3.32}$$

Note that $a_{kl,1}, a_{kl,2} = O(n^2)$, $a_{kl,1}, a_{kl,2}$ are real if both $k, l > m$, and $a_{kk,2}$ are purely imaginary. In particular $a_{kk,2} = 0$ if $k > m$. We have

$$\begin{aligned} A^{(2)} &= \kappa c_m(Q^2, \mathrm{Im} \sum_{k,l \in I_m} \zeta_k \zeta_l) = \mathrm{Im} \sum_{k,l \in I_m} \{a_{kl,1} z_k z_l + a_{kl,2} z_k \bar{z}_l\} \\ &= b_0(t) + \mathrm{Im}(A_1^{(2)}), \end{aligned} \tag{3.33}$$

where

$$b_0(t) = \sum_{k<m} b_{0k} |z_k|^2, \quad b_{0k} := \mathrm{Im}\, a_{kk,2}, \quad \widetilde{b}_0(t) := \int_T^t b_0(s) ds, \tag{3.34}$$

$$A_1^{(2)} := \sum_{k,l \in I_m} a_{kl,1} z_k z_l + \sum_{k \neq l} a_{kl,2} z_k \bar{z}_l. \tag{3.35}$$

Note $|b_{0k}| \lesssim n^2 \left\| u_k^- \right\|_{L^2_{loc}} = O(n^4)$ for $k < m$ by Lemmas 2.4 and 2.5.

We shall integrate $A_1^{(2)}$ by parts. Note that for all $\lambda_k + \lambda_l = -i(\omega_k + \omega_l) + O(n^4)$ and $\lambda_k + \bar{\lambda}_l = -i(\omega_k - \omega_l) + O(n^4)$. Therefore, $\lambda_k + \lambda_l = O(1)$ for all $k, l \in I_m$ and $\lambda_k + \bar{\lambda}_l = O(1)$ for all $k, l \in I_m$ and $k \neq l$. We then write

$$\begin{aligned} A_1^{(2)} &= \sum_{k,l \in I_m} a_{kl,1} e^{(\lambda_k + \lambda_l)t} r_k r_l + \sum_{k \neq l} e^{(\lambda_k + \bar{\lambda}_l)t} a_{kl,2} r_k \bar{r}_l \\ &= \sum_{k,l \in I_m} \frac{a_{kl,1}}{\lambda_k + \lambda_l} \left[\frac{d}{dt}(z_k z_l) - e^{(\lambda_k + \lambda_l)t} \frac{d}{dt}(r_k r_l) \right] \\ &\quad + \sum_{k \neq l} \frac{a_{kl,2}}{\lambda_k + \bar{\lambda}_l} \left[\frac{d}{dt}(z_k \bar{z}_l) - e^{(\lambda_k + \bar{\lambda}_l)t} \frac{d}{dt}(r_k \bar{r}_l) \right]. \end{aligned} \tag{3.36}$$

Now, define

$$\begin{aligned} a^{(2)}(t) &:= \mathrm{Im} \sum_{k,l \in I_m} \frac{a_{kl,1}}{\lambda_k + \lambda_l} z_k z_l + \mathrm{Im} \sum_{k \neq l} \frac{a_{kl,2}}{\lambda_k + \bar{\lambda}_l} z_k \bar{z}_l \\ A_{2,rm} &:= \mathrm{Im} \sum_{k,l \in I_m} \frac{a_{kl,1} e^{(\lambda_k + \lambda_l)t}}{\lambda_k + \lambda_l} \frac{d}{dt}(r_k r_l) + \mathrm{Im} \sum_{k \neq l} \frac{a_{kl,2} e^{(\lambda_k + \bar{\lambda}_l)t}}{\lambda_k + \bar{\lambda}_l} \frac{d}{dt}(r_k \bar{r}_l). \end{aligned} \tag{3.37}$$

We shall get

$$\mathrm{Im}(A_1^{(2)}) = \frac{d}{dt}a^{(2)}(t) - A_{2,rm}(t). \tag{3.38}$$

So, we have $A^{(2)} = \frac{d}{dt}a^{(2)}(t) + b_0(t) - A_{2,rm}(t)$. Therefore,

$$a(t) = a^{(2)}(t) + b(t), \tag{3.39}$$

where $b(t)$ satisfies

$$\dot b = b_0 + c_m(Q, \mathrm{Im}(F - F_1 + \dot\theta h)) - A_{2,rm}, \quad b(T) = a(T) - a^{(2)}(T). \tag{3.40}$$

Moreover, let $a_{kl,3} := 2a_{kl,1}(\lambda_k+\lambda_l)^{-1}$ and $a_{kl,4} := 2a_{kl,2}(\lambda_k+\bar\lambda_l)^{-1}$. Since $a_{kl,1}$ and $a_{kl,2}$ are of order n^2, so are $a_{kl,3}$ and $a_{kl,4}$. Moreover, $a_{kl,3}, a_{kl,4}$ are purely imaginary for $k, l \in I_{>m}$. Using (3.27), $a_{kl,1} = a_{lk,1}$ and $a_{kl,2} = -\bar a_{lk,2}$, we obtain

$$A_{2,rm} = \mathrm{Im} \sum_{k,l\in I_m} a_{kl,3} z_k Z_l + \mathrm{Im} \sum_{k\neq l} a_{kl,4} Z_k \bar z_l. \tag{3.41}$$

It worths noting that the benefits from using r_k instead of p_k in (3.37) is that we do not have terms of order zz_k for $k \in I_{<m}$ in (3.41). This is very essential in the normal forms.

3.4 Decomposition of η

We shall single out the main terms in η. Recall from (3.23) that

$$\partial_t \eta - \mathcal{L}\eta = P_c^\sharp i^{-1}(F + \dot\theta(a\partial_E Q + \zeta + \eta)). \tag{3.42}$$

In the vector form, we have

$$\partial_t[\eta] = \mathbf{L}[\eta] + P_c^\sharp J\dot\theta[\eta] + P_c^\sharp J[(F + \dot\theta(a\partial_E Q + \zeta))]. \tag{3.43}$$

We first deal with the non-localized linear term $J\dot\theta[\eta]$ using Lemma 2.15, following Buslaev-Perelman [4], also see [5, 7].[4] We need to revise their original statement and proof to take care of eigenvalues near the continuous spectrum.

Recall $P_\pm$ are defined in subsection 2.6. Taking projection $P_\pm$ of (3.43), and using

$$P_\pm J \mp iP_\pm = P_\pm(P_\pm J \mp iP_\pm) = P_\pm[K_\pm - (P_{R\pm}J \mp iP_{R\pm})] = P_\pm K_\pm, \tag{3.44}$$

we get

$$\partial_t P_\pm[\eta] = \mathbf{L}P_\pm[\eta] \pm i\dot\theta P_\pm[\eta] + P_\pm K_\pm \dot\theta[\eta] + P_\pm J[(F + \dot\theta(a\partial_E Q + \zeta))]. \tag{3.45}$$

Denote

$$\eta_\pm := e^{\mp i\theta} P_\pm[\eta]. \tag{3.46}$$

[4]The term $i\dot\theta\eta$ is not a problem in [31] in which $\mathcal{L}$ is factorized in the form $\mathcal{L} = U^{-1}JAU$ for some scalar self-adjoint operator A. Such factorization does not exist for linearized operators near excited states. In [33], the term $i\dot\theta\eta$ is removed by introducing $\tilde\eta = P_c^\sharp e^{i\theta}\eta$ and using Strichartz estimates to control the (small) commutator term. This last method is not suitable for L^p-decay approach since the commutator term, although smaller, has the same decay rate as η itself. The approach of Buslaev-Perelman has the further benefit of being applicable to the large soliton case.

We have

$$\partial_t \eta_\pm = \mathbf{L}\eta_\pm + e^{\mp i\theta} P_\pm \left[K_\pm \dot\theta[\eta] + J[(F + \dot\theta(a\partial_E Q + \zeta))] \right]. \tag{3.47}$$

Recall that $[\zeta_k] = (z_k \Phi_k + \bar z_k \bar\Phi_k)/2$. Note the term $e^{\mp i\theta} P_\pm J \dot\theta[\zeta]$ is not localized. However, by formula (2.35)

$$P_c^\sharp J \Phi_k = P_c^\sharp \Phi_k', \quad P_c^\sharp J \bar\Phi_k = P_c^\sharp \bar\Phi_k' \quad \Phi_k' = \left[\begin{smallmatrix} -2i \\ -2 \end{smallmatrix}\right] \bar u_k^+ \tag{3.48}$$

and note Φ_k' is localized. Thus we can rewrite the linear terms in (3.47) as

$$F_{L\pm} := e^{\mp i\theta} \dot\theta \left\{ K_\pm[\eta] + J[a\partial_E Q] + \textstyle\sum_{j\in I_m} (z_j \Phi_j' + \bar z_j \bar\Phi_j') \right\}, \tag{3.49}$$

where all functions are localized, and (3.47) becomes

$$\partial_t \eta_\pm = \mathbf{L}\eta_\pm + P_\pm \left[e^{\mp i\theta} J[F] + F_{L\pm} \right]. \tag{3.50}$$

In other words, for some $t_0 \ge 0$ and for all $t \ge t_0$, we have

$$\eta_\pm(t) = e^{\mathbf{L}(t-t_0)} \eta_\pm(t_0) + \int_{t_0}^t e^{\mathbf{L}(t-s)} P_\pm \{ e^{\mp i\theta} J[F] + F_{L\pm} \}(s) ds. \tag{3.51}$$

We will decompose $\eta_\pm$ as follows. Denote

$$\begin{aligned} \eta_{\pm,1}^{(3)}(t) &:= e^{\mathbf{L}(t-t_0)} \eta_\pm(t_0), \\ \eta_{\pm,4}^{(3)}(t) &:= \int_{t_0}^t e^{\mathbf{L}(t-s)} P_\pm \{ F_{L\pm} + e^{\mp i\theta} J[F - F_1] \}(s) ds. \end{aligned} \tag{3.52}$$

Then, we have

$$\eta_\pm(t) = \eta_{\pm,1}^{(3)}(t) + \eta_{\pm,4}^{(3)}(t) + \int_{t_0}^t e^{\mathbf{L}(t-s)} P_\pm \{ e^{\mp i\theta} J[F_1] \}(s) ds. \tag{3.53}$$

We shall integrate the last term in (3.53). Recall that $F_1 = \kappa Q(2|\zeta|^2 + \zeta^2)$ is the main term in F with

$$\zeta = \sum_{k\in I_m} \zeta_k = \sum_{k\in I_m} (z_k \bar u_k^+ + \bar z_k u_k^-), \quad u_k^+ = \phi_k + O_{L^2_r}(n^2), \quad u_k^- = O_{L^2_{\mathrm{loc}}}(n^2). \tag{3.54}$$

So,

$$F_1 = \sum_{k,l\in I_m} F_{kl}[z_k z_l + 2 z_k \bar z_l] + \sum_{k,l\in\Omega_m} \widetilde F_{kl} z_k z_l, \quad F_{kl} = \kappa Q \phi_k \phi_l, \quad \widetilde F_{kl} = O_{L^\infty_{3r}}(n^3). \tag{3.55}$$

In other words, we can write

$$F_1 = \kappa \sum_{k,l\in\Omega_m} z_k z_l \Phi_{kl}, \tag{3.56}$$

for some localized functions Φ_{kl} which can be computed explicitly. In particular, $\operatorname{Re}\Phi_{kl} = O(n)$ and $\operatorname{Im}\Phi_{kl} = O(n^3)$ for all $k, l \in \Omega_m$.

To integrate $P_\pm e^{\mp i\theta} J[F_1]$ in the $\eta_\pm$ equation, we want to integrate terms of the form

$$I_\pm(t) = \int_{t_0}^t e^{(t-s)\mathbf{L}} e^{-i\omega s} P_\pm f(s)\, ds, \tag{3.57}$$

where $\omega \in \mathbb{R}$, $f(s) \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and $\dot{f}(s)$ decays faster than f. We re-write $I_\pm$ as

$$I_\pm(t) = e^{t\mathbf{L}} \int_{t_0}^{t} e^{-s(\mathbf{L}+i\omega)} P_\pm f(s) ds. \tag{3.58}$$

Denote $R = \lim_{\varepsilon\to 0+}(\mathbf{L} + i\omega - \varepsilon)^{-1}$. Integration by parts gives

$$I_\pm(t) = -e^{-i\omega t} RP_\pm f(t) + e^{(t-t_0)\mathbf{L}} e^{-i\omega t_0} RP_\pm f(t_0) + \int_{t_0}^{t} e^{(t-s)\mathbf{L}} RP_\pm e^{-i\omega s} \dot{f}(s) ds. \tag{3.59}$$

The choice of the sign of ε ensures that $e^{t\mathbf{L}} RP_\pm$ has singular decay estimate according to Lemma 2.13. We can now identify the main term of $\eta_\pm$. Since $i^{-1}F_1 = -i\kappa \sum z_k z_l \Phi_{kl}$ with summation over $k, l \in \Omega_m$,

$$J[F_1] = -\operatorname{Re} \textstyle\sum i\kappa z_k z_l \Phi_{kl} \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right] = -\operatorname{Re} \textstyle\sum f_{kl}(s) e^{-i(\omega_k+\omega_l)s}, \tag{3.60}$$

where $f_{kl} = i\kappa p_k p_l \Phi_{kl} \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$. We decompose $P_\pm = \Pi_\pm P_c^\sharp$ since $\Pi_\pm$ does not commute with Re. Denote $R_{kl} = \lim_{\varepsilon\to 0+}(\mathbf{L} + i(\omega_k + \omega_l) - \varepsilon)^{-1} P_c^\sharp$ and $\omega_{kl} = \omega_k + \omega_l$. We get

$$\int_{t_0}^{t} e^{(t-s)\mathbf{L}} P_\pm e^{\mp i\theta(s)} J[F_1] ds = \eta_\pm^{(2)} + \eta_{\pm,2}^{(3)} + \eta_{\pm,3}^{(3)} \tag{3.61}$$

where

$$\begin{aligned}
\eta_\pm^{(2)} &= e^{\mp i\theta(t)} \Pi_\pm \operatorname{Re} \textstyle\sum_{k,l\in\Omega_m} R_{kl} e^{-i\omega_{kl} t} f_{kl}(t) \\
\eta_{\pm,2}^{(3)} &= -e^{(t-t_0)\mathbf{L}} e^{\mp i\theta(t_0)} \Pi_\pm \operatorname{Re} \textstyle\sum_{k,l\in\Omega_m} R_{kl} e^{-i\omega_{kl} t_0} f_{kl}(t_0) \\
\eta_{\pm,3}^{(3)} &= -\int_{t_0}^{t} e^{(t-s)\mathbf{L}} e^{\mp i\theta(s)} \Pi_\pm \textstyle\sum_{k,l\in\Omega_m} \big(\operatorname{Re} R_{kl} e^{-i\omega_{kl} s} \dot{f}_{kl} \mp i \operatorname{Re} R_{kl} e^{-i\omega_{kl} s} \dot{\theta} f_{kl} \big)(s) ds.
\end{aligned} \tag{3.62}$$

Observe that

$$\| |\dot{f}_{kl}| + |\dot{\theta} f_{kl}| \|_{L_r^2} \lesssim n|\dot{\theta}|\beta^2 + n\beta \max |\dot{p}_k|, \quad \beta = \max |p_k|. \tag{3.63}$$

Now, let

$$\eta_\pm^{(3)}(t) := \sum_{j=1}^{4} \eta_{\pm,j}^{(3)}(t), \quad \eta^{(j)} := e^{i\theta}\eta_+^{(j)} + e^{-i\theta}\eta_-^{(j)}, \quad j = 2, 3. \tag{3.64}$$

Then, from (3.53) and (3.62), we obtain the decomposition of $\eta_\pm$ and η as

$$\eta_\pm = \eta_\pm^{(2)} + \eta_\pm^{(3)}, \quad [\eta] = e^{i\theta}\eta_+ + e^{-i\theta}\eta_- = \eta^{(2)} + \eta^{(3)}. \tag{3.65}$$

We now compute the explicit form of $\eta^{(2)}$ which will be used in the computation of the key coefficients in the normal forms of z_k. By (3.65), (3.62), $\Pi_+ + \Pi_- = P_c^\sharp$, and (3.55),

$$\begin{aligned}
\eta^{(2)} &= e^{i\theta}\eta_+^{(2)} + e^{-i\theta}\eta_-^{(2)} \\
&= \operatorname{Re} \textstyle\sum_{k,l\in\Omega_m} R_{kl} e^{-i(\omega_k+\omega_l)t} f_{kl}(t) \\
&= \operatorname{Re} \sum_{k,l\in I_m} \Big\{ R_{kl} z_k z_l \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] F_{kl} + 2R_{kl^*} z_k \bar{z}_l \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] F_{kl} \Big\} + \sum_{k,l\in\Omega_m} z_k z_l R_{kl} O_{L_r^2}(n^3).
\end{aligned} \tag{3.66}$$

Recall $F_{kl} = \kappa Q \phi_k \phi_l$. Thus the first sum contains terms of order $O(nz^2)$.

3.5 Decomposition of F

We now decompose F into appropriate terms of the same order. We write

$$F = F_1 + F_2 + \cdots + F_5, \tag{3.67}$$

where

$$\begin{aligned}
&F_1 = \kappa Q(2|\zeta|^2 + \zeta^2),\\
&F_2 = 2\kappa Q\partial_E Q b(2\zeta + \bar{\zeta}) + 3\kappa Q\partial_E Q^2 b^2 + \kappa(\zeta + b\partial_E Q)^2(\bar{\zeta} + b\partial_E Q),\\
&F_3 = 2\kappa Q\partial_E Q a^{(2)}(2\zeta + \bar{\zeta}), \quad F_4 = 2\kappa Q[(\zeta + \bar{\zeta})\eta + \zeta\bar{\eta}],\\
&F_5 = \kappa Q\left[2|\eta_a|^2 + \eta_a^2\right] + 2\kappa Q\partial_E Q b(2\eta_a + \bar{\eta}_a)\\
&\qquad + \kappa(a\partial_E Q + h)^2(a\partial_E Q + \bar{h}) - \kappa(\zeta + b\partial_E Q)^2(\bar{\zeta} + b\partial_E Q),
\end{aligned} \tag{3.68}$$

with $\eta_a = \eta + a^{(2)}\partial_E Q$. Note that F_1 consists of terms of order nz^2; F_2, F_3 and F_4 consist of terms no smaller than n^2z^3; and F_5 higher order terms.

3.6 Basic estimates and normal forms

In this subsection, we first give some basic estimates in Lemmas 3.4, 3.5 and 3.6. We then give the normal forms of the equations of z_k and b in Lemmas 3.7 and 3.8.

Lemma 3.4 (Basic Estimates) *Suppose, for a fixed time, for some $\beta \ll n \le n_0$ and $p \ge 5$,*

$$\begin{aligned}
&\|Q\| = n, \qquad \|\eta\|_{L^2\cap L^p} \ll 1, \qquad \|\eta\|_{L^2_{\text{loc}}} \le n,\\
&\max_{j\neq m}|z_j| \le \beta, \qquad |a| \le C\beta^2.
\end{aligned} \tag{3.69}$$

For all $1 \le r \le 2$, denote

$$\begin{aligned}
&X := n\beta\|\eta\|_{L^2_{\text{loc}}} + n\|\eta\|^2_{L^2_{\text{loc}}} + \left\|\eta^3\right\|_{L^1_{\text{loc}}},\\
&\widetilde{X} := \beta^2\|\eta\|_{L^2_{\text{loc}}} + n\|\eta\|^2_{L^2_{\text{loc}}} + \left\|\eta^3\right\|_{L^1_{\text{loc}}}, \quad Y(r,p) := n\|\eta\|^2_{L^p} + \left\|\eta^3\right\|_{L^r}.
\end{aligned} \tag{3.70}$$

We have

$$\begin{aligned}
&\|F_5\|_{L^1_{\text{loc}}} \lesssim n\beta^4 + \widetilde{X}, \quad \|F_3 + F_4 + F_5\|_{L^1_{\text{loc}}} \lesssim n^2\beta^3 + X,\\
&\|F - F_1\|_{L^1_{\text{loc}}} \lesssim \beta^3 + X, \qquad \|F\|_{L^1_{\text{loc}}} \lesssim n\beta^2 + X,\\
&|F_\theta| \lesssim \beta^2 + n^{-1}X, \quad \|F - F_1\|_{L^r} \lesssim \beta^3 + n\beta\|\eta\|_{L^2_{\text{loc}}} + Y(r,p),\\
&\|F\|_{L^r} \lesssim n\beta^2 + n\beta\|\eta\|_{L^2_{\text{loc}}} + Y(r,p).
\end{aligned} \tag{3.71}$$

Proof. The first five estimates of (3.71) in L^1_{loc} can be found in [30, Lemma 3.2]. Although [30] is for $m = 0$ case, for L^1_{loc} bounds the new non-localized terms for $m > 0$ are similarly estimated.

For the last two L^r-estimates of (3.71), the only non-localized terms of F are of order $(u_k^- z_k)^3$, $(u_k^- z_k)^2\eta$, $u_k^- z_k\eta^2$, and η^3 for $k < m$. Since $|(u_k^- z_k)^2\eta| + |u_k^- z_k\eta^2| \lesssim |u_k^- z_k|^3 + |\eta|^3$, they are bounded by $|z_k|^3\left\|u_k^-\right\|^3_{L^{3r}} + \left\|\eta^3\right\|_{L^r} \lesssim \beta^3 + \left\|\eta^3\right\|_{L^r}$. $\square$

Define

$$z_L = \left(\sum_{k=0}^{m-1}|z_k|^2\right)^{1/2}, \quad z_H = \left(\sum_{k=m+1}^{K}|z_k|^2\right)^{1/2}. \tag{3.72}$$

If $m = 0$, we set $z_L = 0$. For $\frac{9}{2} < p < 6$, denote

$$\hat{X} = \hat{X}_p := n^4 z_L \|\eta\|_{L^p}^2 + n^6 z_L^2 \|\eta\|_{L^p} + m \cdot n^{\frac{6(6-p)}{p}} \|\eta\|_{L^p}^3. \tag{3.73}$$

Note $\hat{X} = 0$ if $m = 0$. Let

$$D = 6Kc_{\max}\gamma_0^+/\gamma_0 = O(1) \tag{3.74}$$

where $c_{\max} = \max_k 2\int \phi_k^4$ and

$$\gamma_0^+ = \max_{k,l,m\in I,|s|<s_0} \lim_{r\to 0+} \operatorname{Im}\left(\phi_k\phi_l\phi_m, \frac{1}{H_0 + e_k - e_l - e_m - s - ri} P_c^{H_0}\phi_k\phi_l\phi_m\right). \tag{3.75}$$

Note that $(Q_{k,n}, \partial_E Q_{k,n})^{-1} = 2\kappa\int \phi_k^4 + o(1)$.

Lemma 3.5 *Assume as in the Lemma 3.4, then for all $k \neq m$, we have*

$$\begin{aligned} &|Z_k| \lesssim n\beta^2 + \hat{X}_p + X, \quad \text{if} \quad k < m, \quad |Z_k| \lesssim n\beta^2 + X, \quad \text{if} \quad k > m, \\ &|R_k| \lesssim \beta^3 + \hat{X}_p + X, \quad \text{if} \quad k < m, \quad |R_k| \lesssim \beta^3 + X, \quad \text{if} \quad k > m. \end{aligned} \tag{3.76}$$

Here Z_k is defined in (3.25) *and R_k is part of Z_k,*

$$R_k := -2c_k\left[(u_k^+, F - F_1) + (u_k^-, \overline{F} - \overline{F}_1) + \left\{(u_k^+, h) + (u_k^-, \overline{h}) + (\bar{u}_k, \partial_E Q)a\right\} F_\theta\right]. \tag{3.77}$$

Proof. Recall (3.25) that

$$Z_k := -2c_k\left\{(u_k^+, F) + (u_k^-, \overline{F}) + \left[(u_k^+, h) + (u_k^-, \overline{h}) + (\bar{u}_k, \partial_E Q)a\right]\dot{\theta}\right\}. \tag{3.78}$$

For $m < k \le K$, since u_k^+, u_k^- are both real and localized, $P_k\eta = 0$, using Lemma 2.6 we have

$$|(u_k^+, \eta) + (u_k^-, \bar{\eta})| = 2|(u_k^-, \bar{\eta})| \le Cn^2 \|\eta\|_{L^2_{\text{loc}}}. \tag{3.79}$$

Therefore,

$$\begin{aligned} |Z_k| &\le \|F\|_{L^1_{loc}} + |\dot{\theta}|[|a| + |z| + n^2 \|\eta\|_{L^2_{\text{loc}}}] \\ &\lesssim n\beta^2 + X + [\beta^2 + n^{-1}X](\beta + \|\eta\|_{L^2_{loc}}) \lesssim n\beta^2 + X. \end{aligned} \tag{3.80}$$

Now, we consider the case when $k < m$. We first consider the term $2c_k[(u_k^+, F) + (u_k^-, \overline{F})]$. As we already see in the proof of Lemma 3.4, the only non-localized terms in F are bounded by $|\eta^3| + \sum_{j,l,h<m}|u_j^- u_l^- u_h^-|z_L^3$. Thus for $k < m$, using Hölder's inequality and Lemma 2.5,

$$\begin{aligned} &|[(u_k^+, F) + (u_k^-, \overline{F})]| \lesssim \|F\|_{L^1_{loc}} + (|u_k^-|, |\eta^3| + \textstyle\sum_{j,l,h<m}|u_j^- u_l^- u_h^-|z_L^3) \\ &\lesssim n\beta^2 + \hat{X}_p + X. \end{aligned} \tag{3.81}$$

On the other hand, using (2.38), we have

$$|(u_k^+, \eta) + (u_k^-, \bar{\eta})| = |(\sigma_1\bar{\Phi}_k, J[\eta])| \lesssim \|\eta\|_{L^2_{\text{loc}}}, \quad (k < m). \tag{3.82}$$

Then, it follows from Lemmas 3.4 and 2.6 that

$$|[(u_k^+, h) + (u_k^-, \bar{h}) + (\bar{u}_k, \partial_E Q)a]F_\theta| \lesssim [|z| + n^{-1}|a| + \|\eta\|_{L^2_{loc}}]|F_\theta| \lesssim \beta^3 + X. \tag{3.83}$$

This completes the proof of the estimates of Z_k. The estimates of R_k are proved similarly. $\square$

Lemma 3.6 *Assume as in the Lemma 3.4, then we have*

$$|\dot{b}| \le C[n^4 z_L^2 + n\beta^3 + nX + n^2\beta\hat{X}]. \tag{3.84}$$

Above $\hat{X} = \hat{X}_p$ is defined in (3.73) *and can be omitted if $m = 0$.*

Proof. Recall (3.40) that

$$\dot{b} = b_0 + c_m(Q, \mathrm{Im}(F - F_1 + \dot{\theta}h)) - A_{2,rm}. \tag{3.85}$$

It follows from (3.34), (3.41) and Lemma 3.5 that

$$|b_0| \le Cn^4 z_L^2, \quad |A_{2,rm}| \le n^2\beta[n\beta^2 + X + \hat{X}]. \tag{3.86}$$

On the other hand, we have

$$|c_m(Q, \mathrm{Im}(F - F_1 + \dot{\theta}h))| \lesssim n \|F - F_1\|_{L^1_{\mathrm{loc}}} + n^3\beta^2 + |\dot{\theta}|[n^3\beta + n\|\eta\|_{L^2_{\mathrm{loc}}}] \lesssim n\beta^3 + nX. \tag{3.87}$$

So, (3.84) follows. □

Lemma 3.7 (Normal form of z_k) *Fix $0 \le m \le K$ and $0 < n_1 \sim n \le n_0$. Let $Q = Q_{m,n_1}$ and $\mathcal{L} = \mathcal{L}_{m,n_1}$. Suppose ψ is decomposed as in* (3.17) *with respect to $\mathcal{L}$, and for some $0 < \beta \ll n$*

$$\|\eta\|_{L^2_{\mathrm{loc}}} \le \beta, \quad \|\eta\|_{L^2 \cap L^p} \ll 1, \quad \max_{k \ne m} |z_k| \le \beta, \quad |a| \le C\beta^2. \tag{3.88}$$

Then there exist functions q_k, g_k, Y_k and constants D_{kl} for $l \ne m$ such that

$$\begin{aligned} &\dot{q}_k - \mathrm{Re}(\lambda_k)q_k = \textstyle\sum_{l>m} D_{kl}|q_l|^2 q_k + Y_k q_k + g_k, \quad \text{with} \quad |q_k - p_k| \lesssim n\beta^2, \\ &|D_{kl}| \le Dn^2, \quad \mathrm{Re}(D_{kl}) \le -\gamma_0 n^2, \quad \forall k, l > m, \quad \text{and} \\ &|\mathrm{Re}(Y_k)| \lesssim n^2 z_L^2, \quad (k > m); \quad |\mathrm{Re}(Y_k)| \lesssim n^2\beta^2, \quad (k < m). \end{aligned} \tag{3.89}$$

Recall $\mathrm{Re}\,\lambda_k \gtrsim n^4$ *if $k < m$ and* $\mathrm{Re}\,\lambda_k = 0$ *if $k > m$. Moreover, we have*

$$\begin{aligned} &|g_k| \lesssim n\beta^4 + n^4\beta z_L^2 + n^3\beta\|\eta\|_{L^2_{\mathrm{loc}}} + n\beta\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + n\beta\hat{X}_p + \widetilde{X}, \quad (k > m), \\ &|g_k| \lesssim n^5\beta^2 + n^4\beta z_L^2 + n\beta^4 + n^3\beta\|\eta\|_{L^2_{\mathrm{loc}}} + n\beta\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + \hat{X}_p + \widetilde{X}, \quad (k < m). \end{aligned} \tag{3.90}$$

Above $\hat{X}_p$ is defined in (3.73) *and can be omitted if $m = 0$.*

In case $m = 0$, Lemma 3.7 is identical to [30, Lemma 3.4]. The main difference in case $m > 0$ is that u_l^- are not localized and $u_l^\pm$ are complex for $l < m$. For those new terms involving z_l with $l < m$, we either integrate them by parts and use equations of r_l, as in (3.36), or include them in the error terms. The proof is skipped and can be found in [21].

Lemma 3.8 (Normal form of b) *Assume as in Lemma 3.7. Then there exist functions $\widetilde{b}, g_b$ and numbers B_{kl} for $k, l \in I_m$ such that*

$$\begin{aligned} &\dot{\widetilde{b}} = b_0 + \sum_{k,l \in I_{>m}} B_{kl}|z_k|^2|z_l|^2 + g_b, \quad |b - \widetilde{b}| \le Cn\beta[\beta^2 + n\|\eta\|_{L^2_{\mathrm{loc}}}], \\ &|g_b| \le C[n^3\beta^4 + n^5\beta z_L^2 + n^2\beta^2 z_L^2 + n\beta^5 + n^2 z_L\|\eta\|_{L^2_{\mathrm{loc}}} \\ &\qquad + n^2\|\eta\|_{L^2_{\mathrm{loc}}}^2 + n\|\eta^3\|_{L^1_{\mathrm{loc}}} + n\beta^2\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + n\beta\hat{X}_p]. \end{aligned} \tag{3.91}$$

Above b_0 is define in (3.34) *and can be omitted if $m = 0$. Moreover, we also have $|B_{kl}| \le Cn^2$ and $B_{kl} = -\frac{c_m}{2}\,\mathrm{Re}\,D_{kl} + O(n^4)$ where D_{kl} is defined in Lemma 3.7 and $c_m = (Q_m, R_m)^{-1} = O(1) > 0$. Moreover,* $\max_{kl}(|B_{kl}|)/(K^{-1}\gamma_0 n^2) \le \frac{D}{2}$.

The proof is again skipped, see [21].

4 Converging to an excited state

In this and the next sections, we study the dynamics when the solution is in a neighborhood of some excited states Q_1 at $t = 0$. We want to show that the solution either converges to an excited state, or exits the neighborhood eventually. In the first case, the ground state component is always bounded by other states. In the second case, the ground state component becomes significant after some time, denoted t_c below. In this section we study the dynamics for $t < t_c$. In next section we study the dynamics for $t > t_c$ if t_c is finite.

Denote $x_j(t) = (\phi_j, \psi(t))$ and $\xi(t) = P_c^{H_0}\psi(t)$. The assumption of Theorem 1.1 states that, at time $t = 0$,

$$|x_1(0)| = n, \quad \|\textstyle\sum_{j\neq 1} x_j(0)\phi_j + \xi(0)\|_{H^1\cap L^1} \le \rho_0, \quad \rho_0 = n^{1+\delta}. \tag{4.1}$$

Denote

$$T_e := \sup_{T>0} \left\{ T : \ \frac{1}{\varepsilon_3} \|\psi(t) - x_1(t)\phi_1\|_{L^2} \le |x_1(t)| \in ((0.9)n, (1.1)n), \quad 0 \le \forall\, t \le T \right\}. \tag{4.2}$$

Above $\varepsilon_3 > 0$ is the small constant in Lemma 3.3 and $T_e > 0$ by (4.1). T_e is the time the solution exits the neighborhood of first excited state family. Note that (4.1)–(4.2) are in terms of the orthogonal coordinates. For most of this section we will use linearized coordinates which depend on the choice of Q, but (4.1)–(4.2) are independent of such a choice.

From Lemma 3.3 and the definition of T_e, for each $0 \le T < T_e$, we can find a unique $n(T) = n(\psi(T)) \in (0, n_0)$ such that the solution $\psi(t)$ can be decomposed as

$$\psi(t) = [Q + a(t)\partial_E Q + \zeta(t) + \eta(t)]e^{-iEt+i\theta}, \qquad \forall\, 0 \le t < T_e, \tag{4.3}$$

with $a(T) = 0$, where $Q = Q_{1,n(T)}$, $\partial_E Q = \partial_E Q_{1,n(T)}$ and $E = E_{1,n(T)}$. The components ζ and η are in the corresponding spectral subspaces with respect to $Q_{1,n(T)}$. Moreover we decompose

$$\zeta = \textstyle\sum_{j\neq 1}\zeta_j, \quad \zeta_j = \bar z_j u_j^- + z_j \bar u_j^+, \quad [\eta] = e^{i\theta}\eta_+ + e^{-i\theta}\eta_-. \tag{4.4}$$

Define

$$\rho(t) := \frac{1}{n}(\Delta t + \gamma_0 t)^{-1/2}, \quad \Delta t := (n\rho_0)^{-2}, \quad \rho(0) = \rho_0, \tag{4.5}$$

where γ_0 is given in (1.17), and let

$$t_c := \sup_{0<T\le T_e} \{T : |z_0(t)| \le \varepsilon_4 n^{-1}\rho(t)^2, \ \ 0 \le t \le T\}, \tag{4.6}$$

where $\varepsilon_4 > 0$ is a small constant to be chosen in (4.49), and z_0 is the coefficient of ζ_0 in (4.4) with respect to $Q_{1,n(T)}$. If there does not exist any T satisfying the right side of (4.6), we let $t_c = 0$.

Be definition $t_c \le T_e$ could be finite or infinite and is independent of the choice of Q in (4.3). If it is finite, it is the first time that z_0 becomes large enough, and will not be destroyed by other components in the future. The subscript $_c$ means "change" (of behavior). The function $\rho(t)$ is an upper bound for higher bound states for $0 \le t \le t_c$.

If $t_c = 0$, we may skip most of this section and go directly to Lemma 4.6 and section 5.

We will bound η in L^p and L^2_{loc}, with fixed p satisfying

$$\frac{27}{5} < p < 6, \quad \sigma = \sigma(p) = \frac{3p-9}{2p}, \quad \frac{2}{3} < \sigma < \frac{3}{4}. \tag{4.7}$$

From now on let $0 \le T < t_c$ and ψ be decomposed as in (4.3) with respect to $Q_{1,n(T)}$. We start with the following lemma.

Lemma 4.1 (Initial estimates) *Fix $\frac{27}{5} < p < 6$ with $\sigma(p) = \frac{3p-9}{2p}$. We have*

$$\sum_{k \neq 1} |z_k(0)|^2 \le \tfrac{9}{8}\rho_0^2, \quad \left\| e^{\mathbf{L}t} \eta_\pm(0) \right\|_{L^p} \langle t \rangle^{\sigma(p)} + \left\| e^{\mathbf{L}t} \eta_\pm(0) \right\|_{L^2_{\rm loc}} \langle t \rangle^{7/6} \le C_2 \rho_0 \tag{4.8}$$

for $t \ge 0$, for some $C_2 > 0$ uniformly in $n = n(T)$, $0 \le T < T_e$.

Proof. Let $\psi' := e^{-i\theta(0)}\psi(0) - Q$. From (4.3) at $t = 0$, we have

$$a(0)\partial_E Q + \zeta(0) + \eta(0) = \psi' = e^{-i\theta(0)} \left(\textstyle\sum_{j=0}^K x_j(0)\phi_j + \xi(0) \right) - Q. \tag{4.9}$$

For $k \neq 1$, applying the projection P_k on this equation, we get

$$|z_k(0)| \le |2c_k|[|(u_k^+, \psi')| + |(u_k^-, \overline{\psi'})|] \le (1 + o(1))[|x_k(0)| + n^3]. \tag{4.10}$$

Thus $\sum_{k\neq 1} |z_k(0)|^2 \le \frac{9}{8}\rho_0^2$ by (4.1). Moreover, since ψ' is localized and $\|\psi'\|_{H^1 \cap L^1} \lesssim \rho_0$, using Lemma 2.16, we get the estimates of $\eta_\pm(0)$ for $t > 1$ by Lemma 2.11 and for $0 \le t \le 1$ by Lemma 2.8. $\square$

Recall $\eta^{(3)}$ and z_H are defined in (3.64) and (3.72). We now define

$$M_T := \sup_{0\le t\le T} \max \left\{ \begin{array}{l} \rho(t)^{-1} z_H(t), \quad 2D^{-1}\rho^{-2}(t)|a(t)|, \\ \left[n^{2\sigma-1}\rho(t)^{2\sigma-2\alpha} + 2C_2\rho_0\langle t\rangle^{-\sigma(p)} \right]^{-1} \|\eta(t)\|_{L^p}, \\ \left[n^{-\alpha/2}\rho^3 + n^{4/5}\rho^{7/3} + 2C_2\rho_0\langle t\rangle^{-7/6} \right]^{-1} \|\eta^{(3)}(t)\|_{L^2_{\rm loc}} \end{array} \right\}. \tag{4.11}$$

Above $\alpha > 0$ is a small constant to be chosen. We can choose $\alpha = 0.01$.

Clearly $M_0 \le 3/2$ if n is sufficiently small. By continuity we have $M_T \le 2$ for $T > 0$ sufficiently small. Our main result in this section is the following proposition, which implies $M_T \le 3/2$ for all $T < t_c$ by a continuity argument.

Proposition 4.2 *Suppose that for some $T \in [0, t_c)$, M_T is well-defined and $M_T \le 2$. Then we have $M_T \le 3/2$ and $n(T)/n \in (\frac{3}{4}, \frac{5}{4})$.*

The proof of Proposition 4.2 is decomposed to Lemmas 4.3–4.5.

Note that $T < t_c$ and $M_T \le 2$ imply

$$\begin{aligned} &|z_0(t)| \le \varepsilon_4 n^{-1}\rho^2(t), \quad z_H(t) \le 2\rho(t), \quad |a(t)| \le D\rho(t)^2, \\ &\|\eta(t)\|_{L^p} \le 2n^{2\sigma-1}\rho(t)^{2\sigma-2\alpha} + 4C_2\rho_0\langle t\rangle^{-\sigma}, \\ &\|\eta^{(3)}(t)\|_{L^2_{\rm loc}} \le 2n^{-\alpha/2}\rho^3 + 2n^{4/5}\rho^{7/3} + 4C_2\rho_0\langle t\rangle^{-7/6}. \end{aligned} \tag{4.12}$$

Since $[\eta] = \eta^{(2)} + \eta^{(3)}$ and $\|\eta^{(2)}\|_{L^2_{\rm loc}} \lesssim n\rho^2$ by its definition, we get

$$\|\eta(t)\|_{L^2_{\rm loc}} \lesssim n\rho(t)^2 + \rho_0\langle t\rangle^{-7/6}. \tag{4.13}$$

It is sometimes convenient to use

$$\rho_0 \langle t\rangle^{-1/2} \lesssim \rho(t) \lesssim n^{-1} \langle t\rangle^{-1/2}, \quad \|\eta\|_{L^p} + \|\eta(t)\|_{L^2_{\rm loc}} \lesssim \rho. \tag{4.14}$$

Lemma 4.3 *Recall X, $\tilde{X}$, F and F_1 are defined in (3.70), (3.21), and (3.68), with $\frac{27}{5} < p < 6$. Assume $M_T \le 2$, then we have*

$$\begin{aligned} \widetilde{X} &\lesssim n\rho^4 + \rho_0\rho(t)^2\langle t\rangle^{-7/6} + n\rho_0^2\langle t\rangle^{-7/3}, \\ X &\lesssim n^2\rho^3 + n\rho_0\rho(t)\langle t\rangle^{-7/6} + n\rho_0^2\langle t\rangle^{-7/3}, \end{aligned} \tag{4.15}$$

and, with $o(1)$ denoting small positive constants which go to 0 as $n + \|\psi_0\|_{H^1} \to 0$,

$$\begin{aligned} \|F\|_{L^{p'}} &\lesssim n\rho^2 + o(1)\rho_0^2\langle t\rangle^{-1.4}, \\ \|F - F_1\|_{L^{\frac{9}{8}} \cap L^{\frac{3}{2}}} &\lesssim \rho^3 + n^{0.64}\rho^{2.54} + \rho_0^{7/4}\langle t\rangle^{-5/4}. \end{aligned} \tag{4.16}$$

Proof. By Hölder's inequality for $p \ge 9/2$, and $\|\eta\|_{L^2\cap L^p} \ll 1$,

$$\begin{aligned} &\|\eta^3\|_{L^1_{loc}} \le \|\eta\|_{L^2_{loc}}^{\frac{2p-6}{p-2}} \|\eta\|_{L^p}^{\frac{p}{p-2}}, \quad \|\eta^3\|_{L^1} \le o(1)\,\|\eta\|_{L^p}^{\frac{p}{p-2}}, \\ &\|\eta^3\|_{L^{p'}} \le o(1)\,\|\eta\|_{L^p}^{\frac{p+2}{p-2}}, \quad \|\eta^3\|_{L^{9/8}\cap L^{3/2}} \le o(1)\,\|\eta\|_{L^p}^{\frac{11p}{9(p-2)}}. \end{aligned} \tag{4.17}$$

From (3.70) with $\beta = \rho$ and n replaced by $n(T) \sim n$,

$$\widetilde{X} \lesssim \rho^2\,\|\eta\|_{L^2_{loc}} + X_1, \quad X \lesssim n\rho\,\|\eta\|_{L^2_{loc}} + X_1, \quad X_1 = n\,\|\eta\|_{L^2_{loc}}^2 + \|\eta^3\|_{L^1_{loc}}. \tag{4.18}$$

Using $(4.12)_2$, (4.13), and $(4.17)_1$, one gets for $\frac{27}{5} < p < 6$ that

$$X_1 \lesssim n^2\rho^4 + \rho_0\rho^2\,\langle t\rangle^{-7/6} + n\rho_0^2\,\langle t\rangle^{-7/3}. \tag{4.19}$$

One gets (4.15) from the above two equations.

To bound $F = \kappa Q(2|h_\sigma|^2 + h_\sigma^2) + \kappa|h_\sigma|^2 h_\sigma$ in $L^{p'}$ with $h_\sigma = a\partial_E Q + \zeta + \eta$, since $\|a\partial_E Q\| \lesssim n^{-1}\rho^2$, $\|\zeta\|_{L^q} \lesssim \rho$ for $q \ge 2$, and $\|\eta\|_{L^p} \le \rho$, by $(4.17)_2$ and (4.12) we get

$$\|F\|_{L^{p'}} \lesssim n\rho^2 + o(1)\,\|\eta\|_{L^p}^{\frac{p+2}{p-2}} \lesssim n\rho^2 + o(1)\rho_0^2\,\langle t\rangle^{-1.4510}. \tag{4.20}$$

Similarly, to bound $F - F_1$ with $F_1 = \kappa Q(2|\zeta|^2 + \zeta^2)$, by (4.17) we have

$$\|F - F_1\|_{L^{\frac{9}{8}}\cap L^{\frac{3}{2}}} \lesssim \rho^3 + n\rho\,\|\eta\|_{L^2_{loc}} + o(1)\,\|\eta\|_{L^p}^{\frac{11p}{9(p-2)}}. \tag{4.21}$$

By (4.12), $\rho \le n^{-1}\,\langle t\rangle^{-1/2}$, and $\frac{27}{5} < p < 6$, it is bounded by

$$\begin{aligned} &\lesssim \rho^3 + n\rho[n\rho^2 + \rho_0\langle t\rangle^{-7/6}] + [n^{0.6471}\rho^{2.5494} + \rho_0^{1.8333}\,\langle t\rangle^{-1.2941}] \\ &\lesssim \rho^3 + n^{0.64}\rho^{2.54} + \rho_0^{7/4}\,\langle t\rangle^{-5/4}. \end{aligned} \tag{4.22}$$

□

Lemma 4.4 (Dispersion estimates) *Assume $M_T \le 2$, then for all $0 \le t \le T$, we have*

$$\begin{aligned} \|\eta(t)\|_{L^p} &\le \frac{3}{2}n^{2\sigma-1}\rho(t)^{2\sigma-2\alpha} + 3C_2\rho_0\langle t\rangle^{-\sigma}, \\ \|\eta^{(3)}(t)\|_{L^2_{loc}} &\le \frac{3}{2}[n^{-\alpha}\rho^3 + n^{4/5}\rho^{7/3}] + 3C_2\rho_0\langle t\rangle^{-7/6}. \end{aligned} \tag{4.23}$$

Proof. We first prove the L^p-bound. Since $[\eta] = e^{i\theta}\eta_+ + e^{-i\theta}\eta_-$, it suffices to estimate $\|\eta_\pm\|_{L^p}$. By (3.50) with $t_0 = 0$, and by Lemmas 2.11 and 2.16,

$$\|\eta_\pm\|_{L^p} \lesssim \left\|e^{t\mathbf{L}}\eta_\pm(0)\right\|_{L^p} + \int_0^t \alpha_p(t-s)[\|F_{L\pm}\|_{L^{p'}} + \|F\|_{L^{p'}}](s)ds. \tag{4.24}$$

By Lemma 4.1,

$$\left\|e^{t\mathbf{L}}\eta_\pm(0)\right\|_{L^p} \le C_2\rho_0\langle t\rangle^{-\sigma}. \tag{4.25}$$

By (3.71), Lemma 4.3, and (4.14),

$$|\dot\theta| = |F_\theta| \lesssim \rho^2 + n^{-1}X \lesssim \rho(t)^2 + \rho_0\rho(t)\langle t\rangle^{-7/6} + \rho_0^2\langle t\rangle^{-7/3} \lesssim \rho(t)^2. \tag{4.26}$$

By (3.49), (4.14), and Lemma 2.15,

$$\|F_{L\pm}\|_{L^{p'}} \lesssim |F_\theta|(\|\eta\|_{L^p} + n^{-1}|a| + |z|) \lesssim \rho^2\cdot\rho = \rho^3. \tag{4.27}$$

By Lemma 4.3, $\|F\|_{L^{p'}} \lesssim n\rho^2 + \rho_0^2\langle t\rangle^{-7/5}$. Thus the integral in (4.24) is bounded by

$$\lesssim \int_0^t \alpha_p(t-s)[n\rho^2(s) + \rho_0^2\langle s\rangle^{-7/5}]ds \lesssim \rho_0^{2\alpha}n^{2\sigma-1}\rho(t)^{2\sigma-2\alpha} + \rho_0^2\langle t\rangle^{-\sigma}. \tag{4.28}$$

Here we have used (4.5), $n\rho^2(s) \sim n^{-1}(\Delta t + s)^{-1}$, and $\forall 0 < \alpha < \sigma < 1$

$$\int_0^t |t-s|^{-\sigma}(\Delta t + s)^{-1}ds \lesssim (\Delta t)^{-\alpha}(\Delta t + t)^{-\sigma+\alpha}. \tag{4.29}$$

Combining (4.25) and (4.28), we get the first estimate of Lemma 4.4.

We next prove the second estimate. Recall that $\eta_\pm^{(3)} = \sum_{j=1}^4 \eta_{\pm,j}^{(3)}$, where $\eta_{\pm,j}^{(3)}$ are defined in (3.52) and (3.62) with $t_0 = 0$. By Lemmas 4.1 and 2.13, we get

$$\left\|\eta_{\pm,1}^{(3)}\right\|_{L^2_{loc}} \le C_2\rho_0\langle t\rangle^{-7/6}, \quad \left\|\eta_{\pm,2}^{(3)}\right\|_{L^2_{loc}} \le Cn\rho_0^2\langle t\rangle^{-3/2}. \tag{4.30}$$

For $\eta_{\pm,3}$, by Lemma 3.5, (4.14), and (4.15),

$$\max|p_k| \lesssim n\rho^2 + \hat X_p + X \lesssim n\rho^2. \tag{4.31}$$

By (3.63), (4.26) and the above,

$$\||\dot f_{kl}| + |\dot\theta f_{kl}|\|_{L^2_r} \lesssim n|\dot\theta|\rho^2 + n\rho\max|\dot p_k| \lesssim n\rho^2\rho^2 + n\rho(n\rho^2) \lesssim n^2\rho^3. \tag{4.32}$$

It follows from Lemma 2.13 that

$$\left\|\eta_{\pm,3}^{(3)}\right\|_{L^2_{\rm loc}} \le C\int_0^t \langle t-s\rangle^{-3/2}n^2\rho^3(s)ds \le Cn^2\rho^3(t). \tag{4.33}$$

Here we have used, for $a, b > 1$ and $S \ge 1$,

$$\int_0^t \langle t-s\rangle^{-a}(S+s)^{-b}ds \lesssim S^{1-b}(S+t)^{-a} + (S+t)^{-b}, \tag{4.34}$$

which is bounded by $(S+t)^{-b}$ if $a \ge b$.

For $\eta_{\pm,4}$, by Lemma 2.11, we have

$$\left\|\eta^{(3)}_{\pm,4}\right\|_{L^2_{\rm loc}} \le C\int_0^t \alpha_\infty(t-s)[\|F_{L\pm}\|_{L^{9/8}\cap L^{3/2}} + \|F-F_1\|_{L^{9/8}\cap L^{3/2}}](s)ds, \tag{4.35}$$

where $\alpha_\infty(t) = t^{-1/2}\langle t\rangle^{-2/3}$. It follows from (4.34) that

$$\int_0^t \alpha_\infty(t-s)\rho(s)^r ds \lesssim \rho(t)^r + n^{1/3}\rho_0^{r-2}\rho(t)^{7/3}, \quad r>2. \tag{4.36}$$

As for (4.27), we have $\|F_{L\pm}\|_{L^{9/8}\cap L^{3/2}} \lesssim \rho^3$. By Lemma 4.3, $\|F-F_1\|_{L^{9/8}\cap L^{3/2}} \lesssim \rho^3 + n^{0.64}\rho^{2.54} + \rho_0^{7/4}\langle t\rangle^{-5/4}$. Thus

$$\begin{aligned}\left\|\eta^{(3)}_{\pm,4}\right\|_{L^2_{\rm loc}} &\lesssim (\rho^3 + n^{1/3}\rho_0\rho^{7/3}) + (n^{0.64}\rho^{2.54} + n^{0.97}\rho_0^{0.54}\rho^{7/3}) + \rho_0^{7/4}\langle t\rangle^{-5/4} \\ &\lesssim \rho^3 + o(1)n^{4/5}\rho^{7/3} + \rho_0^{7/4}\langle t\rangle^{-5/4}.\end{aligned} \tag{4.37}$$

Summing (4.30), (4.33) and (4.37), we get the bound of $\|\eta^{(3)}_\pm\|_{L^2_{\rm loc}}$ in the lemma. □

Lemma 4.5 (Bound states estimates) *Assume $M_T \le 2$, then for all $0\le t\le T$, we have*

$$z_H(t) \le \frac{3}{2}\rho(t), \quad |a(t)| \le \frac{3}{4}D\rho(t)^2, \quad |n(t)-n| \le \frac{1}{4}n. \tag{4.38}$$

Proof. For $1<k\le K$, from Lemma 3.7, we have a perturbation q_k of p_k such that

$$\dot q_k = \sum_{l\neq 1} D_{kl}|q_l|^2 q_k + Y_k q_k + g_k, \tag{4.39}$$

where

$$\begin{aligned}&|q_k - p_k| \lesssim Cn\rho^2, \quad |\operatorname{Re}(Y_k)| \le Cn^2 z_L^2 \le C\rho^4(t), \\ &|g_k| \lesssim n\rho^4 + n^3\rho\,\|\eta\|_{L^2_{\rm loc}} + n\rho\|\eta^{(3)}\|_{L^2_{loc}} + \widetilde X + n\rho\hat X.\end{aligned} \tag{4.40}$$

From (3.73) and $\|\eta\|_{L^p}\le\rho$, we have $\hat X \lesssim \rho^3$. Thus, from (4.12), (4.13) and Lemma 4.3, we get

$$|g_k| \lesssim o(1)n^2\rho^3 + n\rho_0\rho\langle t\rangle^{-7/6} + n\rho_0^2\langle t\rangle^{-7/3}. \tag{4.41}$$

Since $\rho_0 = n^{1+\delta}$ and $0<\delta<\frac{3}{2}$, it follows that

$$\int_0^{n^{-3}\wedge T} |g_k|(t)dt \le Cn\rho_0; \quad |g_k|(t) \le o(1)n^2\rho^3(t), \quad \forall t\ge n^{-3}. \tag{4.42}$$

Now, from (4.39), we get

$$\frac{d}{dt}|q_k| = \sum_{l\neq 1}\operatorname{Re}(D_{kl})|q_l|^2|q_k| + (\operatorname{Re} Y_k)|q_k| + \operatorname{Re}(\frac{\bar q_k}{|q_k|}g_k). \tag{4.43}$$

for all $0\le t\le n^{-3}$, by integrating this equation on $(0,t)$, we see that $|q_k(t)-q_k(0)| \ll \rho_0$. Using $z_H = (\sum_{k>1}|p_k|^2)^{1/2}$, $z_H(0)\le\sqrt{9/8}\rho_0$ and $|q_k-p_k|\lesssim n\rho^2$, we get

$$z_H(t) \le 1.1\rho_0, \quad \forall\ 0\le t\le n^{-3}. \tag{4.44}$$

Now, let $f_H = (|q_2|^2 + \cdots + |q_K|^2)^{1/2}$, from (4.43) and (3.89), in particular $D_{k0}|q_0|^2 \lesssim n^2(n^{-1}\rho^2)^2 = \rho^4$, we get

$$\dot{f}_H \le -\frac{\gamma_0 n^2}{2} f_H^3 + C[f_H \rho^4 + \textstyle\sum_{k=2}^K |g_k|]. \tag{4.45}$$

By (4.12) and (4.42), we get

$$\dot{f}_H \le -\frac{\gamma_0 n^2}{2} f_H^3 + o(1) n^2 \rho(t)^3, \quad n^{-3} \le t \le t_c. \tag{4.46}$$

Let $g(t) := \frac{7}{5}\rho(t)$. We have $f_H(n^{-3}) < g(n^{-3})$ and $\dot{g} = -\frac{\gamma_0 n^2}{2}\frac{25}{49} g^3$, thus $\dot{f}_H(t) < \dot{g}(t)$ if $f_H(t) = g(t)$. By comparison principle,

$$f_H(t) \le g(t) = \frac{7}{5}\rho(t), \quad (n^{-3} \le t \le T), \tag{4.47}$$

which together with (4.44) give the first estimate of the Lemma.

For the second estimate, recall that $a = a^{(2)} + b$ with $|a^{(2)}| \le Cn^2\rho^2(t)$. From Lemma 3.8, there is a perturbation $\widetilde{b}$ such that

$$\frac{d}{dt}\widetilde{b} = b_0 + \hat{b}_0 + \sum_{1<l,k\le K} B_{kl}|z_l|^2|z_k|^2 + g_b, \tag{4.48}$$

where g_b and B_{kl} are defined in Lemma 3.8 and $\hat{b}_0 = B_{00}|z_0|^4 + 2\sum_{1<k\le K} B_{k0}|z_0|^2|z_k|^2$. We have $|b - \widetilde{b}| \le Cn^2\rho^2$ and $|b_0| + |\hat{b}_0| \lesssim n^4|z_0|^2 \lesssim \varepsilon_4^2 n^2 \rho^4$. By Lemma 3.8, (4.12), (in particular $|z_0| \le \varepsilon_4 n^{-1}\rho^2$ and this is where we choose ε_4), (4.13), Lemma 4.3, (4.19) and $\hat{X} \lesssim n^4\rho^3 + \|\eta\|_{L^p}^3$,

$$\begin{aligned} |g_b| &\lesssim n^3\rho^4 + n\rho^5 + \varepsilon_4 n\rho^2 \|\eta\|_{L^2_{loc}} + n\rho^2\|\eta^{(3)}\|_{L^2_{loc}} + nX_1 + n\rho\hat{X} \\ &\lesssim o(1)n^2\rho(t)^4 + \widetilde{g}_b, \qquad \widetilde{g}_b = n^2\rho_0^2\langle t\rangle^{-7/3} + n\rho_0\rho^2\langle t\rangle^{-7/6}. \end{aligned} \tag{4.49}$$

Then, for $t \ge \Delta t = n^{-2}\rho_0^{-2}$, we have $\rho(t) \sim n^{-1}t^{-1/2}$ and

$$\int_t^T |\widetilde{g}_b|(s)ds \lesssim \int_t^\infty [n^4 s^{-7/3} + s^{-7/6-1}]ds \lesssim n^4 t^{-4/3} + t^{-7/6} \lesssim n^2\rho(t)^2. \tag{4.50}$$

For $0 \le t \le \Delta t$, we have $\rho(t) \sim \rho_0$ and

$$\int_t^T |\widetilde{g}_b|(s)ds \le \left(\int_t^{\Delta t} + \int_{\Delta t}^\infty\right)|\widetilde{g}_b|(s)ds \lesssim \int_t^{\Delta t} n^2\rho_0^2\langle s\rangle^{-7/6}ds + n^2\rho_0^2 \lesssim n^2\rho_0^2. \tag{4.51}$$

Using $\int_t^\infty n^2\rho^4 ds \lesssim \rho(t)^2$, we get have

$$\int_t^T |b_0 + g_b|(s)ds \le o(1)\rho(t)^2, \ \forall\, t \in [0, T). \tag{4.52}$$

Integrating (4.48) on (t, T) and using $\max_{kl}(|B_{kl}|)/(K^{-1}\gamma_0 n^2) \le \frac{D}{2}$, we get

$$|\widetilde{b}(t)| \le |\widetilde{b}(T)| + \frac{D}{2}\rho^2(t) + o(1)\rho^2(t) \le |\widetilde{b}(T)| + \frac{5}{9}D\rho^2(t). \tag{4.53}$$

Now, since $a(T) = 0$, we get

$$|\widetilde{b}(T)| = |a(T) - b(T)| + |b(T) - \widetilde{b}(T)| \le |a^{(2)}(T)| + Cn^2\rho(T)^2 \lesssim n^2\rho(t)^2. \tag{4.54}$$

Thus we have $|\widetilde{b}(t)| \le |\widetilde{b}(T)| + |\widetilde{b}(t) - \widetilde{b}(T)| \le \frac{5}{8}D\rho(t)^2$ and

$$|a(t)| \le |a^{(2)}(t)| + |\widetilde{b}(t)| + |\widetilde{b}(t) - b(t)| \le \frac{3}{4}D\rho(t)^2. \tag{4.55}$$

Finally, Lemma 3.3 shows $|n(T) - n(t)| \lesssim n^{-1}|a(t)| + n^3 \ll n$ and the last claim of the Lemma. $\square$

The proof of Lemma 4.4 and Lemma 4.5 complete the proof of Proposition 4.2.

We now distinguish the two cases that $t_c = \infty$ and $t_c < \infty$.

Suppose $t_c = \infty$. By Lemma 3.3 (iii) we have for any $t < T < \infty$

$$|n(t)^2 - n(T)^2| \lesssim |a_{n(T)}(t)| \lesssim \rho^2(t), \tag{4.56}$$

which shows that $n(t)$ converges to some $n_\infty \sim n$ as $t \to \infty$. Furthermore $n(t) \sim n(0) \sim n_\infty$ and $|n(t) - n_\infty| \lesssim n^{-1}\rho^2(t)$. Together with the estimate $M_T \le 3/2$ we have shown the main theorem in the case the solution converges to an excited state.

In the case $t_c < \infty$, by continuity we also have $M_{t_c} \le 3/2$. we will show that the solution escapes from the first excited state family in the next section. We prepare it with the following lemma, whose proof is the same as that for $\eta_\pm(t)$ in Lemma 4.4 with the nonlinear terms set to zero for $t_c < s < t$.

Lemma 4.6 *Suppose $t_c < \infty$. Let $\Delta t = n^{-2}\rho_0^{-2}$ and $\eta_\pm(t) = e^{\mp i\theta(t)}P_\pm[\eta(t)]$ where $\eta(t)$ is as in* (4.3) *with respect to $Q_{1,n(t_c)}$. Then for all $t \ge t_c$, we have*

$$\left\|e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c)\right\|_{L^p} \le \frac{1}{4}\Lambda_1(t), \quad \left\|e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c)\right\|_{L^2_{\rm loc}} \le \frac{1}{4}\Lambda_2(t), \tag{4.57}$$

where for C_2 from Lemma 4.1, some $C_3 > 0$ and $\rho_c = \rho(t_c)$,

$$\begin{aligned}\Lambda_1(t) &= C_3[C_2\rho_0\langle t\rangle^{-\sigma(p)} + n^{2\sigma-1}\rho_0^{2\alpha}\rho(t)^{2\sigma-2\alpha}],\\ \Lambda_2(t) &= C_3[C_2\rho_0\langle t\rangle^{-7/6} + n\rho_c^2\langle t-t_c\rangle^{-7/6} + \rho^3(t) + n^{4/5}\rho^{7/3}(t)].\end{aligned} \tag{4.58}$$

Moreover, with $\sigma_2 := \min(\delta, \frac{3}{2} - \delta, \frac{2+5\delta}{15}) > 0$ and $t_c^+ := t_c + n^{-3}$,

$$\begin{aligned}&\Lambda_1(t) + \Lambda_2(t) \lesssim \rho_c, && (\forall t > t_c),\\ &\Lambda_1 \lesssim \rho_0\,\langle t\rangle^{-\sigma} + n^{1/3}\rho_c^{4/3}, \quad \Lambda_2 \lesssim \rho_0\,\langle t\rangle^{-7/6} + n\rho_c^2, && (t_c < t < t_c^+),\\ &\Lambda_1(t) \lesssim n^{1/3}\rho_c^{4/3}, \quad \Lambda_2(t) \lesssim n^{1+\sigma_2}\rho_c^2, && (t > t_c^+).\end{aligned} \tag{4.59}$$

Proof. From (3.51), we have

$$e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c) = e^{\mathbf{L}t}\eta_\pm(0) + \int_0^{t_c} e^{\mathbf{L}(t-s)}P_\pm\{F_{L\pm} + e^{\mp i\theta}J[F]\}(s)ds. \tag{4.60}$$

We also decompose $\eta_\pm(t_c) = \eta_\pm^{(2)}(t_c) + \eta_\pm^{(3)}(t_c)$ with a similar formula for $e^{\mathbf{L}(t-t_c)}\eta_\pm^{(3)}(t_c)$. We can bound $e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c)$ in L^p and $e^{\mathbf{L}(t-t_c)}\eta_\pm^{(3)}(t_c)$ in L^2_{loc} using the same proof for Lemma 4.4 with the integrand set to zero for $t_c < s < t$. We also have

$$\left\|e^{\mathbf{L}(t-t_c)}\eta_\pm^{(2)}(t_c)\right\|_{L^2_{\rm loc}} \lesssim \langle t-t_c\rangle^{-3/2}n\rho_c^2 \tag{4.61}$$

using the explicit definition of $\eta_{\pm}^{(2)}$ in (3.62) and Lemma 2.13. The above shows (4.57).

We now show (4.59). Its first part is because $\rho_0 \langle t\rangle^{-1/2} \le \rho_c$ for all $t \ge t_c$, which follows from (4.14).

Its second part follows from $\rho(t) \sim \rho_c < \rho_0$.

For the third part with $t > t_c^+$, it suffices to show

$$\rho_0 \langle t\rangle^{-\sigma} \lesssim n^{2\sigma-1}\rho_c^{2\sigma-2\alpha}, \quad \rho_0 \langle t\rangle^{-7/6} \lesssim n^{1+\sigma_2}\rho_c^2. \tag{4.62}$$

If $t_c < \Delta t$, then $\rho \sim \rho_c \sim \rho_0$. Writing all factors as powers of n using $\langle t\rangle^{-1} \le n^3$, (4.62) is reduced to $1+\delta+3\sigma > 2\sigma - 1 + (2\sigma-2\alpha)(1+\delta)$ and $1+\delta+7/2 > 1+\sigma_2+2(1+\delta)$. Both are valid using $2/3<\sigma<3/4$, $0<\delta<3/2$ and $\sigma_2 < 3/2-\delta$.

If $t_c > \Delta t$, then $\rho_c \sim n^{-1}t_c^{-1/2}$, and (4.62) is reduced to $n^{1+\delta}\langle t\rangle^{-\sigma} \lesssim n^{-1+2\alpha}t_c^{-\sigma+\alpha}$ and $n^{1+\delta}\langle t\rangle^{-7/6} \lesssim n^{-1+\sigma_2}t_c^{-1}$, both are correct. □

5 Escaping from an excited state

In this section we study the dynamics near an excited state when $t > t_c$ assuming $t_c < \infty$. We want to show that the solution will escape from the ρ_0-neighborhood of the excited state. Recall $\rho_0 = n^{1+\delta}$ with $0<\delta<3/2$. (We need $\delta \ll 1$ in next section but not here.)

Fix $Q = Q_{1,n(t_c)}$ and decompose $\psi(t)$ for $t_c \le t < T_e$ as in (4.3) and (4.4) with respect to this fixed Q. At $t = t_c$ we have Lemma 4.6 and, by definition of t_c and $M_{t_c} \le 3/2$,

$$|z_0(t_c)| \ge \varepsilon_4 n^{-1}\rho_c^2, \quad z_H(t_c) \le \frac{3}{2}\rho_c, \quad |a(t_c)| \le \frac{3}{4}D\rho_c^2, \quad \rho_c := \rho(t_c). \tag{5.1}$$

Let

$$\gamma(t) := |q_0(t)| + n^5|q_0(t)|^{1/2} + \rho_c, \tag{5.2}$$

where $q_0(t)$ is the perturbation of $p_0(t)$ defined in Lemma 3.7. It will be shown to be an upper bound for bound states.[5] We have defined $\gamma(t)$ in terms of $|q_0|$ instead of $|z_0|$ so that it is non-decreasing in t (for $t > t_c^+ := t_c + n^{-3}$).

Define

$$t_o := \sup\left\{t \ge t_c : z_L(s) < 2n^{1+\delta}, \ \forall\, s \in [t_c, t)\right\}. \tag{5.3}$$

The time t_o is the time that z_L becomes powerful enough in orthogonal coordinates. The subscript $_o$ means "out" (of the neighborhood). It follows from Proposition 5.1 below that $t_o < T_e$ and hence the decompositions (4.3) and (4.4) are valid at least slightly beyond t_o.

Recall

$$\frac{27}{5} < p < 6, \quad \sigma = \sigma(p) = \frac{3p-9}{2p}, \quad \frac{2}{3} < \sigma < \frac{3}{4}. \tag{5.4}$$

The main result of the section is the following proposition.

[5] The term $n^5|q_0|^{1/2}$ is included in γ so that $z_H \lesssim \gamma$. Explicitly: The bound of $\|\eta\|_{L^p}$ includes $n^{11}|q_0|$, see (5.32). By (5.21), the bound of $\|\eta^3\|_{L^{9/8}\cap L^{3/2}}$ and hence $\|\eta^{(3)}\|_{L^2_{loc}}$ contains $n^{18}z_L^m$ where $m \to 11/6$ as $p \to 6$. To bound z_H by γ, we need $\|\eta\|_{L^2_{loc}} \lesssim n\gamma^2$ for (5.49) and $\gamma = |q_0| + \rho_c$ is insufficient.

Proposition 5.1 *There exist constants $C_3, D_1 > 0$, uniform in n, (with C_3 greater than that in Lemma 4.6), such that for all $t_c \le t \le t_o$, we have*

$$
\begin{aligned}
&|q_0(t) - q_0(s)| \le \frac{1}{10}\varepsilon_4 n^{-1}\rho_c^2, \quad (t_c \le s \le t \le t_c^+ := t_c + n^{-3}),\\
&\frac{|q_0(t)|}{|q_0(s)|} \in [e^{\frac{1}{2}(\mathrm{Re}\,\lambda_0)(t-s)}, e^{\frac{3}{2}(\mathrm{Re}\,\lambda_0)(t-s)}], \quad (t_c^+ \le s < t),\\
&z_H(t) \le \sqrt{\frac{6D}{\gamma_0}}\gamma(t), \quad |a(t)| \le D_1\gamma^2,\\
&\|\eta(t)\|_{L^p} \le n^{\sigma_1}\gamma(t)^2 + \frac{1}{2}\Lambda_1(t), \quad \sigma_1 = 4\sigma - 3 - \alpha,\\
&\left\|\eta^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le C_3 n^5\gamma(t)^2 + C_3\gamma(t)^3 + \frac{1}{2}\Lambda_2(t),
\end{aligned}
\tag{5.5}
$$

where $\alpha > 0$ is so small that $-\frac{1}{3} + 2\alpha < \sigma_1 = \frac{3(p-6)}{p} - \alpha < 0$, and $\Lambda_1(t)$ and $\Lambda_2(t)$ are defined in (4.57). *In particular, $t_0 \le T_e$ and for some constants c_1 and c_2,*

$$
t_c + c_1 n^{-4}\log\frac{2\rho_0}{z_L(t_c)} \le t_o \le t_c + c_2 n^{-4}\log\frac{2\rho_0}{z_L(t_c)}. \tag{5.6}
$$

The main term in the integrand of η is of order nz^2. In the first term of its L^p-bound we lose some powers of n due to integration over a time interval of order n^{-4}. On the other hand, the first term γ^3 of $\|\eta(t)\|_{L^2_{\mathrm{loc}}}$ estimate is optimal and comes from recent time terms of order z^3 in the integrand.

Proof. The lemma clearly holds true for $t = t_c$. By a continuity argument, it suffices to prove the lemma with additional weaker assumptions:

$$
\begin{aligned}
&|q_0(t) - q_0(s)| \le \frac{1}{2}\varepsilon_4 n^{-1}\rho_c^2, \quad (t_c \le s \le t \le t_c^+),\\
&\frac{|q_0(t)|}{|q_0(s)|} \in [e^{\frac{1}{4}(\mathrm{Re}\,\lambda_0)(t-s)}, e^{2(\mathrm{Re}\,\lambda_0)(t-s)}], \quad (t_c^+ \le s < t),\\
&z_H(t) \le 2\sqrt{\frac{6D}{\gamma_0}}\gamma(t), \quad |a(t)| \le 2D_1\gamma^2,\\
&\|\eta(t)\|_{L^p} \le 2n^{\sigma_1}\gamma(t)^2 + 2\Lambda_1(t),\\
&\left\|\eta^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le 2C_3 n^5\gamma(t)^2 + 2C_3\gamma(t)^3 + 2\Lambda_2(t).
\end{aligned}
\tag{5.7}
$$

At least for t near t_c, the assumptions of Lemma 3.7 are satisfied and hence $|z_0| \le |q_0| + |p_0 - q_0| \le \gamma + Cn\gamma^2 = (1+o(1))\gamma$. Together with (5.7) and $[\eta] = \eta^{(2)} + \eta^{(3)}$, the assumptions of Lemmas 3.4–3.7 are valid until $t = t_o$ with $\beta = (1+o(1))\gamma(t)$, and

$$
\begin{aligned}
|z_0(t)| &\le (1+o(1))\gamma(t),\\
\|\eta(t)\|_{L^2_{loc}} &\le Cn\gamma^2(t) + \Lambda_2(t),\\
\|\eta(t)\|_{L^2_{loc}\cap L^p} &\le \gamma(t).
\end{aligned}
\tag{5.8}
$$

Here we have used (4.59).

It is convenient to have an upper bound of γ in terms of $|q_0|$. Clearly

$$\gamma^2(t) \sim |q_0|^2 + n^{10}|q_0| + \rho_c^2 \lesssim \varepsilon_4^{-1} n|q_0(t)| + \varepsilon_4^{-1} n|z_0(t_c)|. \tag{5.9}$$

Since $|z_0(t_c)| \le |q_0(t_c)| + Cn\gamma(t_c)^2 \le |q_0(t)| + Cn\gamma(t)^2$, we get

$$\gamma^2(t) \lesssim \varepsilon_4^{-1} n|q_0(t)|. \tag{5.10}$$

Thus we get an improved z_0 estimate,

$$|z_0| \le |q_0| + Cn\gamma^2 \le (1+o(1))|q_0|. \tag{5.11}$$

We can also derive from (5.7) and $|z_0(t_c)| \ge \varepsilon_4 n^{-1}\rho_c^2$ that, for any $t_c \le s < t < t_o$,

$$|q_0(s)| \le \frac{6}{5}|q_0(t)|e^{-\frac{1}{4}(\operatorname{Re}\lambda_0)(t-s)}. \tag{5.12}$$

We now give error estimates. For $X_1 = n\|\eta\|_{L^2_{loc}}^2 + \|\eta^3\|_{L^1_{loc}}$, using (5.7), (5.8), and Hölder inequality, we have

$$X_1 \lesssim n(n^2\gamma^4 + \Lambda_2^2) + (n\gamma^2 + \Lambda_2)^A (n^{\sigma_1}\gamma^2 + \Lambda_1)^B, \tag{5.13}$$

with $A = \frac{2p-6}{p-2}$ and $B = \frac{p}{p-2}$. We claim that

$$X_1(t) \lesssim \begin{cases} n\gamma^2, & (\forall t > t_c), \\ n\rho_0^2 \langle t\rangle^{-7/6} + n^{2.8}\gamma^4, & (t_c < t < t_c^+), \\ n^{2.8}\gamma^4, & (t > t_c^+). \end{cases} \tag{5.14}$$

The first estimate is because $\Lambda_1 + \Lambda_2 \lesssim \rho_c$. The last estimate is, using $(4.59)_3$ and $1.4 < A < 1.5 < B < 1.6$ with $A + B = 3$,

$$X_1(t) \lesssim n^3\gamma^4 + (n\gamma^2)^A (n^{1/3}\gamma^{4/3})^B = n^3\gamma^4 + (n\gamma)^{2A/3} n\gamma^4 \lesssim n^{2.8}\gamma^4. \tag{5.15}$$

When $t_c < t < t_c^+$, using $\rho \sim \rho_c < \rho_0$, $(4.59)_2$, $\sigma_1 > -1/3$, and the previous estimate,

$$\begin{aligned} X_1(t) &\lesssim n^3\gamma^4 + n\rho_0^2 \langle t\rangle^{-7/3} + (\rho_0 \langle t\rangle^{-7/6} + n\gamma^2)^A (\rho_0 \langle t\rangle^{-\sigma} + n^{1/3}\gamma^{4/3})^B \\ &\lesssim n\rho_0^2 \langle t\rangle^{-7/6} + n^{2.8}\gamma^4. \end{aligned} \tag{5.16}$$

For $\widetilde{X}$ and X defined in (3.70), we have

$$\begin{aligned} \widetilde{X} &\le \gamma^2 \|\eta\|_{L^2_{loc}} + X_1 \le n\gamma^4 + \gamma^2\Lambda_2 + X_1(t), \\ X &\le n\gamma \|\eta\|_{L^2_{loc}} + X_1 \le n^2\gamma^3 + n\gamma\Lambda_2 + X_1(t). \end{aligned} \tag{5.17}$$

For $\hat{X}_p$ defined in (3.73) we have

$$\begin{aligned} \hat{X}_p &= n^4 z_L \|\eta\|_{L^p}^2 + n^6 z_L^2 \|\eta\|_{L^p} + n^{6(6-p)/p} \|\eta\|_{L^p}^3 \\ &\lesssim n^4 z_L (n^{2\sigma_1}\gamma^4 + \Lambda_1^2) + n^6 z_L^2 (n^{\sigma_1}\gamma^2 + \Lambda_1) + n^{6(6-p)/p}(n^{3\sigma_1}\gamma^6 + \Lambda_1^3). \end{aligned} \tag{5.18}$$

Using Young's inequality on $n^4 z_L \Lambda_1^2 + n^6 z_L^2 \Lambda_1$, and $6(6-p)/p + 3\sigma_1 = \sigma_1 - 2\alpha > -1/2$, we get

$$\hat{X}_p \lesssim n^{3/2}\gamma^4 + n^{8.5} z_L^3 + n^{6(6-p)/p}\Lambda_1^3. \tag{5.19}$$

From (3.22), (3.26), Lemmas 3.4, 3.5 and (5.7), (5.17) and $(4.59)_2$, we get

$$\begin{aligned}|\dot\theta| &\lesssim \beta^2 + n^{-1}X \lesssim \gamma^2 + n^{-1}(n^2\gamma^3 + n\gamma\Lambda_2 + X_1) \lesssim \gamma^2,\\ |\dot p_k| &\lesssim n^4 z_L + n\beta^2 + \hat X_p + X \lesssim n^4 z_L + n\gamma^2 + X_1 \lesssim n^4 z_L + n\gamma^2.\end{aligned} \tag{5.20}$$

We now estimate the main terms. By Hölder inequality,

$$\left\|\eta^3\right\|_{L^{p'}} \le \|\eta\|_{L^2}^{\frac{2(p-4)}{p-2}} \|\eta\|_{L^p}^{\frac{p+2}{p-2}}, \quad \left\|\eta^3\right\|_{L^{9/8}\cap L^{3/2}} \le \|\eta\|_{L^2}^{\frac{2(2p-9)}{2(p-2)}} \|\eta\|_{L^p}^{\frac{11p}{9(p-2)}}. \tag{5.21}$$

Using $36/7 < p < 6$ and $-\frac12 < \sigma_1 = 4\sigma - 3 - \alpha = 3 - \frac{18}{p} - \alpha < 0$,

$$(n^{4\sigma-3-\alpha}\gamma^2)^{\frac{p+2}{p-2}} \le (n^{4\sigma-3-\alpha}\gamma^2)^{\frac{11p}{9(p-2)}} \le o(1)\gamma^3, \tag{5.22}$$

for $\alpha > 0$ sufficiently small. By Lemma 3.4 and $\|\eta\|_{L^2} \le o(1)$, we get

$$\begin{aligned}\|F\|_{L^{p'}} &\lesssim n\gamma^2 + X + n\|\eta\|_{L^p}^2 + \left\|\eta^3\right\|_{L^{p'}} \lesssim n\gamma^2 + \delta_2,\\ \|F - F_1\|_{L^{9/8}\cap L^{3/2}} &\lesssim \gamma^3 + X + n\|\eta\|_{L^p}^2 + \left\|\eta^3\right\|_{L^{9/8}\cap L^{3/2}} \lesssim \gamma^3 + \delta_2,\\ \delta_2(t) &:= n\gamma(t)\Lambda_2(t) + n\Lambda_1^2(t).\end{aligned} \tag{5.23}$$

In deriving the above estimates most terms in X_1 are controlled by δ_2 except

$$n^A\gamma^{2A}\Lambda_1^B \le (n^{A-B/2}\gamma^{2A})(n^{B/2}\Lambda_1^B) \lesssim (n^{A-B/2}\gamma^{2A})^{2/(2-B)} + (n^{B/2}\Lambda_1^B)^{2/B} \lesssim \gamma^3 + n\Lambda_1^2. \tag{5.24}$$

Estimates (5.5) now follows from Lemmas 5.2 and 5.3 below.

In particular, taking $s = t_c^+$ and $t = t_o$, $(5.5)_2$ together with $\operatorname{Re}\lambda_0 \sim n^{-4}$ and $|z_0| = (1+o(1))|q_0|$ imply (5.6). □

Lemma 5.2 (Dispersion estimates) *For all $t_c \le t \le t_o$, we have*

$$\|\eta(t)\|_{L^p} \le [n^{\sigma_1}\gamma^2 + \Lambda_1](t), \quad \left\|\eta^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \le [C_3 n^5\gamma^2 + C_3\gamma^3 + \Lambda_2](t). \tag{5.25}$$

Note that $\Lambda_j(t)$ may compete with the main terms for t near t_c but decay rapidly.

Proof. We first estimate $\|\eta(t)\|_{L^p}$. It suffices to estimate $\eta_\pm$ with

$$\eta_\pm(t) = e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c) + \int_{t_c}^t e^{\mathbf{L}(t-s)}P_\pm\{F_{L\pm} + e^{\mp i\theta}J[F]\}ds. \tag{5.26}$$

By Lemma 2.11, we have

$$\|\eta_\pm(t)\|_{L^p} \lesssim \left\|e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c)\right\|_{L^p} + \int_{t_c}^t \alpha_p(t-s)\{\|F_{L\pm}\|_{p'} + \|F\|_{p'}\}(s)ds. \tag{5.27}$$

By Lemma 4.6, we have $\left\|e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c)\right\|_{L^p} \le \frac14\Lambda_1(t)$. By (3.49) and (5.20), we get

$$\|F_{L\pm}\|_{L^{p'}\cap L^{9/8}\cap L^{3/2}} \lesssim |\dot\theta|[\|\eta\|_{L^p} + n^{-1}|a| + |z|] \lesssim \gamma^2\cdot\gamma. \tag{5.28}$$

From this, (5.23), (5.27), and $X_1 \ll n\rho_c^2$, we get

$$\int_{t_c}^t \alpha_p(t-s)[\|F_{L\pm}\|_{L^{p'}} + \|F\|_{L^{p'}}](s)ds \lesssim \int_{t_c}^t \alpha_p(t-s)(n\gamma(s)^2 + \delta_2(s))ds. \tag{5.29}$$

Recall $\gamma^2 \sim |q_0|^2 + n^{10}|q_0| + \rho_c^2$. By (5.7), $\operatorname{Re}\lambda_0 \sim n^4$ and $\int^t |t-s|^{-\sigma} e^{-a(t-s)} ds \lesssim a^{\sigma-1}$,

$$\begin{aligned} \int_{t_c}^t \alpha_p(t-s) n|q_0|^2(s) ds &\le \int_{t_c}^t \alpha_p(t-s) n|q_0|(t)^2 e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t-s)} ds \\ &\le Cn^{4\sigma-3}|q_0|^2(t). \end{aligned} \tag{5.30}$$

The integral of $nn^{10}|q_0|$, part of δ_2, is bounded in the same way by $Cn^{4(\sigma-1)+11}|q_0|(t)$.

For ρ_c^2, we have

$$\int_{t_c}^t \alpha_p(t-s) n\rho_c^2 ds \lesssim n\rho_c^2 \langle t-t_c\rangle^{1-\sigma} = n^{4\sigma-3-\alpha/2} \cdot \rho_c^2 n^{\alpha/2} T^{1-\sigma} \tag{5.31}$$

where $\alpha > 0$ is to be chosen and $T = n^4 \langle t - t_c\rangle$. Let $A = \frac{1}{8} n^{-4} \operatorname{Re}\lambda_0$ which is of order 1. If $AT \le 10 \log \frac{1}{n}$, then $n^{\alpha/2} T^{1-\sigma} = o(1)$ if n is sufficiently small. If $AT \ge 10 \log \frac{1}{n}$, then by (5.12)

$$\rho_c^2 T^{1-\sigma} \le Cn|q_0(t_c)|T^{1-\sigma} \le Cn|q_0(t)|e^{-2AT} T^{1-\sigma}. \tag{5.32}$$

Since $e^{-AT} \le n^{10}$ and $e^{-AT} T^{1-\sigma} \le C$, it is bounded by $Cn^{11}|q_0(t)|$.

Using (4.59), the error term $\delta_2(t) = n\gamma(t)\Lambda_2(t) + n\Lambda_1^2(t)$ is bounded by $n^{7/3}\rho_c^2$ when $t > t_c^+$ and by $n^{7/3}\rho_c^2 + n\rho_0^2 \langle t\rangle^{-7/6}$ when $t < t_c^+$. The term $n^{7/3}\rho_c^2$ is smaller than the main term $n\gamma^2$ in (5.29) and can be absorbed, while

$$\int_{t_c}^{t_c^+} n\rho_0^2 \langle t\rangle^{-7/6} dt \lesssim n\rho_c^2 \tag{5.33}$$

which can be checked using $\rho_c \sim \rho_0$ for $t_c < \Delta t$ and $\rho_c \sim n^{-1} t_c^{-1/2}$ for $t_c > \Delta t$.

Thus the integral in (5.27) is bounded by $n^{\sigma_1}\gamma^2$ with $\sigma_1 = 4\sigma - 3 - \alpha$, and we have shown the first estimate of (5.25) for $\|\eta\|_{L^p}$.

Next, we estimate $\|\eta^{(3)}\|_{L^2_{loc}}$. Decompose $\eta_\pm^{(3)} = \sum_{j=1}^4 \eta_{\pm,j}^{(3)}$, where $\eta_{\pm,j}^{(3)}$ are defined explicitly in (3.52) and (3.62) with $t_0 = t_c$. From Lemmas 2.13 and 4.6, we get

$$\left\|\eta_{\pm,1}^{(3)}\right\|_{L^2_{\text{loc}}} \le \frac{1}{4}\Lambda_2(t), \quad \left\|\eta_{\pm,2}^{(3)}\right\|_{L^2_{\text{loc}}} \le \frac{1}{4} C_3 n\rho_c^2 \langle t-t_c\rangle^{-3/2} \le \frac{1}{4}\Lambda_2(t). \tag{5.34}$$

By (3.63) and (5.20), we have

$$\begin{aligned} \Big\| |\dot f_{kl}| + |\dot\theta f_{kl}| \Big\|_{L^2_r} &\lesssim n|\dot\theta|\gamma^2 + n\gamma|\dot p| \lesssim n(\gamma^2)\gamma^2 + n\gamma(n^4\gamma + n\gamma^2) \\ &\lesssim n^5\gamma^2 + n^2\gamma^3. \end{aligned} \tag{5.35}$$

By Lemma 2.13 again and $\gamma(s) \lesssim \gamma(t)$ for $s < t$, we obtain

$$\left\|\eta_{\pm,3}^{(3)}\right\|_{L^2_{\text{loc}}} \lesssim \int_{t_c}^t \langle t-s\rangle^{-3/2} [n^5\gamma^2 + n^2\gamma^3](s) ds \lesssim [n^5\gamma^2 + n^2\gamma^3](t). \tag{5.36}$$

Finally, $\left\|\eta_{\pm,4}^{(3)}\right\|_{L^2_{\text{loc}}}$ is bounded by $\int_{t_c}^t \alpha_\infty(t-s) I_4(s) ds$ by Lemma 2.11, with

$$I_4 = \|F_{L\pm}\|_{L^{9/8}\cap L^{3/2}} + \|F - F_1\|_{L^{9/8}\cap L^{3/2}} \lesssim \gamma^3 + \delta_2 \tag{5.37}$$

by (5.28) and $(5.23)_2$. Using $\delta_2(t) = n\gamma(t)\Lambda_2(t) + n\Lambda_1^2(t)$ and the explicit form of Λ_j in (4.58) together with the integral bound (4.34), we get

$$\begin{aligned} \left\|\eta_{\pm,4}^{(3)}\right\|_{L^2_{\rm loc}} &\lesssim \int_{t_c}^{t} \alpha_\infty(t-s)[\gamma^3 + \delta_2](s)ds \\ &\lesssim \gamma^3(t) + n\rho_0^2 \langle t\rangle^{-7/6} + n^{5/3}\rho^{7/3} \lesssim \gamma^3(t) + o(1)\Lambda_2(t). \end{aligned} \tag{5.38}$$

Summing the above estimates, we get the second estimate of (5.25) for $\|\eta^{(3)}\|_{L^2_{\rm loc}}$. $\square$

Lemma 5.3 (Bound states estimates) *There is a uniform in n constant $D_1 > 0$ such that for all $t_c \le t \le t_o$, we have*

$$\begin{aligned} &|q_0(t) - q_0(t_c)| \le \frac{1}{10}\varepsilon_4 n^{-1}\rho_c^2, \quad (t_c \le t \le t_c^+), \\ &\frac{|q_0(t)|}{|q_0(s)|} \in [e^{\frac{1}{2}(\operatorname{Re}\lambda_0)(t-s)}, e^{\frac{3}{2}(\operatorname{Re}\lambda_0)(t-s)}], \quad (t_c^+ \le s < t), \\ &z_H(t) \le \sqrt{\frac{6D}{\gamma_0}}\gamma(t), \quad |a(t)| \le D_1\gamma(t)^2. \end{aligned} \tag{5.39}$$

Proof. First we estimate $q_0(t)$. From Lemma 3.7, we have

$$\dot q_0(t) = (\operatorname{Re}\lambda_0)q_0 + \widetilde{Y}_0 q_0 + g_0, \quad |q_0 - p_0| \lesssim n\gamma^2, \quad |\operatorname{Re}(\widetilde{Y}_0)| \le Cn^2\gamma^2 \ll n^4. \tag{5.40}$$

Here $\widetilde{Y}_0 = Y_0 + \sum_{l\neq 1} D_{0l}|q_l|^2$. Moreover, from (3.90), (5.17) and (5.10), we have

$$|g_0| \le C[n^5\gamma^2 + n\gamma^4 + n^3\gamma\|\eta\|_{L^2_{\rm loc}} + n\gamma\left\|\eta^{(3)}\right\|_{L^2_{\rm loc}} + \hat X_p + \widetilde{X}] \le o(1)n^4|q_0| + \delta_3, \tag{5.41}$$

where $\delta_3 = C(n^{6(6-p)/p}\Lambda_1^3 + \gamma^2\Lambda_2 + X_1)$. If $t < t_c^+$, by $(4.59)_2$, $(5.14)_2$ and (5.33),

$$\begin{aligned} \delta_3(t) &\lesssim n\rho_0^2\langle t\rangle^{-7/6} + n\rho_c^4 + n\gamma^2\rho_c^2 + n^{2.8}\gamma^4, \\ |q_0(t) - q_0(t_c)| &\le \int_{t_c}^{t_c^+} Cn^4|q_0| + \delta_3(s)ds \le o(1)(|q_0(t_c)| + \varepsilon_4 n^{-1}\rho_c^2), \end{aligned} \tag{5.42}$$

This shows the $q_0(t)$-estimate for $t < t_c^+$. Suppose now $t_c^+ < t$. By $(4.59)_3$, $(5.14)_3$, and (5.10),

$$\delta_3(t) \lesssim n^{6(6-p)/p}(n^{1/3}\rho_c^{4/3})^3 + \gamma^2 n^{1+\sigma_2}\rho_c^2 + n^{2.8}\gamma^4 \ll n^4|q_0|. \tag{5.43}$$

Since $\operatorname{Re}\lambda_0 > 0$ is of order n^4, Eq. (5.40) gives

$$0 < \frac{1}{2}(\operatorname{Re}\lambda_0)|q_0| \le \frac{d}{dt}|q_0| \le \frac{3}{2}(\operatorname{Re}\lambda_0)|q_0|, \tag{5.44}$$

which implies the estimate of $|q_0(t)|$ for $t > t_c^+$.

Next, we estimate $z_H(t)$. For any $k > 1$, by Lemma 3.7, we have

$$\frac{d}{dt}q_k = \sum_{l>1} D_{kl}|q_l|^2 q_k + Y_k q_k + g_k, \quad |q_k - p_k| \le Cn\gamma^2. \tag{5.45}$$

Moreover, we have

$$|D_{kl}| \le Dn^2, \quad |\operatorname{Re}(Y_k)| \le Dn^2|z_0|^2, \quad \operatorname{Re}(D_{kl}) \le -\frac{\gamma_0}{2}n^2, \quad \forall l > 1. \tag{5.46}$$

So, we have

$$\frac{d}{dt}(|q_k|) \le -\frac{\gamma_0 n^2}{2}\sum_{l>1}|q_l|^2|q_k| + 2Dn^2|q_0|^2|q_k| + |g_k|. \tag{5.47}$$

Let $f(t) = (\sum_{l>1}|q_l|^2)^{1/2}$. We have $f(t_c) \lesssim \rho_c$ and

$$\dot f(t) \le -\frac{\gamma_0 n^2}{2} f^3 + 2Dn^2|q_0|^2 f(t) + \sum_{k>1}|g_k|. \tag{5.48}$$

On the other hand, from (3.90), we have

$$|g_k| \le C[n\gamma^4 + n^4\gamma^3 + n^3\gamma\,\|\eta\|_{L^2_{\mathrm{loc}}} + n\gamma\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + n\gamma\hat X_p + \widetilde X] \le o(1)n^2\gamma^3 + \delta_4, \tag{5.49}$$

where $\delta_4 = C(n\gamma n^{6(6-p)/p}\Lambda_1^3 + n\gamma\Lambda_2 + X_1)$. If $t \le t_c^+$, by $(4.59)_2$, $(5.14)_2$ and (5.33)

$$\begin{aligned} &\delta_4(t) \lesssim n\rho_0^2\langle t\rangle^{-7/6} + n^2\rho_c^2\gamma + n^{2.8}\gamma^4 \\ &|f(t) - f(t_c)| \le \int_{t_c}^{t_c^+} Cn^2\rho_c^3 + \delta_4(s)ds \le Cn\rho_c^2 \ll \rho_c. \end{aligned} \tag{5.50}$$

Thus $f(t) \lesssim \rho_c$ for $t < t_c^+$. When $t_c^+ < t$, since $\delta_4(t) \le n^2\gamma^5 + n^{2+\sigma_2}\gamma^3 + n^{2.8}\gamma^4 \ll n^2\gamma^3$, for $\widetilde\gamma = (\frac{16D}{3\gamma_0})^{1/2}\gamma$,

$$\dot f(t) \le \frac{\gamma_0 n^2}{4}[\widetilde\gamma^3 - f^3], \quad (t > t_c^+). \tag{5.51}$$

Since $\gamma(t)$ is nondecreasing and $f(t_c^+) < \widetilde\gamma(t_c^+)$, by comparison we get

$$f(t) \le \widetilde\gamma(t), \quad \forall\ t > t_c^+. \tag{5.52}$$

Thus $z_H(t) \le f(t) + |f(t) - z_H(t)| \le \widetilde\gamma(t) + Cn\gamma^2(t) < \sqrt{\frac{6D}{\gamma_0}}\gamma(t)$.

Finally, we estimate $a(t)$. By (3.39) and Lemma 3.8, $a = a^{(2)} + (b - \widetilde b) + \widetilde b$, where

$$|a^{(2)}| \lesssim n^2\gamma^2, \quad |\widetilde b - b| \le Cn\gamma[\gamma^2 + n\,\|\eta\|_{L^2_{\mathrm{loc}}}] \le Cn^2\gamma^2, \tag{5.53}$$

and

$$\frac{d}{dt}\widetilde b = b_0 + \sum_{k,l\ne 1} B_{kl}|z_k|^2|z_l|^2 + g_b. \tag{5.54}$$

Using $a(t_c) = 0$,

$$\begin{aligned} |a(t) - 0| &\le |a^{(2)}(t)| + |a^{(2)}(t_c)| + |(b-\widetilde b)(t)| + |(b-\widetilde b)(t_c)| + |\widetilde b(t) - \widetilde b(t_c)| \\ &\le Cn^2\gamma^2(t) + \int_{t_c}^t |\frac{d}{dt}\widetilde b|. \end{aligned} \tag{5.55}$$

From (3.34), $b_0(t) = b_{00}|z_0(t)|^2$ with $b_{00} = 2\,\mathrm{Im}\,\kappa c_0(Q^2, \bar u_0^+ u_0^-)$ and $|b_{00}|n^{-4} \le C_4$ for some explicit $C_4 = O(1)$. We also have $|B_{kl}||z_k|^2|z_l|^2 \lesssim n^2\gamma^4$ and

$$\begin{aligned} |g_b| &\le C[n^3\gamma^4 + n^2\beta^2|z_0|^2 + n\beta^5 + n^2|z_0|\,\|\eta\|_{L^2_{\mathrm{loc}}} + nX_1 + n\gamma^2\left\|\eta^{(3)}\right\|_{L^2_{\mathrm{loc}}} + n\gamma\hat X_p] \\ &\le o(1)n^4|z_0|^2 + Cn^2\gamma^4 + \delta_5, \end{aligned} \tag{5.56}$$

where $\delta_5 = nX_1 + (n^2 z_L + n\gamma^2)\Lambda_2 + n\gamma n^{6(6/p-1)}\Lambda_1^3$. Thus

$$|a(t)| \le Cn^2\gamma^2(t) + \int_{t_c}^t (C_4 + o(1))n^4|q_0(s)|^2 + Cn^2\gamma^4(s) + \delta_5(s)ds. \tag{5.57}$$

By (5.7),

$$\int_{t_c}^t (C_4 + o(1))n^4|q_0(s)|^2 ds \le \frac{6}{5}C_4 n^4|q_0(t)|^2 \int_{t_c}^t e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t-s)}ds \le \frac{24C_4n^4}{5\operatorname{Re}\lambda_0}|q_0(t)|^2. \tag{5.58}$$

Moreover, by the definition of γ,

$$\int_{t_c}^t Cn^2\gamma^4(s)ds \lesssim \int_{t_c}^t [n^2|q_0|^4 + n^{22}|q_0|^2](s)ds + n^2\rho_c^4(t-t_c). \tag{5.59}$$

The integral is bounded by $n^{-2}|q_0|^4 + n^{18}|q_0|^2 = o(1)|q_0|^2$ similarly as in (5.58), while the last term is bounded by $n^2\rho_c^4 Cn^{-4}\log\frac{|z_0|(t)}{\varepsilon_4 n\rho_c^2} = o(1)\rho_c^2$. Thus this term is $o(1)\gamma^2$.

For the error term $\int_{t_c}^t \delta_5(s)ds$, if $t \le t_c^+$, by $(4.59)_2$ and $(5.14)_2$ we have

$$\begin{aligned}\delta_5(s) &\le n^2\rho_0^2\langle t\rangle^{-7/6} + n^{3.8}\gamma^4 + (n^2|q_0(t_c)| + n\gamma^2)(\rho_0\langle t\rangle^{-7/6} + n\rho_c^2)\\ &\quad + n\gamma(t_c)(\rho_0^3\langle t\rangle^{-3\sigma} + n\rho_c^4)\\ &\le n^2\rho_0^2\langle t\rangle^{-7/6} + o(1)n^4\gamma^2.\end{aligned} \tag{5.60}$$

Thus, using (5.33), we have $\int_{t_c}^t \delta_5(s)ds \le o(1)n\gamma(t_c)^2$. If $t > t_c^+$, by $(4.59)_3$ and $(5.14)_3$ we have $\delta_5(s) \le n^{3.8}\gamma^4 + n^2\gamma n^{1+\sigma_2}\rho_c^2 + n\gamma n\rho_c^4 = o(1)(n^2\gamma^4 + n^4\gamma^2)$, which is dominated by other terms in (5.57).

In conclusion, we have shown

$$|a(t)| \le D_1\gamma^2(t), \quad D_1 := \frac{5C_4n^4}{\operatorname{Re}\lambda_0} = O(1). \tag{5.61}$$

This completes the proof of the Lemma 5.3. □

The above finishes the proof of Proposition 5.1.

We now prove the following out-going estimate of η at t_o.

Lemma 5.4 *For some $C_5 > 0$, for all $t \ge t_o$, we have*

$$\begin{aligned}\left\|e^{(t-t_o)\mathbf{L}}\eta_\pm(t_o)\right\|_{L^p} &\le \widetilde{\Lambda}_1(t) := \Lambda_1(t) + C_5n^{-2}\rho_0(n^{-4} + t - t_o)^{-\sigma},\\ \left\|e^{(t-t_o)\mathbf{L}}\eta_\pm(t_o)\right\|_{L^2_{loc}} &\le \widetilde{\Lambda}_2(t) := \Lambda_2(t) + C_5n\rho_0^2\langle t-t_o\rangle^{-7/6}\\ &\quad + C_5\rho_0^3\langle t-t_o\rangle^{-1/6}n^{-4}(t-t_o+n^{-4})^{-1}\\ &\quad + C_5n^{-3}(n^{7/3}\rho_c + \rho_c^2)(t-t_o+n^{-4})^{-7/6}.\end{aligned} \tag{5.62}$$

Proof. For all $t \ge t_o$, we have

$$e^{\mathbf{L}(t-t_o)}\eta_\pm(t_o) = e^{\mathbf{L}(t-t_c)}\eta_\pm(t_c) + \int_{t_c}^{t_o} e^{\mathbf{L}(t-s)}P_\pm\{F_{L\pm} + Je^{i\theta}[F]\}ds. \tag{5.63}$$

We first bound it in L^p. By Lemma 4.6, the first term is bounded in L^p by $\Lambda_1(t)$. The second term is bounded in L^p as in (5.29) by

$$\lesssim \int_{t_c}^{t_o} \alpha_p(t-s)[\|F_{L\pm}\|_{L^{p'}} + \|F\|_{L^{p'}}]ds \lesssim \int_{t_c}^{t_o} \alpha_p(t-s)[n\gamma^2(s) + \delta_2(s)]ds. \tag{5.64}$$

Note $n\gamma^2 + \delta_2 \sim n|q_0|^2 + n^{11}|q_0| + n\rho_c^2 + \delta_2$. By (5.7),

$$\int_{t_c}^{t_o} \alpha_p(t-s)n|q_0|^2(s)ds \le \frac{6}{5}\int_{t_c}^{t_o} \alpha_p(t-s)n\rho_0^2 e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t_o-s)}ds. \tag{5.65}$$

Using

$$\int^{t_o} |t-s|^{-\sigma}e^{-(t_o-s)/T}ds \lesssim \int_{t_o-T}^{t_o} |t-s|^{-\sigma}e^{-(t_o-s)/T}ds \lesssim \int_{t_o-T}^{t_o} |t-s|^{-\sigma}ds \lesssim T(t-t_o+T)^{-\sigma} \tag{5.66}$$

with $T = 4/\operatorname{Re}\lambda_0 \sim n^{-4}$, (5.65) is bounded by $Cn^{-3}\rho_0^2(t-t_o+n^{-4})^{-\sigma}$.

Similarly $\int_{t_c}^{t_o} \alpha_p(t-s)n^{11}|q_0|(s)ds$ is bounded by $n^{11}\rho_0 n^{-4}(t-t_o+n^{-4})^{-\sigma}$.

Let t_k denote the first time in $[t_c, t_o)$ so that $|q_0(t)| = \rho_c$. When $t > t_k$, the integrand ρ_c^2 is dominated by $|q_0|^2$ and can be absorbed. By (5.7), $t_o - t_k \gtrsim n^{-4}\log\frac{2\rho_0}{\rho_c}$. We have

$$\int_{t_c}^{t_k} \alpha_p(t-s)n\rho_c^2 ds \lesssim n\rho_c^2|t_k-t_c||t-t_k|^{-\sigma}. \tag{5.67}$$

Using

$$\rho_c^2 \lesssim \varepsilon_4^{-1}nq_0(t_c) \lesssim \frac{6}{5}\varepsilon_4^{-1}n\rho_c e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t_k-t_c)}, \tag{5.68}$$

and $n^4|t_k-t_c|e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t_k-t_c)} \le C$, the integral in (5.67) is bounded by $Cn^{-2}\rho_c|t-t_k|^{-\sigma}$.

Using (4.59), the error term $\delta_2(t)$ is bounded by $n^{7/3}\rho_c^2$ when $t > t_c^+$ and by $n^{7/3}\rho_c^2 + n\rho_0^2\langle t\rangle^{-7/6}$ when $t < t_c^+$. The term $n^{7/3}\rho_c^2$ is much smaller than the main terms and can be absorbed, while by (5.33),

$$\int_{t_c}^{t_c^+} \alpha_p(t-s)n\rho_0^2\langle s\rangle^{-7/6}ds \lesssim n\rho_c^2|t-t_c|^{-\sigma}. \tag{5.69}$$

Summing the above estimates gives the first estimate of Lemma 5.4.

For the second estimate, we have $\eta_\pm(t_o) = \eta_\pm^{(2)}(t_o) + \eta_\pm^{(3)}(t_o)$. By (3.52), (3.62) and (3.64) with t_0 replaced by t_c, we have for $\tau = t - t_o \ge 0$

$$e^{\mathbf{L}\tau}\eta_\pm(t_o) = e^{\mathbf{L}\tau}\eta_\pm^{(2)}(t_o) + \sum_{j=1}^{4} e^{\mathbf{L}\tau}\eta_{\pm,j}^{(3)}(t_o), \tag{5.70}$$

with

$$e^{\mathbf{L}\tau}\eta_{\pm,1}^{(3)}(t_o) = e^{(t-t_c)\mathbf{L}}\eta_\pm(t_c), \quad e^{\mathbf{L}\tau}\eta_{\pm,2}^{(3)}(t_o) = -e^{(t-t_c)\mathbf{L}}\eta_\pm^{(2)}(t_c), \tag{5.71}$$

$$e^{\mathbf{L}\tau}\eta_{\pm,3}^{(3)}(t_o) = -\int_{t_c}^{t_o} e^{(t-s)\mathbf{L}}e^{\mp i\theta(s)}\Pi_\pm \textstyle\sum_{k,l\in\Omega_m}\big(\operatorname{Re}R_{kl}e^{-i\omega_{kl}s}\dot f_{kl} \mp i\operatorname{Re}R_{kl}e^{-i\omega_{kl}s}\dot\theta f_{kl}\big)(s)ds,$$

$$e^{\mathbf{L}\tau}\eta_{\pm,4}^{(3)}(t_o) = \int_{t_c}^{t_o} e^{(t-s)\mathbf{L}}P_\pm\{F_{L\pm} + Je^{\mp i\theta}[F-F_1]\}ds. \tag{5.72}$$

From the explicit definition of $\eta_\pm^{(2)}(t_o)$ in (3.62) and Lemma 2.13 we obtain

$$\left\| e^{\mathbf{L}\tau}\eta_\pm^{(2)}(t_o) \right\| \le Cn\rho_0^2 \langle t - t_o\rangle^{-3/2}. \tag{5.73}$$

By Lemma 4.6,

$$\left\| e^{\mathbf{L}\tau}\eta_{\pm,1}^{(3)}(t_o) \right\|_{L^2_{\mathrm{loc}}} \le \frac{1}{4}\Lambda_2(t), \quad \left\| e^{\mathbf{L}\tau}\eta_{\pm,2}^{(3)}(t_o) \right\|_{L^2_{\mathrm{loc}}} \le C_3 n\rho_c^2 \langle t - t_c\rangle^{-3/2}. \tag{5.74}$$

As in (5.36) and (5.38), we obtain

$$\left\| e^{\mathbf{L}\tau}(\eta_{\pm,3}^{(3)} + \eta_{\pm,4}^{(3)})(t_o) \right\|_{L^2_{\mathrm{loc}}} \lesssim \int_{t_c}^{t_o} \alpha_\infty(t-s)[n^5\gamma^2 + \gamma^3 + \delta_2](s)ds \le I_1 + I_2 + I_3, \tag{5.75}$$

where I_j are integrals over the same time interval with the following integrands

$$(n^5|q_0|^2 + n^{15}|q_0| + |q_0|^3 + n^{15}|q_0|^{3/2}), \quad (n^{7/3}\rho_c^2 + \rho_c^3)1_{[t_c,t_k]}, \quad n\rho_0^2 \langle s\rangle^{-7/6} 1_{[t_c,t_c^+]}. \tag{5.76}$$

Then

$$\begin{aligned} I_1(t) &\lesssim \int_{t_c}^{t_o} \alpha_\infty(t-s)\rho_0^3 e^{-\frac{1}{4}\operatorname{Re}\lambda_0(t_o-s)}ds \lesssim \rho_0^3 \int_{t_o-n^{-4}}^{t_o} \langle t-s\rangle^{-7/6} ds \\ &\le \rho_0^3 \langle t-t_o\rangle^{-1/6} n^{-4}(t-t_o+n^{-4})^{-1}. \end{aligned} \tag{5.77}$$

With constant $\varepsilon = n^{7/3}\rho_c^2 + \rho_c^3$, using (5.68) and $n^4(t_k - t_c)e^{-\operatorname{Re}\frac{1}{4}\lambda_0(t_k-t_c)} \le C$,

$$\begin{aligned} I_2(t) &\lesssim \int_{t_c}^{t_k} \alpha_\infty(t-s)\varepsilon ds \le \varepsilon(t-t_k)^{-1/6}(t_k-t_c)(t-t_c)^{-1} \\ &\le \varepsilon(t-t_k)^{-1/6}(t-t_c)^{-1}n^{-4}n^4(t_k-t_c)n\rho_c^{-1}e^{-\operatorname{Re}\frac{1}{4}\lambda_0(t_k-t_c)} \\ &\le \varepsilon_4^{-1}n^{-3}(n^{7/3}\rho_c + \rho_c^2)(t-t_k)^{-1/6}(t-t_c)^{-1}. \end{aligned} \tag{5.78}$$

Finally, $I_3(t) \lesssim \int_{t_c}^{t_c^+} \alpha_\infty(t-s)n\rho_0^2 \langle s\rangle^{-7/6} ds \le (t-t_c)^{-7/6}n\rho_0^2$. Summing the estimates we get the second part of the Lemma. □

6 Dynamics away from bound states

In this section, we study the dynamics of the solution $\psi(t)$ for $t_o \le t \le t_i$, where t_o is the time it leaves $2\rho_0$ neighborhood of first excited states, and t_i is the time it enters the ρ_0-neighborhood of ground states, to be defined in (6.73). In this time interval we use orthogonal coordinates and decompose

$$\psi(t) = \sum_{j=0}^{K} x_j(t)\phi_j + \xi(t), \quad \xi(t) \in \mathbf{E}_c^{H_0}, \quad (t \ge t_o). \tag{6.1}$$

We first estimate $x_j(t_o)$ and $\xi(t_o)$ in Lemma 6.1, for which we recall some definitions. Recall that $\Delta t = n^{-2}\rho_0^{-2} = n^{-2(2+\delta)}$, $0 < \alpha \ll 1$ is fixed and $0 < \delta \le \frac{1}{10}$. Moreover,

$\frac{27}{5} < p < 6$ is fixed, $\frac{2}{3} < \sigma = \frac{3(p-3)}{2p} < \frac{3}{4}$, and $\sigma' := \frac{3(p-2)}{2p} > \sigma$. Recall from Lemma 5.4 that $\widetilde{\Lambda}_2 = \widetilde{\Lambda}_{2,1} + \widetilde{\Lambda}_{2,2}$ with

$$\begin{aligned} \widetilde{\Lambda}_{2,1}(t) &:= \Lambda_2(t) + C_5 n\rho_0^2 \langle t-t_o\rangle^{-7/6} + C_5 n^{-3}(n^{7/3}\rho_c + \rho_c^2)(t-t_o+n^{-4})^{-7/6}, \\ \widetilde{\Lambda}_{2,1}(t) &:= C_5\rho_0^3 \langle t-t_o\rangle^{-1/6} n^{-4}(t-t_o+n^{-4})^{-1}. \end{aligned} \tag{6.2}$$

We also define

$$\Lambda_3(t) := 3\widetilde{\Lambda}_2(t) + C_6 n^3(1+t-t_o)^{-3/2}, \quad \Lambda_4(t) := \sum_{j=1}^{3} \Lambda_{4,j}(t), \tag{6.3}$$

where C_6 is some uniform constant defined in (6.14) and

$$\begin{aligned} &\Lambda_{4,1} := C_6 n^{-1+(4+2\delta)\alpha}(\Delta t + t)^{-\sigma+\alpha}, \quad \Lambda_{4,2} := C_6\rho_0(1+t-t_o)^{-\sigma}, \\ &\Lambda_{4,3} := C_6 n^{-1+\delta}(n^{-4}+t-t_o)^{-\sigma}. \end{aligned} \tag{6.4}$$

Note that $\Lambda_{4,1}$ is the second term in Λ_1 and comes from the out-going estimate at t_c; $\Lambda_{4,3}$ is from the out-going estimate at t_o and $\Lambda_{4,2}$ is from (6.14). Also note that

$$\Lambda_3(t) \le 3C_6 n^3, \quad \Lambda_4(t) \le 2C_6 n^{\frac{5p-18}{p}+\delta} + C_6\rho_0\langle t-t_o\rangle^{-\sigma}, \quad \frac{5p-18}{p} > \frac{5}{3}. \tag{6.5}$$

Lemma 6.1 *At $t = t_o$ we have*

$$(1.9)n^{1+\delta} \le |x_0| \le (2.1)n^{1+\delta}, \quad (\sum_{k>1} |x_k|^2)^{\frac{1}{2}} \le 6\sqrt{\frac{D}{\gamma_0}}\rho_0, \quad (0.9)n \le |x_1| \le (1.1)n. \tag{6.6}$$

Moreover, we have for all $t \ge t_o$

$$\left\| e^{-i(t-t_o)H_0}\xi(t_o)\right\|_{L^2_{\mathrm{loc}}} \le \Lambda_3(t), \quad \left\| e^{-i(t-t_o)H_0}\xi(t_o)\right\|_{L^p} \le \Lambda_4(t). \tag{6.7}$$

Proof. For all $0 \le t \le t_o$, we have

$$\psi = [Q + a(t)\partial_E Q + \zeta + \eta]^{-iEt+i\theta} = \sum_{j=0}^{K} x_j\phi_j + \xi. \tag{6.8}$$

Here $Q = Q_{1,n(t_c)}$. Recall $n(t_c) = n + O(n^{1+2\delta})$ by substituting (6.8) with $t = 0$ into $n = |(\phi_1, \psi_0)|$. For $j \ne 1$, taking the inner product of (6.8) at $t = t_o$ with ϕ_j we get

$$|x_j(t_o)| = O(n^3) + (1 + O(n^2))|z_j(t_o)|, \quad (j \ne 1). \tag{6.9}$$

We also have

$$|x_1(t_o)| = (\phi_1, Q) + O(n^3) = n(t_c) + O(n^3) = n + O(n^{1+2\delta}). \tag{6.10}$$

Since $|z_0(t_o)| = (1+o(1))2\rho_0$ and $z_H(t_o) \le \sqrt{6D/\gamma_0}(1+o(1))|z_0(t_o)|$, we have (6.6).

Next, we shall prove (6.7). Denote $\theta_* := iEt_o - i\theta(t_o)$ and

$$x^* = e^{\theta_*}\sum_{j=0}^{K} x_j(t_o)\phi_j, \quad \xi^* = e^{\theta_*}\xi(t_o), \quad \eta^* = \eta(t_o). \tag{6.11}$$

From (6.8), we get

$$\xi^* = P_c^{H_0}\{Q + a(t_o)\partial_E Q + \zeta(t_o) + \eta^* - x^*\}. \tag{6.12}$$

We write $\xi^* = \xi_1^* + \xi_2^* + \xi_3^*$ where

$$\begin{aligned} \xi_1^* &:= P_c^{H_0}\Big\{Q + a(t_o)\partial_E Q + \sum_{j\neq 1} z_j(t_o)\bar u_j^+ + \sum_{j>1} \bar z_j(t_o) u_j^- - x^*\Big\}, \\ \xi_2^* &:= P_c^{H_0}[\bar z_0(t_o)u_0^-], \quad \xi_3^* := P_c^{H_0}\eta^*. \end{aligned} \tag{6.13}$$

From the explicit formulae of $Q, \partial_E Q, u_j^+$, we see that ξ_1^* is localized and $\|\xi_1^*\| \lesssim n^3 + n|a(t_o)| + \max_{j\neq 1}|z_j|n^2 \lesssim n^3$. Therefore, for all $t \ge t_o$, $\tau = t - t_o$, by Lemma 2.5 and Lemma 2.10, we have a uniform constant $C_6 > \max\{C_3, C_5\}$ such that

$$\begin{aligned} \left\|e^{-i\tau H_0}\xi_1^*\right\|_{L^2_{loc}} &\le \frac{1}{2}C_6 n^3(1+\tau)^{-3/2}, & \left\|e^{-i\tau H_0}\xi_1^*\right\|_{L^p} &\le \frac{1}{2}C_6 n^3(1+\tau)^{-\sigma'}, \\ \left\|e^{-i\tau H_0}\xi_2^*\right\|_{L^2_{loc}} &\le \frac{1}{2}C_6 n^{3+\delta}(1+\tau)^{-3/2}, & \left\|e^{-i\tau H_0}\xi_2^*\right\|_{L^p} &\le \frac{1}{2}C_6 \rho_0(1+\tau)^{-\sigma}. \end{aligned} \tag{6.14}$$

Here for $\tau < 1$ we have used $\left\|e^{-i\tau H_0}\xi_2^*\right\|_{L^p} \lesssim \|\xi_2^*\|_{H^1} \lesssim |z_0(t_o)|$. Next, we estimate $e^{-i\tau H_0}\xi_3^*$ in L^2_{loc} and L^p. Note $[e^{-i\tau(H_0-E)}\xi_3^*] = e^{\tau J(H_0-E)}[\xi_3^*]$. Recall that

$$\mathbf{L} = J(H_0 - E) - W, \quad [\eta^*] = e^{i\theta(t_o)}\eta_+^* + e^{-i\theta(t_o)}\eta_-^*, \quad \eta_\pm^* = \eta_\pm(t_o), \tag{6.15}$$

for some localized potential W of order n^2. By Duhamel's principle, we have

$$e^{\tau J(H_0-E)}[\xi_3^*] = P_c^{H_0} e^{\tau \mathbf{L}}[\eta^*] + \int_0^\tau e^{J(H_0-E)(\tau - s)} P_c^{H_0} W e^{\mathbf{L}s}[\eta^*] ds. \tag{6.16}$$

From Lemma 5.4, we get

$$\begin{aligned} \left\|e^{\tau J(H_0-E)}[\xi_3^*]\right\|_{L^p} &\le \sum_\pm \left\|e^{\tau \mathbf{L}}\eta_\pm^*\right\|_{L^p} + Cn^2 \sum_\pm \int_0^\tau |\tau - s|^{-\sigma'} \left\|e^{\mathbf{L}s}\eta_\pm^*\right\|_{L^2_{loc}} ds \\ &\le 2\widetilde\Lambda_1(t) + Cn^2 \sum_\pm \int_0^\tau |\tau - s|^{-\sigma'} \widetilde\Lambda_2(s + t_o) ds. \end{aligned} \tag{6.17}$$

Using the fact that

$$\int_0^t (t-s)^{-\beta_1}(\epsilon^{-1} + s)^{-\beta_2} \le C\epsilon^{\beta_2 - 1}(\epsilon^{-1} + t)^{-\beta_1}, \quad 0 < \beta_1 < 1 < \beta_2, \tag{6.18}$$

we have

$$n^2 \int_0^\tau |\tau - s|^{-\sigma'} \widetilde\Lambda_2(s + t_o) ds \le Cn^\delta \rho_0 \langle t - t_o\rangle^{-\sigma'} + C\rho_0(\Delta t + t)^{-\sigma'} \tag{6.19}$$

which is $o(1)\rho_0\langle t - t_o\rangle^{-\sigma}$. From this and (6.17), we get

$$\left\|e^{\tau J(H_0-E)}[\xi_3^*]\right\|_{L^p} \le 2\widetilde\Lambda_1 + o(1)\rho_0\langle t - t_o\rangle^{-\sigma}. \tag{6.20}$$

Similarly from (6.16) with $\tilde\alpha_\infty(t) = \min\{t^{-3/2},\, t^{-9/10}\}$,

$$\begin{aligned} \left\|e^{\tau J(H_0-E)}[\xi_3^*]\right\|_{L^2_{\rm loc}} &\le \sum_\pm \left\|e^{\tau \mathbf{L}}\eta_\pm^*\right\|_{L^2_{\rm loc}} + Cn^2 \sum_\pm \int_0^\tau \tilde\alpha_\infty(\tau - s) \left\|e^{\mathbf{L}s}\eta_\pm^*\right\|_{L^2_{loc}} ds \\ &\le 2\widetilde\Lambda_2(t) + Cn^2 \sum_\pm \int_0^\tau \tilde\alpha_\infty(\tau - s)\widetilde\Lambda_2(s + t_o) ds \le 3\widetilde\Lambda_2(t). \end{aligned} \tag{6.21}$$

So, (6.7) follows from (6.14), (6.20), and (6.21). This completes the proof of Lemma 6.1. □

For $j \in \{0, 1, \cdots, K\}$, let $f_j := |\mu_j(t)|^2$, where μ_j is the perturbation of x_j defined in Lemma 3.2. Since $\frac{d}{dt}|\mu|^2 = 2\,\mathrm{Re}\,\bar{\mu}\dot{\mu}$ and c_l^j are all purely imaginary, from (3.10) we have

$$\dot{f}_j = \sum_{a,b=0}^{K} 2(\mathrm{Re}\, d_{ab}^j) f_a f_b f_j + 2\,\mathrm{Re}\,\bar{\mu}_j g_j. \tag{6.22}$$

Let

$$f = \sum_{l=1}^{K} f_l, \quad h = \sum_{l=1}^{K} 2^{-l} f_l, \quad \gamma \ := \min\{\gamma_{ab}^0, \text{ for } a, b \geq 1\} > 0. \tag{6.23}$$

Then, from (6.22), Lemma 3.2 and as in [30, (4.58)], we have

$$\frac{d}{dt}(f_0 + f)(t) \leq 2(K+1) \max_l |\bar{\mu}_l g_l|, \quad \frac{d}{dt}(f_0 + h)(t) \geq -2(K+1) \max_l |\bar{\mu}_l g_l|. \tag{6.24}$$

Moreover, we have the following lemma.

Lemma 6.2 *Assume as in Lemma 3.2. We have*

$$\dot{f}_0 \geq 2\gamma f^2 f_0 + 2\,\mathrm{Re}\,\bar{\mu}_0 g_0, \quad \dot{f} \leq -4\gamma f_0 f^2 + \sum_{l=1}^{K} 2\,\mathrm{Re}\,\bar{\mu}_l g_l. \tag{6.25}$$

Proof. From (6.22) and Lemma 3.2 in particular (3.12), we have

$$\dot{f}_0 - 2\,\mathrm{Re}\,\bar{\mu}_0 g_0 = \sum_{a,b=0}^{K} 2\,\mathrm{Re}(d_{ab}^0) f_a f_b f_0 = \sum_{a,b=0}^{K} [2(2-\delta_a^b)\gamma_{ab}^0 - 4(2-\delta_0^b)\gamma_{0b}^a] f_a f_b f_0. \tag{6.26}$$

Note that $\gamma_{0b}^a = 0$ for any a and b. Thus

$$\dot{f}_0 - 2\,\mathrm{Re}\,\bar{\mu}_0 g_0 = \sum_{a,b=1}^{K} 2(2-\delta_a^b)\gamma_{ab}^0 f_a f_b f_0 \geq 2\gamma f^2 f_0. \tag{6.27}$$

This proves the first part of (6.25). For the second part,

$$\begin{aligned} \dot{f} - 2\sum_{l=1}^{K} \mathrm{Re}\,\bar{\mu}_l g_l &= \sum_{l=1}^{K} \sum_{a,b=0}^{K} 2[(2-\delta_a^b)\gamma_{ab}^l - 2(2-\delta_l^b)\gamma_{lb}^a] f_a f_b f_l, \\ &= \sum_{b=0}^{K} \sum_{a,l=1}^{K} 2[(2-\delta_a^b)\gamma_{ab}^l - 2(2-\delta_l^b)\gamma_{lb}^a] f_a f_b f_l + \sum_{l=1}^{K} \sum_{b=0}^{K} -4(2-\delta_l^b)\gamma_{lb}^0 f_0 f_b f_l. \end{aligned} \tag{6.28}$$

By switching a and l in the terms with factor γ_{ab}^l, the summands in the first sum become $-2(2-\delta_l^b)\gamma_{lb}^a f_a f_b f_l \leq 0$. The summands of the second sum are also nonpositive. Keeping only terms with $b > 0$ in the second sum, we get

$$\dot{f} - 2\sum_{l=1}^{K} \mathrm{Re}\,\bar{\mu}_l g_l \leq -4 \sum_{b,l=1}^{K} (2-\delta_l^b)\gamma_{lb}^0 f_0 f_b f_l \leq -4\gamma f_0 f^2.$$

This proves the second part of (6.25). □

The following proposition estimates the solution in a time interval containing $[t_o, t_i]$.

Proposition 6.3 *Let $\delta_6(t) := \rho_0^2 \langle t - t_o \rangle^{-\frac{6}{p}}$. For all $t \in [t_o, t_o + \frac{6}{\gamma} n^{-2(2+\delta)}]$, we have*

$$\begin{aligned} &\frac{n}{5} \le \max_j |x_j| \le (\textstyle\sum_{j=0}^K |x_j(t)|^2)^{\frac{1}{2}} \le 2n, \\ &\|\xi(t)\|_{L^2_{loc}} \le n^{3-\alpha} + \delta_6(t), \quad \|\xi(t)\|_{L^p} \le n^{3-\alpha} |t - t_o|^{\frac{6-p}{2p}} + \frac{3}{2} \Lambda_4(t). \end{aligned} \tag{6.29}$$

Proof. Since (6.29) holds at $t = t_o$, we then prove it by using the continuity argument. So, we can assume the following weaker estimates: For $t_o \le t \le t_o + \frac{6}{\gamma} n^{-2(2+\delta)}$,

$$\begin{aligned} &\frac{n}{10} \le \max_j |x_j| \le (\textstyle\sum_{j=0}^K |x_j(t)|^2)^{\frac{1}{2}} \le 3n, \\ &\|\xi(t)\|_{L^2_{loc}} \le 2[n^{3-\alpha} + \delta_6(t)] \le n^2, \\ &\|\xi(t)\|_{L^p} \le 2n^{3-\alpha} |t - t_o|^{\frac{6-p}{2p}} + 3\Lambda_4(t) \le n^{2.7} + 3\Lambda_4(t). \end{aligned} \tag{6.30}$$

In particular $\|\xi(t)\|_{L^2_{loc}} + \|\xi(t)\|_{L^p} \ll n$. The proof of Proposition 6.3 follows from Lemma 6.4 and Lemma 6.6 below. □

Lemma 6.4 *For all $t \in [t_o, t_o + \frac{6}{\gamma} n^{-2(2+\delta)}]$, we have*

$$\|\xi(t)\|_{L^2_{loc}} \le n^{3-\alpha} + \delta_6(t), \quad \|\xi(t)\|_{L^p} \le n^{3-\alpha} |t - t_o|^{\frac{6-p}{2p}} + \frac{3}{2} \Lambda_4(t). \tag{6.31}$$

Proof. For all $t - t_o \le C n^{-2(2+\delta)}$, by (6.30), we have

$$\|\xi(t)\|_{L^p} \lesssim n^{3-\alpha-\frac{2(2+\delta)(6-p)}{2p}} + \Lambda_4(t) \le C[n^{\frac{(5+\delta)p-6(2+\delta)}{p}-\alpha} + 3\Lambda_4(t)]. \tag{6.32}$$

We have

$$\xi(t) = e^{-iH_0(t-t_o)} \xi(t_o) + \int_{t_o}^t e^{-iH_0(t-s)} P_c i^{-1} G(s) ds. \tag{6.33}$$

So, we have

$$\|\xi(t)\|_{L^p} \le \Lambda_4(t) + C \int_{t_o}^t |t - s|^{-\frac{3(p-2)}{2p}} \|G(s)\|_{L^{p'}} ds. \tag{6.34}$$

Note that $\|G\|_{L^{p'}} \lesssim \|G_3\|_{L^{p'}} + \left\|G - G_3 - \kappa \xi^2 \bar{\xi}\right\|_{L^{p'}} + \left\|\kappa \xi^2 \bar{\xi}\right\|_{L^{p'}}$ and $\|G_3\|_{L^{p'}} \lesssim n^3$. On the other hand, from Lemma 3.1, (6.30) and (6.32), we get

$$\left\|G - G_3 - \kappa \xi^2 \bar{\xi}\right\|_{L^1 \cap L^{p'}} \lesssim n^2 \|\xi\|_{L^2_{loc}} \lesssim [n^{5-\alpha} + n^2 \delta_6(t)]. \tag{6.35}$$

On the other hand, using Hölder's inequality, we get

$$\left\|\kappa \xi^2 \bar{\xi}\right\|_{L^{p'}} \le \|\xi\|_{L^2}^{\frac{2(p-4)}{p-2}} \|\xi\|_{L^p}^{\frac{p+2}{p-2}}, \quad \left\||\xi|^2 \xi\right\|_{L^1} \le \|\xi\|_{L^2}^{\frac{2(p-3)}{p-2}} \|\xi\|_{L^p}^{\frac{p}{p-2}}. \tag{6.36}$$

From this, (6.32) and since $0 < \delta \le \frac{1}{10}$, we get

$$\left\|\kappa \xi^2 \bar{\xi}\right\|_{L^{p'}} \le \|\xi\|_{L^2}^{\frac{2(p-4)}{p-2}} \|\xi\|_{L^p}^{\frac{p+2}{p-2}} \le o(1)[n^{5-2\alpha} + \Lambda_4(t)^{\frac{p+2}{p-2}}]. \tag{6.37}$$

By (6.5), (6.35), and (6.37), we have

$$\|G(s)\|_{L^{p'}} \le C[n^3 + o(1)\widetilde{\delta}_2(t)], \quad \widetilde{\delta}_2(t) := [\rho_0 \langle t - t_o \rangle^{-\sigma}]^{\frac{p+2}{p-2}}. \tag{6.38}$$

Therefore, using $\sigma\frac{p+2}{p-2} > 1$,

$$\begin{aligned}\|\xi(t)\|_{L^p} &\le \Lambda_4(t) + C\int_{t_o}^t |t-s|^{-(\frac{3}{2}-\frac{3}{p})}[n^3 + o(1)\widetilde{\delta}_2(s)]ds \\ &\le Cn^3|t-t_o|^{\frac{6-p}{2p}} + \frac{3}{2}\Lambda_4(t).\end{aligned} \tag{6.39}$$

So, we have proved the estimate of $\|\xi(t)\|_{L^p}$.

We now estimate $\|\xi(t)\|_{L^2_{loc}}$. By (3.5), (3.6), (6.30) and Lemma 6.1, we have

$$\left\|\xi_1^{(3)}(t)\right\|_{L^2_{loc}} \le \Lambda_3(t), \quad \left\|\xi_2^{(3)}(t)\right\|_{L^2_{loc}} \lesssim n^3(1+t-t_o)^{-3/2}. \tag{6.40}$$

By (3.6) and the estimate of $\max_j |\dot{u}_j|$ in Lemma 3.1, we get

$$\left\|\xi_3^{(3)}(t)\right\|_{L^2_{loc}} \lesssim \int_{t_o}^t |1+t-s|^{-3/2} n^5 ds \lesssim n^5. \tag{6.41}$$

For $\xi_4^{(3)}(t)$, bounding its integrand by either L^∞ or L^p-norm and using (6.35), we have

$$\begin{aligned}\left\|\xi_4^{(3)}(t)\right\|_{L^2_{loc}} &\lesssim \int_{t_o}^t \min\{|t-s|^{-3/2}, |t-s|^{-\frac{3(p-2)}{2p}}\} \left\|G - G_3 - \kappa\xi^2\bar{\xi}\right\|_{L^1\cap L^{p'}} ds \\ &\lesssim \int_{t_o}^t \min\{|t-s|^{-3/2}, |t-s|^{-\frac{3(p-2)}{2p}}\}[n^{5-\alpha} + n^2\delta_6(s)]ds \\ &\lesssim n^{5-\alpha} + n^2\delta_6(t).\end{aligned} \tag{6.42}$$

For $\xi_5^{(3)}(t)$, bounding its integrand in either $L^{\frac{2p}{p-4}}$ or L^p, we have

$$\left\|\xi_5^{(3)}(t)\right\|_{L^2_{loc}} \le C\int_{t_o}^t \min\{|t-s|^{-\frac{6}{p}}, |t-s|^{-\frac{3(p-2)}{2p}}\} \left\||\xi|^2\xi\right\|_{L^{\frac{2p}{p+4}}\cap L^{p'}} ds. \tag{6.43}$$

By (6.36), $\frac{p+2}{p-2} > 2$ and $2 < \frac{6p}{p+4} < p$ because $\frac{27}{5} < p < 6$,

$$\left\||\xi|^2\xi\right\|_{L^{\frac{2p}{p+4}}\cap L^{p'}} \le C\|\xi\|_{L^p\cap L^2}\|\xi\|_{L^p}^2 \le o(1)\|\xi\|_{L^p}^2. \tag{6.44}$$

Therefore, by (6.30),

$$\begin{aligned}\left\|\xi_5^{(3)}(t)\right\|_{L^2_{loc}} &\le o(1)\int_{t_o}^t \min\{|t-s|^{-\frac{6}{p}}, |t-s|^{-\frac{3(p-2)}{2p}}\}[n^{5.4} + \Lambda_4^2(s)]ds \\ &\le o(1)[n^{5.4} + \Lambda_4(t)^2 + \delta_7(t)],\end{aligned} \tag{6.45}$$

where

$$\delta_7(t) := \rho_0^2\langle t-t_o\rangle^{-\frac{6}{p}} + n^{\frac{-2+2\delta}{3}}(n^{-4} + t - t_o)^{-\frac{6}{p}}, \tag{6.46}$$

and we have used $\frac{2}{3} < \sigma < \frac{3}{4}$, (6.4), and (4.34) with $a = 6/p < b = 2\sigma - 2\alpha$, (or $b = 2\sigma$).

Collecting all of the estimates of $\xi_j^{(3)}$ with $j = 1, 2, 3, 4$, we have

$$\left\|\xi^{(3)}(t)\right\|_{L^2_{loc}} \le \Lambda_3(t) + Cn^5 + o(1)[\Lambda_4(t)^2 + \delta_7(t)]. \tag{6.47}$$

By (6.3), we have $\Lambda_3(t) \lesssim n^3$ and $\Lambda_4(t)^2 + \delta_7(t) \le n^3 + \delta_6(t)$. Thus

$$\|\xi(t)\|_{L^2_{loc}} \le \left\|\xi^{(2)}(t)\right\|_{L^2_{loc}} + \left\|\xi^{(3)}(t)\right\|_{L^2_{loc}} \le Cn^3 + o(1)\delta_6(t). \tag{6.48}$$

This completes the proof of the lemma. $\square$

Lemma 6.5 *For $t \in [t_o, t_o + \frac{6}{\gamma} n^{-2(2+\delta)}]$, the error terms $g_j(t)$ in (3.10) satisfy*

$$|g_j(t)| \leq o(1) n^{6.7+\delta} + C n^2 g(t), \tag{6.49}$$

where

$$g(t) := \Lambda_3(t) + o(1)[n^{1+3\delta} \langle t - t_o \rangle^{-\frac{p\sigma}{p-2}} + \Lambda_4^2(t) + \delta_7(t)] \tag{6.50}$$

satisfies

$$\int_{t_o}^{\infty} g(s) ds \leq o(1) n^{-\frac{2}{3}}; \quad g(t) \leq o(1) n \rho_0^2, \ \forall \ t \geq t_o + n^{-3}. \tag{6.51}$$

Proof. Recall (3.11),

$$|g_j(t)| \lesssim n^7 + n^2 \left\| \xi^{(3)} \right\|_{L^2_{loc}} + n \|\xi\|^2_{L^2_{loc}} + \|\xi\|^{\frac{2(p-3)}{p-2}}_{L^2_{loc}} \|\xi\|^{\frac{p}{p-2}}_{L^p}. \tag{6.52}$$

From (6.30) and (6.47), we get

$$\begin{aligned} n^2 \left\| \xi^{(3)} \right\|_{L^2_{loc}} &\leq n^2 \Lambda_3 + C n^7 + o(1) n^2 [\Lambda_4^2 + \delta_7], \\ n \|\xi\|^2_{L^2_{loc}} &\leq C[n^{7-2\alpha} + n \delta_6(t)^2], \end{aligned} \tag{6.53}$$

and, using $[n^{2.7} + \Lambda_{4,1} + \Lambda_{4,3}]^{\frac{p}{p-2}} \leq o(1) n^{\frac{5+3\delta}{2}}$,

$$\begin{aligned} \|\xi\|^{\frac{2(p-3)}{p-2}}_{L^2_{loc}} \|\xi\|^{\frac{p}{p-2}}_{L^p} &\lesssim [n^{3-\alpha} + \delta_6(t)]^{\frac{2(p-3)}{p-2}} [n^{2.7} + \Lambda_4]^{\frac{p}{p-2}} \\ &\leq o(1)[n^{\frac{2(p-3)(3-\alpha)}{p-2}} + \delta_6^{\frac{2(p-3)}{p-2}}][n^{\frac{5+3\delta}{2}} + \rho_0^{3/2} \langle t - t_o \rangle^{-\frac{p\sigma}{p-2}}] \\ &\leq o(1)[n^{6.7+\delta} + \rho_0^3 \langle t - t_o \rangle^{-\frac{p\sigma}{p-2}}]. \end{aligned} \tag{6.54}$$

Summing the estimates we get (6.49). The estimates (6.51) follow from direct checking. □

Lemma 6.6 *For all $t \in [t_o, t_o + \frac{6}{\gamma} n^{-2(2+\delta)}]$, we have*

$$\frac{1}{5} n \leq \max_j |x_j(t)| \leq (\textstyle\sum_{j=0}^K |x_j(t)|^2)^{\frac{1}{2}} \leq 2n. \tag{6.55}$$

Proof. From the first equation of (6.24), (6.51) and $\delta \leq \frac{1}{10}$, we get

$$\begin{aligned} (f_0 + f)(t) &\leq (f_0 + f)(t_o) + C n \max_j \int_{t_o}^{t} |g_j(s)| ds \\ &\leq (f_0 + f)(t_o) + C[o(1) n^{7.7+\delta}(t - t_o) + n^3 \int_{t_o}^{t} g(s) ds] \\ &\leq (f_0 + f)(t_o) + o(1) \rho_0^2 \leq [1 + o(1)](f_0 + f)(t_o). \end{aligned} \tag{6.56}$$

By (3.11), (6.30), we have $[1 - o(1)] \sum_j |x_j|^2 \leq f_0 + f$. By Lemma 6.1, we get $(f_0 + f)(t_o) \leq 2n^2$. It follows from (6.56) that $(\sum_{j=0}^K |x_j(t)|^2)^{\frac{1}{2}} \leq 2n$.

Similarly, by integrating the second equation of (6.24), we obtain

$$(f_0 + h)(t) \geq [1 - o(1)](f_0 + h)(t_o). \tag{6.57}$$

By (3.11), (6.30) and the definition of f_0, h, we get

$$(f_0 + h)(t) \leq [\textstyle\sum_{k=0}^{K} 2^{-k} + o(1)] \max_j |x_j(t)|^2. \tag{6.58}$$

Therefore,

$$2 \max |x_j(t)|^2 \geq [1 - o(1)](f_0 + h)(t_o) \geq [1 - o(1)]\frac{1}{2}|x_1(t_o)|^2. \tag{6.59}$$

Hence $\max_j |x_j(t)|^2 \geq \frac{n^2}{25}$ for all $t \in [t_o, t_o + \frac{6}{\gamma}n^{-2(2+\delta)}]$. $\square$

Proposition 6.7 *There exists t_i such that $t_o + \frac{\delta}{10\tilde{\gamma}}n^{-4}\log\frac{1}{n} \leq t_i \leq t_o + \frac{7}{\gamma}n^{-4-2\delta}$ and*

$$\frac{n}{5} \leq |x_0(t_i)| \leq 2n, \quad (0.9)\rho_0 \leq (\sum_{j=1}^{K} |x_j(t_i)|^2)^{1/2} \leq (1.1)\rho_0. \tag{6.60}$$

Above $\widetilde{\gamma} = \max\{1, (d^l_{ab})_- : \ \forall a, b, l = 0, \ldots, K\}$ and $d^l_{ab} = O(1)$ are given in (3.12).

Proof. By Lemma 6.6, we already have $|x_0| \leq 2n$. The proof is divided into four steps.

Step 1: Let $t_1 := t_o + n^{-3}$. For $t_o \leq t \leq t_1$, for any j, by (6.22), (6.30), (6.49), and (6.51), we get

$$|f_j(t) - f_j(t_o)| \lesssim \int_{t_o}^{t_1} [n^6 + n|g_j(s)|]ds \lesssim n^3 + n\int_{t_o}^{t_1} [n^2 g(s)]ds \leq o(1)\rho_0^2. \tag{6.61}$$

In particular, for $j = 0, 1$, we get

$$[1 - o(1)]f_j(t_o) \leq f_j(t) \leq [1 + o(1)]f_j(t_o), \quad \forall\, t \in [t_o, t_1]. \tag{6.62}$$

By (3.11) and the definitions of f_j, we get

$$[1 - o(1)]|x_j(t_o)| \leq |x_j(t)| \leq [1 + o(1)]|x_j(t_o)|, \ \forall\, t \in [t_o, t_1], \ j = 0, 1. \tag{6.63}$$

Together with (6.6), for $t \in [t_o, t_1]$, we have

$$1.8\rho_0 \leq |x_0(t)| \leq 2.2\rho_0, \quad 0.8n \leq |x_1(t)| \leq 1.2n. \tag{6.64}$$

On the other hand, for $j > 1$, from (6.61), we obtain $f_j(t) \leq f_j(t_o) + o(1)\rho_0^2$ for $t \in [t_o, t_1]$. So, by (3.11), (6.6), and the definition of f_j, we get

$$|x_j(t)| \leq [1 + o(1)]f_j(t)^{1/2} \leq 7\sqrt{\frac{D}{\gamma_0}}\rho_0, \quad \forall\, t \in [t_o, t_1], \ \forall\, j > 1. \tag{6.65}$$

Step 2: Let us define

$$t_2 := \sup\{t \geq t_1 : f_0(s) < \frac{n^2}{10}, \ \forall\, s \ \in [t_1, t]\}. \tag{6.66}$$

By (6.64), $t_2 < t_1$. We shall prove that

$$t_1 < t_2 \leq t_2' := t_1 + a^{-1}\log\frac{n^2}{5f_0(t_1)}, \quad a := 2\gamma[\frac{n^2}{50}]^2. \tag{6.67}$$

For all $t_1 \le t \le t_2$, $f_0(t) < \frac{n^2}{10}$. Note $h(t_1) \ge f_1(t_1)/2 \ge (1+o(1))(0.8n)^2/2 \ge (0.3)n^2$. From (6.24) and Lemma 6.5, we get

$$\begin{aligned} h(t) &\ge (f_0+h)(t_1) - f_0(t) - 2(K+1)\int_{t_1}^{t} \max_j |\mu_j||g_j|(s)ds \\ &\ge (0.3)n^2 - \frac{n^2}{10} - Cn\int_{t_1}^{t}[n^{6.7+\delta} + n^2 g(s)]ds \ge \frac{n^2}{100}. \end{aligned} \tag{6.68}$$

By (6.25), (6.30) and (6.68), we have, for $t \in [t_1, t_2']$,

$$\dot{f_0} \ge 2\gamma f^2 f_0 - 2|\mu_0||g_0| \ge 2\gamma_0(2h)^2 f_0 - 4n|g_0| \ge 2\gamma[\frac{n^2}{50}]^2 f_0 - 4n|g_0|. \tag{6.69}$$

Note the coefficient of f_0 is a. Thus

$$f_0(t) \ge e^{a(t-t_1)}[f_0(t_1) - 4n\int_{t_1}^{t} e^{-a(s-t_1)} g_0(s)ds]. \tag{6.70}$$

On the other hand, from (6.49), we have

$$\begin{aligned} n\int_{t_1}^{t} e^{-a(s-t_1)} g_0(s)ds &\le n\int_{t_1}^{t}[n^{6.7+\delta} + n^2 g(s)]ds \\ &\le n^{7.7+\delta}(t-t_1) + n^3\int_{t_1}^{t} g(s)ds \le o(1)\rho_0^2 \le o(1)f_0(t_1). \end{aligned} \tag{6.71}$$

Therefore,

$$f_0(t) \ge \frac{1}{2}e^{a(t-t_1)} f_0(t_1), \quad \forall t \in [t_1, t_2]. \tag{6.72}$$

This shows $t_2 \le t_2'$ is finite, and $f_0(t_2) = \frac{n^2}{10}$.

Step 3: Define

$$t_i := \sup\{t \ge t_2 : f(s) > \rho_0^2,\ \forall\ s \in [t_2, t)\}. \tag{6.73}$$

From (6.68), we get $t_i > t_2$. We shall prove in Steps 3 and 4 that

$$t_2 + \frac{\delta}{10\tilde{\gamma}} n^{-4}\log\frac{1}{n} \le t_i \le t_3 := t_2 + \frac{6}{\gamma}n^{-4-2\delta}. \tag{6.74}$$

By definition of t_i, we get

$$f(t) > \rho_0^2, \quad \forall\ t \in [t_2, t_i). \tag{6.75}$$

From Lemma 6.2 and (6.75), we have

$$\frac{d}{dt}(f_0(t)) \ge 2\gamma\rho_0^4 f_0(t) - 4n|g_0|,\ \forall\ t \in [t_2, t_i). \tag{6.76}$$

From this and as in (6.72), we also obtain

$$f_0(t) \ge \frac{1}{2}e^{2\gamma\rho_0^4(t-t_2)} f_0(t_2) \ge \frac{n^2}{20},\ \forall\ t \in [t_2, t_i). \tag{6.77}$$

From this, (6.25), and Lemma 6.5, for $t \in [t_1, t_i)$,

$$\begin{aligned} \frac{d}{dt}(f(t)) &\le -4\gamma f_0(t) f(t)^2 + Cn\max_{k>0}|g_k| \\ &\le -\frac{\gamma n^2}{5} f(t)^2 + Cn[n^{6.7+\delta} + n^2 g(s)]. \end{aligned} \tag{6.78}$$

From this and (6.75), (and $\delta \le \frac{1}{10}$), we get

$$\frac{n^2\gamma}{6} - Cn^3\rho_0^{-4}g(t) < \frac{n^2\gamma}{5} - \frac{Cn[n^{6.7+\delta}+n^2 g(s)]}{f^2} \le -\frac{\dot f}{f^2}, \quad \forall t \in [t_2, t_i). \tag{6.79}$$

Note that by (6.51), (6.68), Proposition 6.3 and $\delta \le \frac{1}{10}$, we have $\forall\, t \ge t_2$

$$n^{-1-4\delta}\int_{t_2}^t g(s)ds \le o(1)n^{-1-4\delta}n^{-2(1-\delta)/3} = o(1)n^{-\frac{5(1+2\delta)}{3}} \le o(1)f(t_2)^{-1}. \tag{6.80}$$

Integrating (6.79) in $[t_2, t]$, we get

$$f(t) < [f(t_2)^{-1}/2 + \frac{n^2\gamma}{6}(t-t_2)]^{-1}, \quad \forall\, t \in [t_2, t_i]. \tag{6.81}$$

In particular, $\rho_0^2 < f(t) < [\frac{n^2\gamma}{6}(t-t_2)]^{-1}$, which shows $t_i \le t_3$, and $f(t_i) = \rho_0^2$. From this, (3.11) and (6.77), we get the estimates (6.60). Since

$$t_i - t_o \le (t_i - t_2) + (t_2 - t_1) + (t_1 - t_o) \le \frac{6}{\gamma}n^{-4-2\delta} + Cn^{-4}\log\frac{1}{n} + n^{-3} \tag{6.82}$$

by (6.67) and (6.64), we get the upper bound of $t_i - t_o$ in Prop. 6.7.

Step 4: It remains to show that $t_i \ge t_2 + \frac{\delta}{10\tilde\gamma}n^{-4}\log\frac{1}{n}$. Recall $g(t) \le o(1)n\rho_0^2$ for all $t \ge t_1 = t_o + n^{-3}$ from Lemma 6.5. By (6.22) and Prop. 6.3,

$$\dot f(t) \ge -9\tilde\gamma n^4 f(t) - Cn[n^{6.7+\delta} + n^2 g(t)] \ge -10\tilde\gamma n^4 f(t), \ \forall\, t \in [t_1, t_i], \tag{6.83}$$

where $\tilde\gamma = \max\{1, (d_{ab}^l)_- : \ \forall a, b, l = 0, \dots, K\}$. This implies that

$$t_i - t_2 \ge \frac{n^{-4}}{10\tilde\gamma}\log\frac{f(t_2)}{f(t_i)} \ge \frac{\delta}{10\tilde\gamma}n^{-4}\log\frac{1}{n}. \tag{6.84}$$

For the second inequality we have used $f(t_2) \ge h(t_2) \ge n^2/50$ by (6.68). This completes the proof of Proposition 6.7. $\square$

At $t = t_i$ the solution enters ρ_0-neighborhood of ground states and we change to linearized coordinates. For that purpose we prepare outgoing estimates at $t = t_i$.

Lemma 6.8 *Let t_i be as in Proposition 6.7. For any $t > t_i$, we have*

$$\begin{aligned} \left\| e^{-iH_0(t-t_i)}\xi(t_i) \right\|_{L^2_{\mathrm{loc}}} &\le \frac{1}{2}[\Lambda_{L,1}(t) + \Lambda_{L,2}(t)], \\ \left\| e^{-iH_0(t-t_i)}\xi(t_i) \right\|_{L^p} &\le \frac{1}{2}[\Lambda_{G,1}(t) + \Lambda_{G,2}(t)], \end{aligned} \tag{6.85}$$

where for some constant $C_7 \ge C_6$ and $\sigma' = \frac{3(p-2)}{2p}$,

$$\begin{aligned} \Lambda_{L,1}(t) &:= 2C_7[n^{-1+2\delta}\langle t-t_o\rangle^{-7/6} + \rho(t)^3 + n^{4/5}\rho(t)^{7/3}], \\ \Lambda_{G,1}(t) &:= 2C_7[n^{-1+\delta}\langle t-t_o\rangle^{-\sigma} + n^{-1+2(2+\delta)\alpha}(\Delta t + t)^{-\sigma+\alpha}], \\ \Lambda_{L,2}(t) &:= \frac{2n^{\frac{5p-18+p\delta}{p-2}}(t_i-t_o)}{t-t_o}\langle t-t_i\rangle^{-1/2}, \\ \Lambda_{G,2}(t) &:= 2C_7 n^3(t_i-t_o)(t-t_o)^{-\sigma'}. \end{aligned} \tag{6.86}$$

Proof. Decompose $e^{-i(t-t_i)H_0}\xi(t_i) = \chi(t) + J(t)$, where

$$\chi(t) := e^{-i(t-t_o)H_0}\xi(t_o), \quad J(t) := \int_{t_o}^{t_i} e^{-i(t-s)H_0} P_c G(s) ds. \tag{6.87}$$

Denote $T = t_i - t_o$. By Lemma 6.1 and using $n^{-4}\log\frac{1}{n} \lesssim T \lesssim n^{-2(2+\delta)}$, we have

$$\|\chi(t)\|_{L^p} \le \Lambda_4(t) \le \frac{1}{2}\Lambda_{G,1}(t), \quad \|\chi(t)\|_{L^2_{\text{loc}}} \le \Lambda_3(t) \le \frac{1}{2}\Lambda_{L,1}(t), \tag{6.88}$$

for some C_7. By (6.38), we have

$$\|G(s)\|_{L^{p'}} \le C[n^3 + o(1)\widetilde{\delta}_2(s)] \quad \forall\, s \in [t_o, t_i], \quad \widetilde{\delta}_2(s) = [\rho_0\langle s - t_o\rangle^{-\sigma}]^{\frac{p+2}{p-2}}. \tag{6.89}$$

So, we have (using $\frac{p+2}{p-2} > 2$)

$$\begin{aligned}\|J(t)\|_{L^p} &\le C\int_{t_o}^{t_i} |t-s|^{-\sigma'}\|G(s)\|_{L^{p'}}\, ds \le C\int_{t_o}^{t_i} |t-s|^{-\sigma'}[n^3 + o(1)\widetilde{\delta}_2(s)]ds \\ &\le Cn^3 T(t-t_o)^{-\sigma'} + \rho_0^2(t-t_o)^{-\sigma'} \le \frac{1}{2}\Lambda_{G,2}(t).\end{aligned} \tag{6.90}$$

It remains to estimate $\|J(t)\|_{L^2_{\text{loc}}}$. By (6.35) and (6.36),

$$\|G(s)\|_{L^1\cap L^{p'}} \le Cn^3 + Cn^2\|\xi(s)\|_{L^2_{loc}} + o(1)\|\xi(s)\|_{L^p}^{\frac{p}{p-2}}. \tag{6.91}$$

By (6.30) and (6.5),

$$\|G(s)\|_{L^1\cap L^{p'}} \le o(1)[n^{\frac{5p-18+p\delta}{p-2}} + \rho_0^{3/2}\langle s-t_o\rangle^{-\frac{p\sigma}{p-2}}]. \tag{6.92}$$

Thus

$$\begin{aligned}\|J(t)\|_{L^2_{\text{loc}}} &\le C\int_{t_o}^{t_i} \min\{(t-s)^{-3/2}, (t-s)^{-\sigma'}\}\|G(s)\|_{L^1\cap L^{p'}}\, ds \\ &\le o(1)\int_{t_o}^{t_i} \min\{(t-s)^{-3/2}, (t-s)^{-\sigma'}\}[n^{\frac{5p-18+p\delta}{p-2}} + \rho_0^{3/2}\langle s-t_o\rangle^{-\frac{p\sigma}{p-2}}]ds \\ &\le o(1)n^{\frac{5p-18+p\delta}{p-2}}\frac{T}{t-t_o}\langle t-t_i\rangle^{-1/2} + o(1)\rho_0^{3/2}(t-t_o)^{-1}\langle t-t_i\rangle^{-1/2},\end{aligned} \tag{6.93}$$

which is bounded by $\frac{1}{2}\Lambda_{L,2}(t)$. This completes the proof of the lemma. ☐

7 Converging to a ground state

In this section we consider the solution when it is already inside a neighborhood of the ground states. Although similar to [4, 31, 7, 30], it requires a proof because the dispersive component has much worse estimates. We will however content ourselves with formulating the main proposition and skip the proof.

As in Section 4, for fixed $T \ge t_i$ we decompose $\psi(t)$ as (see (3.17))

$$\psi(t) = [Q_{0,n(T)} + a(t)\partial_E Q_{0,n(T)} + \zeta(t) + \eta(t)]e^{-iEt+i\theta(t)}, \quad t \in [t_i, T]. \tag{7.1}$$

We have $a(T) = 0$, and

$$\zeta = \sum_{j=1}^{K} \zeta_j, \quad \zeta_j = \bar{z}_j u_j^- + z_j u_j^+, \quad [\eta] = \begin{bmatrix} \operatorname{Re} \eta \\ \operatorname{Im} \eta \end{bmatrix} = e^{i\theta}\eta_+ + e^{-i\theta}\eta_-. \tag{7.2}$$

Denote $z_H(t) = (\sum_{j=1}^K |z_j(t)|^2)^{1/2}$. From Lemma 3.3 and Proposition 6.7, (7.1) is valid at least for $T > t_i$ sufficiently close to t_i. It follows from Proposition 7.2 below that (7.1) is valid with suitable estimates for all $T \ge t_i$ and $n(T)$ converges to some $n_+ \sim n$ as $T \to \infty$. We first state the initial estimates at time t_i.

Lemma 7.1 (Initial estimates) *There exists $C_8 > 0$ such that if $T > t_i$ and $n(T)/n(t_i) \in (\frac{1}{2}, \frac{3}{2})$, then*

$$\frac{4}{5}\rho_0 \le z_H(t_i) \le \frac{6}{5}\rho_0, \tag{7.3}$$

and, for $t \ge t_i$,

$$\begin{aligned} \left\| e^{\mathbf{L}(t-t_i)} \eta_\pm(t_i) \right\|_{L^2_{\mathrm{loc}}} &\le \Lambda_L(t) := \Lambda_{L,1}(t) + \Lambda_{L,2}(t) + C_8 n^3 \langle t - t_i \rangle^{-3/2} \\ \left\| e^{\mathbf{L}(t-t_i)} \eta_\pm(t_i) \right\|_{L^p} &\le \Lambda_G(t) := \Lambda_{G,1}(t) + \Lambda_{G,2}(t) + C_8 n^3 \langle t - t_i \rangle^{-\sigma'} \end{aligned} \tag{7.4}$$

where $\Lambda_{L,1}$, $\Lambda_{L,2}$, $\Lambda_{G,1}$ and $\Lambda_{G,2}$ are defined in Lemma 6.8.

We next formulate the main proposition of this section. Denote

$$\begin{aligned} \hat{\rho}(t) &= \rho(t - t_i) = [\rho_0^{-2} + \gamma_0 n^2 (t - t_i)]^{-1/2}, \\ \delta_8(t) &= n^{-\frac{2}{3}(1-\delta)} (t - t_o)^{-6/p} + n^6 \langle t - t_i \rangle^{-6/p} \le o(1) n \hat{\rho}(t)^2, \end{aligned} \tag{7.5}$$

and

$$M_T^* := \sup_{t_i \le t \le T} \max \left\{ \begin{array}{l} \hat{\rho}(t)^{-1} |z_H(t)|, \quad [2D\hat{\rho}(t)]^{-1} |a(t)|, \\ [\Lambda_G(t) + n^{7/9} \hat{\rho}(t)^{5/3}]^{-1} \|\eta\|_{L^p}, \\ [\Lambda_L(t) + \Lambda_G^2(t) + n^{-\alpha} \hat{\rho}(t)^3 + \delta_8(t)]^{-1} \left\| \eta^{(3)} \right\|_{L^2_{\mathrm{loc}}} \end{array} \right\}. \tag{7.6}$$

Proposition 7.2 *Suppose for $T \ge t_i$ we have $n(T)/n(t_i) \in (\frac{1}{2}, \frac{3}{2})$ and $M_T^* \le 3$. Then we have $M_T^* \le \frac{5}{2}$ and $n(T)/n(t_i) \in (\frac{3}{4}, \frac{5}{4})$.*

This Proposition implies Theorem 1.1 in the case $k = 0$, see e.g. [4, 31, 7, 30]. Since the proof is standard, it is skipped and can be found in [21].

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