CONVERGENCE OF ADAPTIVE DISCONTINUOUS GALERKIN APPROXIMATIONS OF SECOND-ORDER ELLIPTIC PROBLEMS

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Abstract. A residual type a posteriori error estimator is introduced and analyzed for a discontinuous Galerkin formulation of a model second-order elliptic problem with Dirichlet-Neumann type boundary conditions. An adaptive algorithm using this estimator together with specific marking and refinement strategies is constructed and shown to achieve any specified error level in the energy norm in a finite number of cycles. The convergence rate is in effect linear with a guaranteed error reduction at every cycle. Results of numerical experiments are presented.

Key words. discontinuous Galerkin methods, a posteriori estimates, convergence of adaptive methods

AMS subject classifications. 65N55, 65F10

1. Introduction. Let $\Omega \subset \mathbf{R}^2$ be a bounded open polygonal domain. We consider the following boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g_D \quad \text{on } \Gamma_D, \tag{1.2}$$

$$\nabla u \cdot n = g_N \quad \text{on } \Gamma_N, \tag{1.3}$$

where $\partial\Omega := \Gamma = \Gamma_D \cup \Gamma_N$ and n is the unit normal vector exterior to Ω. We assume that Γ_D has positive measure, $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$. Assumptions on f, g_D and g_N are given later.

Recently there has been a flurry of activity concerning a posteriori error estimates for the discontinuous Galerkin (DG) method for elliptic as well as other problems. In [7], Becker et al. use a Helmholtz type decomposition of the error to derive estimates in the energy norm. Bustinza et al. [10] use a similar technique to derive estimates for linear and nonlinear elliptic problems for the Local Discontinuous Galerkin (LDG) method (See also the article by P. Castillo [12] in the same issue). Creusé et al. consider the interesting issue of anisotropic elements, i.e. those with large aspect ratio, in the context of the stationary Stokes problem. Houston et al. [15] derive energy norm a posteriori estimates for the hp-version of the DG method for elliptic problems. The fact that the penalty parameter γ appears with a different exponent in their a posteriori estimates provides an interesting alternative to ours. We also mention [21] and [16] for L^2-norm or functional error estimation for the DG method.

In [17] we presented residual type a posteriori estimates in the energy norm for discontinuous Galerkin approximations of a special case of the boundary value problem (1.1)–(1.3) corresponding to $\Gamma_D = \Gamma$ and $g_D = 0$. In the present work, we extend these estimates to encompass the more general mixed boundary conditions (1.2), (1.3) continuing work already begun in [18]. These estimates are used to provide a mesh modification strategy which is then shown to be convergent in the energy norm induced by the bilinear form defining the DG method.

The principal goal of an adaptive algorithm is to achieve a user specified error level in a finite number of cycles. While it is typical for the error to be measured in the energy norm ($\|\nabla e\|$ for the standard Galerkin method for second order elliptic problems and an appropriate energy norm for the discontinuous Galerkin method) other interesting and useful measures of the error, so-called “Quantities Of Interest” (QOI) are emerging, see e.g. [8]. To this date however, no convergence results are known except with respect to the energy norm given that the proofs make use in an essential way of an orthogonality relation and such relations are not known for other QOI’s.

A typical cycle consists of the following basic steps: 1) Given a mesh $\mathcal{T}_H$, calculate the approximation on this mesh. 2) Estimate the error of the approximation at hand using an error estimator. 3) Refine/coarsen $\mathcal{T}_H$ using the information to obtain a new mesh $\mathcal{T}_h$.

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The rigorous treatment of the convergence of adaptive algorithms for elliptic problems can be said to have started with the paper of Babuşka and Vogelius [4] where a detailed treatment of the one-dimensional case was given. In 1996 W. Dörfler [14] gave a convergence proof for the two dimensional case for the standard Galerkin method using linear elements while outlining an extension to quadratic elements. One of the highlights of this work is that bounds on the convergence rate were provided which was not the case for [4]. On the other hand, the initial mesh had to be fine enough to essentially resolve the solution. The latter issue provided the starting point for the work of Morin, Nochetto and Siebert [19], [20] who introduced the concept of *data oscillation* $osc(f, \mathcal{T}_H) = \left(\sum_{K\in\mathcal{T}_H} \|H(f - f_K)\|^2\right)^{1/2}$ to circumvent this requirement. The nagging issue of calculating this quantity accurately on a coarse mesh is not resolved and should be treated within the larger and important framework of accounting for the quadrature errors arising from the implementation of the finite element formulation as well as from the calculation of certain terms in the a posteriori estimators. More recently, Binev, Dahmen and DeVore [9] have proposed a modification of the algorithm in [20] that incorporates coarsening to prove optimal work estimates. More specifically, they have shown that if the solution u can be approximated by a piecewise linear function to an accuracy of $O(n^{-s})$ on a triangulation with n cells then the algorithm constructs an approximation with the same asymptotic accuracy at a cost of $O(n)$ arithmetic operations.

In this paper we take up the issue of convergence of an adaptive algorithm in the context of a discontinuous Galerkin formulation for the problem (1.1)–(1.3). The specific DG method used is of an interior penalty type that can be traced to the work of [5] and [2]. We refer to the survey paper of [3] for a survey and unified view of DG methods. Our main result can be summarized as follows: With $a_h^\gamma(\cdot,\cdot),\ h > 0$ denoting the bilinear form associated with the DG formulation of the problem, we have $a_h^\gamma(e_h, e_h) \le \rho a_H^\gamma(e_H, e_H),\ \rho < 1$. The principal assumptions which enable this result being

(i) The data of the problem f, g_D, g_N belong to the same polynomial spaces which contain the numerical solution.
(ii) The mesh $\mathcal{T}_h$ is not too fine with respect to the mesh $\mathcal{T}_H$.
(iii) The penalty parameter is not too small; specifically it must be larger than a constant depending only on the minimum angle of the triangles and the degree of the polynomials in the discontinuous finite element spaces.
(iv) While the marking strategy used is the one used by Dörfler, the refinement strategy is designed to accomodate the DG approach.

While the assumption on the data of the problem may seem to be restrictive, we should note that for one there are practically important cases satisfying these assumptions e.g. piecewise constant data. Another mitigating argument is that the numerical integration rules cannot distinguish between the data functions and their Lagrange interpolants. Therefore, these assumptions can be relaxed in tandem with an effort to take into account the effect of numerical integration.

The paper is organized as follows: Section 2 is devoted to preliminaries. In addition to establishing notation, we quote a result whose details can be found in [17] and [18] concerning the approximation of discontinuous piecewise polynomial functions by continuous functions of the same type. This result has so far played a key role in the a posteriori estimates as well as of the convergence proof. Section 3 is devoted to the derivation of the residual type a posteriori estimates extending the results of [17] to include the more general mixed boundary conditions (1.2), (1.3). A novel contribution is estimate (3.12) (Theorem 3.2(iv)). It completes the a posteriori error estimates of [17] by providing lower bounds for the gradients of the error. That the jump terms in (3.12) are multiplied by γ^2 is significant in that γ appears with exponent one in the bilinear form. Interestingly, this fact plays an important role in the proof of convergence of the adaptive algorithm. In section 4 we outline the marking and refinement strategies and prove convergence of the adaptive scheme. It is worth noting that the analysis of the DG formulation presents some complications not present in the standard method. One is due to the fact that the energy norm is mesh dependent. Another more basic one is due to the fact that the bilinear form which defines the method is not coercive on the energy space. Both issues are successfully resolved. Let us also note that while the marking and refinement strategies are couched in 2D, we believe that appropriate modifications can be introduced to obtain convergence in 3D as well. In particular, only the refinement strategy, Lemma 4.1 and Corollary 4.1 need be extended. Finally, in section 5 we present the results of some numerical experiments.

2. Preliminaries.

2.1. Notation. For a domain $D \subseteq \mathcal{R}^d$ and integer $m \geq 0$, $H^m(D)$ will denote the (Hilbert) Sobolev space with inner product $(u,v)_{m,D} = \sum_{|\alpha| \leq m} \int_D D^\alpha u \, D^\alpha v \, dx$ and norm $\|u\|_{m,D} = (u,u)_{m,D}^{1/2}$ (cf. [1]). To simplify the notation, we shall drop m when its value is zero. Also, we shall often encounter functions that vanish on Γ_D; so we let $H^1_{0,\Gamma_D} = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_D\}$.

Extensive use will be made of edge/surface integrals. So for a $(d-1)$-dimensional subset e of $\mathcal{R}^d$, we set $< u, v >_e = \int_e u \, v \, ds$ and $|u|_e = < u, u >_e^{1/2}$.

2.2. Triangulations. Let $\mathcal{T}_h = \{K_i : \ i = 1, 2, \cdots, m_h\}$ be a family of star-like partitions (triangulations) of the domain Ω parametrized by $0 < h \leq 1$. We assume that

(i) The elements of $\mathcal{T}_h$ satisfy the minimal angle condition,

(ii) $\mathcal{T}_h$ is locally quasi-uniform, that is if two elements K_j and K_ℓ are adjacent in the sense that $\mu_{d-1}(\partial K_j \cap \partial K_\ell) > 0$ then $\mathrm{diam(K_j)} \approx \mathrm{diam(K_\ell)}$.

We define $\mathcal{E}_h^I$ and $\mathcal{E}_h^B$ to be the set of all interior and boundary edges (faces in the case $d = 3$), respectively:

$$\begin{aligned} \mathcal{E}_h^I &= \{e = \partial K_j \cap \partial K_\ell, \quad \mu_{d-1}(\partial K_j \cap \partial K_\ell) > 0\} \\ \mathcal{E}_h^B &= \{e = \partial K \cap \partial \Omega, \quad \mu_{d-1}(\partial K \cap \partial \Omega) > 0\}, \mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B. \end{aligned}$$

For each $e \in \mathcal{E}_h^I$, we denote the two triangles that "share" it by K^+ and K^- respectively. As to which of the two is K^+ is completely arbitrary but not irrelevant! If $e \in \mathcal{E}_h^B$, then $e = \partial K^+ \cap \partial\Omega \equiv \partial K \cap \partial \Omega$.

We assume that for each $e \in \mathcal{E}_h^B$, either $e \subset \Gamma_D$ or $e \subset \Gamma_N$. We then set $\mathcal{E}_h^B = \mathcal{E}_h^D \cup \mathcal{E}_h^N$ where $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$ are respectively the set of boundary edges on Γ_D and on Γ_N. From the previous assumption, we have $\mathcal{E}_h^D \cap \mathcal{E}_h^N = \emptyset$.

Given a partition or mesh $\mathcal{T}_h$ of Ω, we find it convenient to use the spaces $H^m(\mathcal{T}_h) = \Pi_{K \in \mathcal{T}_h} H^m(K)$. In this context we consider K to be open so that elements of $H^m(\mathcal{T}_h)$ are single-valued. In particular, the "energy space" for the discontinuous Galerkin method for this problem will be $E_h = H^2(\mathcal{T}_h)$.

We shall also use the discontinuous finite element spaces $V_h^r = \Pi_{K \in \mathcal{T}_h} P_{r-1}(K)$, $r \geq 2$ where $P_k(K)$ is the space of polynomials of total degree k defined on K.

It is essential to be able to define values of functions in $H^m(\mathcal{T}_h)$ and V_h^r on the edges e. So for $v \in H^m(\mathcal{T}_h)$, $m \geq 1$, and $e \in \mathcal{E}_h^I \cup \mathcal{E}_h^B$, v_e^+ will denote the trace on e of the restriction of v to K^+. Similarly we define v_e^- for $e \in \mathcal{E}_h^I$.

We also define *jumps* and *averages* of such traces

$$\begin{aligned} &[v]_e = v_e^+ - v_e^-,\ e \in \mathcal{E}_h^I, && [v]_e = v_e^+,\ e \in \mathcal{E}_h^B \\ &\{v\}_e = \tfrac{1}{2}\big(v_e^+ + v_e^-\big),\ e \in \mathcal{E}_h^I, && \{v\}_e = v_e^+,\ e \in \mathcal{E}_h^B. \end{aligned}$$

Finally, for $v \in H^2(\mathcal{T}_h)$ we let $\{\partial_n v\}_e = \{\nabla v\}_e \cdot \mathbf{n}^+ = \frac{1}{2}\big(\nabla v^+ + \nabla v^-\big) \cdot \mathbf{n}^+,\ e \in \mathcal{E}_h^I$ where $\mathbf{n}^+$ is the unit outward normal to K^+ and $[\partial_n v]_e = \nabla v^+ \cdot \mathbf{n}^+ + \nabla v^- \cdot \mathbf{n}^-,\ e \in \mathcal{E}_h^I$.

2.3. Some useful results. We shall make frequent use of the following trace and inverse inequalities, cf. [11],[17],

$$|v|^2_{\partial D} \leq c \left(h_D^{-1} \|v\|_D^2 + h_D \|\nabla v\|_D^2\right) \quad \forall v \in H^1(D), \tag{2.1}$$

where $h_D = \mathrm{diam}(D)$;

$$|v|_{j,D} \leq c h_D^{i-j} |v|_{i,D} \quad \forall v \in P_k(D),\ 0 \leq i \leq j \leq 2. \tag{2.2}$$

We shall also make essential use of the fact that an element of V_h^r can be approximated by *continuous* piecewise polynomial functions, specifically by elements of $V_h^r \cap H^1(\Omega)$; the degree of approximation being controlled, not surprisingly, by the jumps of the discontinuous function. Here we extend the result established in [17] to allow approximation by functions that also satisfy Dirichlet type conditions on the boundary. We also include a significant observation that the approximation result holds in the L^2 norm as well. We omit the proof since its essential points were provided in [17] and [18].

THEOREM 2.1. *Let $\mathcal{T}_h$ be a conforming or nonconforming mesh consisting of triangles when $d = 2$, and tetrahedra when $d = 3$. Then for any $v_h \in V_h^r$ and multiindex α with $|\alpha| = 0, 1$ the following approximation results hold*

(i) Let g be the restriction to Γ of a function in $V_h^r \cap H^1(\Omega)$. Then there exists $\chi \in V_h^r \cap H^1(\Omega)$ satisfying $\chi\big|_\Gamma = g$ and

$$\sum_{K\in\mathcal{T}_h} \|D^\alpha(v_h - \chi)\|_K^2 \le c\left(\sum_{e\in\mathcal{E}_h^I} h_e^{1-2|\alpha|}|[v_h]|_e^2 + \sum_{e\in\mathcal{E}_h^B} h_e^{1-2|\alpha|}|v_h - g|_e^2\right). \tag{2.3}$$

(ii) Let g be the restriction to Γ_D of a function in $V_h^r \cap H^1(\Omega)$. Then there exists $\chi \in V_h^r \cap H^1(\Omega)$ satisfying $\chi\big|_{\Gamma_D} = g$ and

$$\sum_{K\in\mathcal{T}_h} \|D^\alpha(v_h - \chi)\|_K^2 \le c\left(\sum_{e\in\mathcal{E}_h^I} h_e^{1-2|\alpha|}|[v_h]|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{1-2|\alpha|}|v_h - g|_e^2\right). \tag{2.4}$$

(iii) There exists $\chi \in V_h^r \cap H^1(\Omega)$ satisfying

$$\sum_{K\in\mathcal{T}_h} \|D^\alpha(v_h - \chi)\|_K^2 \le c\sum_{e\in\mathcal{E}_h^I} h_e^{1-2|\alpha|}|[v_h]|_e^2. \tag{2.5}$$

for some constant C independent of h and v_h but which may depend on r and the minimal angle θ_0 of the triangles in $\mathcal{T}_h$.

REMARK 2.1. *The proof of this result is constructive and is based on an averaging process. It should also hold for more general partitions of Ω such as quadrilaterals and parallelepipeds.*

3. A posteriori error estimates.

3.1. The discrete problem. In order to construct a weak formulation for the problem (1.1)–(1.3), we introduce the bilinear form $a_h^\gamma : E_h \times E_h \to \mathbf{R}$

$$a_h^\gamma(u, v) = \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \Big(\langle\{\partial_n u\}, [v]\rangle_e + \langle\{\partial_n v\}, [u]\rangle_e - \gamma h_e^{-1}\langle[u], [v]\rangle_e \Big),$$

where $h_e = \operatorname{diam}(e)$ and γ is the interior penalty parameter. We point out that we have adopted the averaged value $\{\partial_n v\}_e$ of the normal derivatives attributed to D. Arnold [2]. The results of this paper do also apply to the so-called Baker formulation for which $\{\partial_n v\}_e = \nabla v_e^+ \cdot n^+$.

The bilinear form a_h^γ is consistent with the BVP (1.1)–(1.3) in the following sense: If $u \in H^2(\Omega)$ satisfies (1.1)–(1.3), then,

$$a_h^\gamma(u, v) = F(v) := (-\Delta u, v)_\Omega - \sum_{e\in\mathcal{E}_h^D} \left\langle u, \partial_n v - \gamma h_e^{-1} v\right\rangle_e + \sum_{e\in\mathcal{E}_h^N} \langle\partial_n u, v\rangle_e, \quad \forall v \in E_h.$$

Thus, we define the discontinuous Galerkin approximation u_h^γ of the solution u of the BVP (1.1)–(1.3) as the element of V_h^r that satisfies

$$a_h^\gamma(u_h^\gamma, v) = F(v) \quad \forall v \in V_h^r \tag{3.1}$$

We thus have the orthogonality relation

$$a_h^\gamma(u - u_h^\gamma, v) = 0 \quad \forall v \in V_h^r, \tag{3.2}$$

which will play an important role in the derivation of the a posteriori estimates as well as the proof of the convergence of the adaptive scheme.

Concerning the continuity and coercivity of the form a_h^γ, we can prove the following result

LEMMA 3.1. *(i)*

$$|a_h^\gamma(u,v)| \le 2\|u\|_{1,h}\|v\|_{1,h}, \qquad \forall u,v \in E_h.$$

(ii) There exist positive constants γ_0 and c_a such that for all $\gamma \ge \gamma_0$

$$a_h^\gamma(v,v) \ge c_a\|v\|_{1,h}^2, \qquad \forall v \in V_h^r,$$

where

$$\|v\|_{1,h} = \left(\sum_{K\in\mathcal{T}_h} \|\nabla v\|_K^2 + \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \Big(h_e \left|\{\partial_n v\}\right|_e^2 + \gamma h_e^{-1}|[v]|_e^2 \Big) \right)^{1/2}.$$

Let us mention here that γ_0 depends only on r and θ_0. Also, the proof of (i) is merely an application of the Cauchy-Schwarz inequality. To prove (ii), we have to use the trace and inverse inequalities (2.1), (2.2).

3.2. A residual-type a posteriori estimate. This section is devoted to the generalization of the residual-type a posteriori estimates given in [17]. The estimators as well as the exposition follow the lines found in Verfürth [23], with the exception of the technical details stemming from the discontinuous nature of V_h^r.

THEOREM 3.1. *Suppose g_D in (1.2) is the restriction to Γ_D of a function in $V_h^r \cap H^1(\Omega)$. Then with $e = u - u_h^\gamma$ there holds*

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 \le c\Big\{ &\sum_{K\in\mathcal{T}_h} h_K^2\|f+\Delta u_h^\gamma\|_K^2 + \sum_{e\in\mathcal{E}_h^I} h_e\big|[\partial_n u_h^\gamma]\big|_e^2 + \sum_{e\in\mathcal{E}_h^N} h_e\big|g_N - \partial_n u_h^\gamma\big|_e^2 \\ &+ \gamma^2 \sum_{e\in\mathcal{E}_h^I} h_e^{-1}\big|[u_h^\gamma]\big|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}_h^D} h_e^{-1}\big|g_D - u_h^\gamma\big|_e^2 \Big\}. \end{aligned} \tag{3.3}$$

In particular, the constant c is independent of the meshsize and γ.

Proof. Integrating by parts, we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} (\nabla e,\nabla\eta)_K = &\sum_{K\in\mathcal{T}_h} (f+\Delta u_h^\gamma,\eta)_K + \sum_{e\in\mathcal{E}_h^I} \Big(\langle\{\partial_n e\},[\eta]\rangle_e + \langle\{\eta\},[\partial_n e]\rangle_e \Big) \\ &+ \sum_{e\in\mathcal{E}_h^D} \langle\partial_n e,\eta\rangle_e + \sum_{e\in\mathcal{E}_h^N} \langle\partial_n e,\eta\rangle_e \quad \forall\eta\in H^1(\mathcal{T}_h). \end{aligned} \tag{3.4}$$

Letting $\eta = e - v_h$, where v_h is piecewise constant on $\mathcal{T}_h$, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 = &\sum_{K\in\mathcal{T}_h} (f+\Delta u_h^\gamma,\eta)_K - \sum_{e\in\mathcal{E}_h^I} \Big(\langle\{\partial_n e\},[u_h^\gamma+v_h]\rangle_e + \langle\{\eta\},[\partial_n u_h^\gamma]\rangle_e \Big) \\ &- \sum_{e\in\mathcal{E}_h^D} \langle\partial_n e, u_h^\gamma+v_h-g_D\rangle_e + \sum_{e\in\mathcal{E}_h^N} \langle g_N - \partial_n u_h^\gamma,\eta\rangle_e. \end{aligned} \tag{3.5}$$

Now, from the orthogonality relation (3.2), for all χ in $V_h^r \cap H^1(\Omega)$ with $\chi\big|_{\Gamma_D} = g_D$, we have

$$
\begin{aligned}
0 = a_h^\gamma(e, u_h^\gamma + v_h - \chi) = &\sum_{K\in\mathcal{T}_h} (\nabla e, \nabla(u_h^\gamma - \chi))_K - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \langle \{\partial_n(u_h^\gamma - \chi)\}, [e]\rangle_e \\
&- \sum_{e\in\mathcal{E}_h^I} \langle \{\partial_n e\}, [u_h^\gamma + v_h]\rangle_e - \sum_{e\in\mathcal{E}_h^D} \langle \partial_n e, u_h^\gamma + v_h - g_D\rangle_e \\
&+ \sum_{e\in\mathcal{E}_h^I} \gamma h_e^{-1} \langle [e], [u_h^\gamma + v_h]\rangle_e + \sum_{e\in\mathcal{E}_h^D} \gamma h_e^{-1} \langle g_D - u_h^\gamma, u_h^\gamma + v_h - g_D\rangle_e
\end{aligned}
$$

Using this relation in (3.5) to eliminate the terms containing $\partial_n e$, we obtain

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 = &\sum_{K\in\mathcal{T}_h} (f + \Delta u_h^\gamma, \eta)_K - \sum_{K\in\mathcal{T}_h} (\nabla e, \nabla(u_h^\gamma - \chi))_K \\
&- \sum_{e\in\mathcal{E}_h^I} \Big(\langle \{\partial_n(u_h^\gamma - \chi)\}, [u_h^\gamma]\rangle_e + \gamma h_e^{-1} \langle [u_h^\gamma], [\eta]\rangle_e + \langle \{\eta\}, [\partial_n u_h^\gamma]\rangle_e \Big) \\
&+ \sum_{e\in\mathcal{E}_h^D} \Big(\langle \partial_n(u_h^\gamma - \chi), g_D - u_h^\gamma\rangle_e + \gamma h_e^{-1} \langle g_D - u_h^\gamma, \eta\rangle_e \Big) \\
&+ \sum_{e\in\mathcal{E}_h^N} \langle g_N - \partial_n u_h^\gamma, \eta\rangle_e . \qquad (3.6)
\end{aligned}
$$

We now obtain bounds for the terms on the right-hand side of (3.6). Those that contain η are bounded by $\frac{1}{2}$ times

$$
\begin{aligned}
&\frac{1}{\epsilon_1} \sum_{K\in\mathcal{T}_h} h_K^2 \|f + \Delta u_h^\gamma\|_K^2 + \frac{1}{\epsilon_2} \sum_{e\in\mathcal{E}_h^I} h_e \big| [\partial_n u_h^\gamma] \big|_e^2 + \frac{1}{\epsilon_3}\gamma \sum_{e\in\mathcal{E}_h^I} h_e^{-1} |[u_h^\gamma]|_e^2 \\
&+\frac{1}{\epsilon_4}\gamma \sum_{e\in\mathcal{E}_h^D} h_e^{-1} |g_D - u_h^\gamma|_e^2 + \frac{1}{\epsilon_5} \sum_{e\in\mathcal{E}_h^N} h_e |g_N - \partial_n u_h^\gamma|_e^2 \\
&+\epsilon_1 \sum_{K\in\mathcal{T}_h} h_K^{-2} \|\eta\|_K^2 + \epsilon_2 \sum_{e\in\mathcal{E}_h^I} h_e^{-1} |\{\partial_n \eta\}|_e^2 + \epsilon_3\gamma \sum_{e\in\mathcal{E}_h^I} h_e^{-1} |[\eta]|_e^2 \\
&+\epsilon_4\gamma \sum_{e\in\mathcal{E}_h^D} h_e^{-1} |\eta|_e^2 + \epsilon_5 \sum_{e\in\mathcal{E}_h^N} h_e^{-1} |\eta|_e^2
\end{aligned} \qquad (3.7)
$$

for any $\epsilon_i > 0$, $i = 1, \ldots, 5$. To estimate the "η" terms in (3.7) we choose as v_h the best piecewise constant approximation of e that gives, using an approximation result of [6], the estimate

$$\|\eta\|_K \le c h_K \|\nabla e\|_K, \ K \in \mathcal{T}_h.$$

Since the mesh is locally quasiuniform, using this approximation result and the trace inequality (2.1), we obtain

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_h} h_K^{-2} \|\eta\|_K^2 &\le c \sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2, \\
\sum_{e\in\mathcal{E}_h^I} h_e^{-1} \big(|\{\partial_n \eta\}|_e^2 + |[\eta]|_e^2 \big) &\le c \sum_{e\in\mathcal{E}_h^I} \sum_{K=K^+,K^-} h_e^{-1} (h_K^{-1} \|\eta\|_K^2 + h_K \|\nabla\eta\|_K^2) \\
&\le c \sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2, \\
\sum_{e\in\mathcal{E}_h^B} h_e^{-1} |\eta|_e^2 &\le c \sum_{e\in\mathcal{E}_h^B} \sum_{K=K^+} h_e^{-1} (h_K^{-1} \|\eta\|_K^2 + h_K \|\nabla\eta\|_K^2) \\
&\le c \sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2.
\end{aligned}
$$

We can now hide the "η" terms in the left-hand side of (3.6) by taking the ϵ's sufficiently small. In particular, we must take $\epsilon_3 \approx 1/\gamma$ and $\epsilon_4 \approx 1/\gamma$.

To obtain (3.3), we also need to estimate the terms containing $u_h^\gamma - \chi$. Indeed these are bounded by

$$\begin{aligned}(3.8)\qquad & \epsilon \sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 + \frac{1}{\epsilon}\sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma-\chi)\|_K^2 + \sum_{e\in\mathcal{E}_h^I} h_e\big|\{\partial_n(u_h^\gamma-\chi)\}\big|_e^2 \\ &+ \sum_{e\in\mathcal{E}_h^I} h_e^{-1}|[u_h^\gamma]|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e\big|\partial_n(u_h^\gamma-\chi)\big|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{-1}|g_D-u_h^\gamma|_e^2.\end{aligned}$$

Using the trace and inverse inequalities, we see that the two terms in (3.8) that contain $\partial_n(u_h^\gamma-\chi)$ are bounded by $\sum_{K\in\mathcal{T}_h}\|\nabla(u_h^\gamma-\chi)\|_K^2$. In view of Theorem 2.1 (ii), the latter is bounded by $\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\ |[u_h^\gamma]|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{-1}|g_D-u_h^\gamma|_e^2$. □

THEOREM 3.2. *The following estimates hold*

- *(i) Suppose that f is a piecewise polynomial on $\mathcal{T}_h$. Then for each $K\in\mathcal{T}_h$,*

$$h_K^2\|f+\Delta u_h^\gamma\|_K^2 \le c\,\|\nabla e\|_K^2. \tag{3.9}$$

- *(ii) For $e=\partial K^+\cap\partial K^-\in\mathcal{E}_h^I$,*

$$h_e\big|[\partial_n u_h^\gamma]\big|_e^2 \le c(\|\nabla e\|_{K^+}^2 + \|\nabla e\|_{K^-}^2). \tag{3.10}$$

- *(iii) Suppose that g_N is a piecewise polynomial on $\mathcal{E}_h^N$. Then for $e=\partial K^+\cap\partial\Omega\in\mathcal{E}_h^N$*

$$h_e\big|g_N-\partial_n u_h^\gamma\big|_e^2 \le c\|\nabla e\|_{K^+}^2. \tag{3.11}$$

- *(iv) Suppose that g_D is the restriction to Γ_D of a function in $V_h^r\cap H^1(\Omega)$. Then there exists γ_1 depending only on r and θ_0 such that for $\gamma\ge\gamma_1$*

$$\gamma^2\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\big|[u_h^\gamma]\big|_e^2 + \gamma^2\sum_{e\in\mathcal{E}_h^D} h_e^{-1}\big|g_D-u_h^\gamma\big|_e^2 \le c\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2. \tag{3.12}$$

The constants c in (3.9)–(3.12) depend on r, θ_0 the degrees of f and g_N but are independent of the meshsize and γ.

Proof. Proofs of assertion (i), (ii), (iii) are similar to the proof of theorem 3.2 in [17].

Concerning assertion (iv), let $u_h^G\in V_h^r\cap H^1(\Omega)$ with $u_h^G\big|_{\Gamma_D}=g_D$ denote the "standard" continuous Galerkin approximation of u given as solution of

$$(\nabla u_h^G,\nabla\chi) = (f,\chi) + \sum_{e\in\mathcal{E}_h^N}\langle\chi,g_N\rangle_e,\quad \forall\chi\in V_h^r\cap H^1_{0,\Gamma_D}. \tag{3.13}$$

It is easily seen that u_h^G satisfies

$$a_h^\gamma(u_h^G,\chi) = (f,\chi) - \sum_{e\in\mathcal{E}_h^D}\langle g_D,\partial_n\chi\rangle_e + \sum_{e\in\mathcal{E}_h^N}\langle\chi,g_N\rangle_e,\quad \forall\chi\in V_h^r\cap H^1_{0,\Gamma_D} \tag{3.14}$$

then one gets the following orthogonality relation for u_h^G

$$a_h^\gamma(u_h^G-u,\chi)=0,\quad \forall\chi\in V_h^r\cap H^1_{0,\Gamma_D}. \tag{3.15}$$

Now for all $\chi \in V_h^r \cap H^1_{0,\Gamma_D}$,

$$\begin{aligned}
a_h^\gamma(u_h^\gamma - u_h^G, u_h^\gamma - u_h^G) =& a_h^\gamma(u - u_h^G, u_h^\gamma - u_h^G) \\
=& a_h^\gamma(u - u_h^G, u_h^\gamma - u_h^G - \chi) \\
=& \sum_{K\in\mathcal{T}_h} (\nabla(u - u_h^G), \nabla(u_h^\gamma - u_h^G - \chi))_K \\
& - \sum_{e\in\mathcal{E}_h^I} \left\langle \left\{\partial_n(u - u_h^G)\right\}, [u_h^\gamma]\right\rangle_e - \sum_{e\in\mathcal{E}_h^D} \left\langle \partial_n(u - u_h^G), u_h^\gamma - g_D \right\rangle_e \\
=& \sum_{K\in\mathcal{T}_h} (\nabla e + \nabla(u_h^\gamma - u_h^G), \nabla(u_h^\gamma - u_h^G - \chi))_K \\
& - \sum_{e\in\mathcal{E}_h^I} \left\langle \{\partial_n e\} + \left\{\partial_n(u_h^\gamma - u_h^G)\right\}, [u_h^\gamma]\right\rangle_e \\
& - \sum_{e\in\mathcal{E}_h^D} \left\langle \partial_n e + \partial_n(u_h^\gamma - u_h^G), u_h^\gamma - g_D \right\rangle_e
\end{aligned}$$

Then integration by parts of $(\nabla e, \nabla(u_h^\gamma - u_h^G - \chi))_K$ gives

$$\begin{aligned}
a_h^\gamma(u_h^\gamma - u_h^G, u_h^\gamma - u_h^G) =& \sum_{K\in\mathcal{T}_h} \Big(\big(f + \Delta u_h^\gamma, u_h^\gamma - u_h^G - \chi\big)_K + \big(\nabla(u_h^\gamma - u_h^G), \nabla(u_h^\gamma - u_h^G - \chi)\big)_K \Big) \\
& - \sum_{e\in\mathcal{E}_h^I} \Big(\left\langle [\partial_n u_h^\gamma], \{u_h^\gamma - u_h^G - \chi\}\right\rangle_e + \left\langle \left\{\partial_n(u_h^\gamma - u_h^G)\right\}, [u_h^\gamma]\right\rangle_e \Big) \\
& - \sum_{e\in\mathcal{E}_h^D} \left\langle \partial_n(u_h^\gamma - u_h^G), u_h^\gamma - g_D \right\rangle_e - \sum_{e\in\mathcal{E}_h^N} \left\langle \partial_n u_h^\gamma - g_N, u_h^\gamma - u_h^G - \chi \right\rangle_e
\end{aligned}$$

In view of the coercivity of a_h^γ on V_h^r, using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
& c_a \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 + (\gamma - \gamma_0) \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|[u_h^\gamma - u_h^G]|_e^2 \\
& \quad \le \frac{\epsilon_1}{2} \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 + \frac{1}{2\epsilon_1} \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G - \chi)\|_K^2 \\
& \quad + \frac{1}{2\epsilon_2\gamma} \sum_{K\in\mathcal{T}_h} h_K^2 \|f + \Delta u_h^\gamma\|_K^2 + \frac{\epsilon_2\gamma}{2} \sum_{K\in\mathcal{T}_h} h_K^{-2} \|u_h^\gamma - u_h^G - \chi\|_K^2 \\
& \quad + \frac{1}{2\epsilon_2\gamma} \sum_{e\in\mathcal{E}_h^I} h_e |[\partial_n u_h^\gamma]|_e^2 + \frac{\epsilon_2\gamma}{2} \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^N} h_e^{-1} |\{u_h^\gamma - u_h^G - \chi\}|_e^2 \\
& \quad + \frac{\epsilon_1}{2} \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e |\left\{\partial_n(u_h^\gamma - u_h^G)\right\}|_e^2 + \frac{1}{2\epsilon_1} \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1} |[u_h^\gamma - u_h^G]|_e^2 \\
& \quad + \frac{1}{2\epsilon_2\gamma} \sum_{e\in\mathcal{E}_h^N} h_e |\partial_n u_h^\gamma - g_N|_e^2, \qquad (3.16)
\end{aligned}$$

where c_a and γ_0 are as in Lemma 3.1. Using the trace and inverse inequalities it follows that

$$\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e |\left\{\partial_n(u_h^\gamma - u_h^G)\right\}|_e^2 \le c_1 \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 \qquad (3.17)$$

$$\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^N} h_e^{-1} |\{u_h^\gamma - u_h^G - \chi\}|_e^2 \le c_2 \sum_{K\in\mathcal{T}_h} h_K^{-2} \|u_h^\gamma - u_h^G - \chi\|_K^2 \qquad (3.18)$$

where c_1 and c_2 depend only on r and θ_0. Now choose $\chi \in V_h^r \cap H^1_{0,\Gamma_D}$ to approximate $u_h^\gamma - u_h^G$ as in Theorem 2.1(ii). Using (3.18) and (3.17) in (3.16) we have

$$
\begin{aligned}
&c_a \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 + (\gamma-\gamma_0) \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|[u_h^\gamma - u_h^G]|_e^2 \\
&\quad\le \frac{\epsilon_1}{2}(1+c_1) \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 + \frac{1}{2\epsilon_2\gamma} \sum_{K\in\mathcal{T}_h} h_K^2 \|f+\Delta u_h^\gamma\|_K^2 \\
&\quad+\frac{1}{2\epsilon_2\gamma} \sum_{e\in\mathcal{E}_h^I} h_e|[\partial_n u_h^\gamma]|_e^2 + \frac{1}{2\epsilon_2\gamma} \sum_{e\in\mathcal{E}_h^N} h_e|\partial_n u_h^\gamma - g_N|_e^2 \\
&\quad+\left(\frac{1+c_3}{2\epsilon_1} + \frac{\epsilon_2(1+c_2)c_3\gamma}{2}\right) \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|[u_h^\gamma - u_h^G]|_e^2,
\end{aligned}
\tag{3.19}
$$

where c_3 is the constant in (2.4). Now choose ϵ_1 small so that $\epsilon_1(1+c_1)\le c_a$ and choose ϵ_2 small so that $\epsilon_2(1+c_2)c_3 \le 1$. Then for $\gamma \ge \gamma_1 := 4\big(\gamma_0 + \dfrac{1+c_3}{2\epsilon_1}\big)$ we will have $\gamma - \gamma_0 - \frac{1+c_3}{2\epsilon_1} - \frac{\gamma}{2} \ge \frac{1}{4}\gamma$. Note that c_a, γ_0, c_1, c_2, c_3 and consequently ϵ_1, ϵ_2, γ_1 depend only on r and θ_0. Thus, from (3.19) we obtain

$$
\begin{aligned}
&\frac{c_a}{2} \sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_h^G)\|_K^2 + \frac{1}{4}\gamma \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|[u_h^\gamma - u_h^G]|_e^2 \\
&\quad\le \frac{1}{2\epsilon_2\gamma}\left(\sum_{K\in\mathcal{T}_h} h_K^2\|f+\Delta u_h^\gamma\|_K^2 + \sum_{e\in\mathcal{E}_h^I} h_e|[\partial_n u_h^\gamma]|_e^2 + \sum_{e\in\mathcal{E}_h^N} h_e|\partial_n u_h^\gamma - g_N|_e^2\right).
\end{aligned}
\tag{3.20}
$$

Using assertions (3.9)–(3.11) in (3.20), we obtain (3.12). This concludes the proof. ☐

REMARK 3.1.

(i) In view of (3.20) one may obtain lower and upper bounds for $\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2$ *with only the 3 terms on the left sides of (3.9), (3.10) and (3.11).*

(ii) Inequality (3.12) is important in that it confirms the right side of (3.3) as both an upper and a lower bound for the error $\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2$ *and thus completes Theorem 3.2 of [17].*

(iii) In [15] the upper bound analogous to (3.3) *contains only* γ *and thus constitues a stronger result than ours in this respect. On the other hand, the fact that the lower bound* (3.12) *contains* γ^2 *is significant and plays an important role in the convergence proof of the adaptive algorithm.*

4. Convergence of the adaptive scheme. In this section we will describe in detail our adaptive algorithm and prove its convergence under appropriate assumptions. The algorithm which is iterative in nature consists in constructing a sequence of meshes and corresponding approximations whereby each cycle consists of the following 4 steps

1. Given a mesh $\mathcal{T}_H$ a DG approximation u_H^γ is constructed by solving (3.1) exactly (to machine precision). In practice however, only an approximate solution is found by a fast iterative method e.g. Multigrid. In that case, the additional errors caused must be taken into account.
2. An (a posteriori) estimation of the error e_H is obtained by calculating e.g. the right side of (3.3) without the terms containing γ, their exclusion being motivated by (3.20).
3. Based on the information supplied by the a posteriori error estimate certain triangles and edges of $\mathcal{T}_H$ are marked for refinement. This is the *marking* strategy. It is patterned after that of [14].
4. The triangles and edges marked for refinement in step 3 lead to a set of triangles to be refined in a specific way. This is the *refinement* strategy and defines the new mesh $\mathcal{T}_h$.

Our convergence result can be summarized as follows: Let $\mathcal{T}_H$ be a mesh with V_H^r denoting the corresponding discontinuous finite element space and let $\mathcal{T}_h$ denote a refinement of $\mathcal{T}_H$ obtained by following the above steps. Let u_h^γ and u_H^γ denote the DG solutions in V_h^r and V_H^r respectively and e_h and e_H the corresponding errors. Then, under certain assumptions on the data of the BVP (1.1)-(1.3) and for γ sufficiently large, there holds

$$a_h^\gamma(e_h, e_h) \le \rho\ a_H^\gamma(e_H, e_H), \qquad 0 < \rho < 1. \tag{4.1}$$

Let us note that such convergence results are based in an essential manner on an orthogonality relation which in this context is written as

$$a_h^\gamma(e_H, e_H) = a_h^\gamma(e_h, e_h) + a_h^\gamma(u_h^\gamma - u_H^\gamma, u_h^\gamma - u_H^\gamma). \tag{4.2}$$

The convergence of the algorithm hinges on obtaining a fixed reduction in the error and this depends in a crucial manner on the nonnegative quantity $a_h^\gamma(u_h^\gamma - u_H^\gamma, u_h^\gamma - u_H^\gamma)$ being sufficiently large with respect to the other two terms in (4.2). However, examples of problems can be constructed, in particular when the solution u is oscillatory, whereby $a_h^\gamma(u_h^\gamma - u_H^\gamma, u_h^\gamma - u_H^\gamma) = 0$ on an arbitrarily long sequence of meshes each one obtained from the previous one by full refinement. See e.g. [19], [20], [14]. It turns out that our assumptions on the data preclude such occurrences resulting in the linear convergence rate (4.1).

Before engaging in the proof of the theorem, We immediately notice a difficulty presented by the fact that we have $a_h^\gamma(e_H, e_H)$ on the left hand side of (4.2) instead of $a_H^\gamma(e_H, e_H)$. Another basic problem is that $a_h^\gamma(\cdot,\cdot)$ is not coercive on the energy space. We will show below that $a_h^\gamma(e_h, e_h)$ behaves like a norm thus giving a meaning to the convergence result $\lim_{h\to 0} a_h^\gamma(e_h, e_h) = 0$ implied by (4.1).

We deal with the first problem by showing that $a_h^\gamma(e_H, e_H)$ is bounded by $a_H^\gamma(e_H, e_H)$ plus a nonnegative quantity that can be absorbed in other terms.

Proposition 4.1. *Suppose the mesh $\mathcal{T}_h$ is not too fine with respect to $\mathcal{T}_H$. Then*

$$\begin{aligned} a_h^\gamma(e_H, e_H) &\le a_H^\gamma(e_H, e_H) + c\gamma \sum_{e\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} h_e^{-1}|\,[e_H]\,|_e^2 \\ &= a_H^\gamma(e_H, e_H) + c\gamma \sum_{e\in\mathcal{E}_H^I} h_e^{-1}|\,[u_H^\gamma]\,|_e^2 + c\gamma \sum_{e\in\mathcal{E}_H^D} h_e^{-1}|g_D - u_H^\gamma|_e^2. \end{aligned} \tag{4.3}$$

Proof. Indeed, we have

$$a_h^\gamma(e_H, e_H) = \sum_{K\in\mathcal{T}_h} \|\nabla e_H\|_K^2 - \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \Big(2\,\langle\{\partial_n e_H\}\,,[e_H]\rangle_e - \gamma h_e^{-1}|\,[e_H]\,|_e^2\Big). \tag{4.4}$$

Since u is smooth and u_H^γ is a polynomial on each $K \in \mathcal{T}_H$, we have $\sum_{K\in\mathcal{T}_h} \|\nabla e_H\|_K^2 = \sum_{K\in\mathcal{T}_H} \|\nabla e_H\|_K^2$. Now if $e \in \mathcal{E}_h^I$ is a "completely" new edge, i.e. is in the interior of some $K \in \mathcal{T}_H$, then $[e_H]\,\big|_e = 0$. Also, since Γ is polygonal, edges $e \in \mathcal{E}_h^D$ are parts of edges in $\mathcal{E}_H^D$. Thus

$$\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \langle\{\partial_n e_H\}\,,[e_H]\rangle_e = \sum_{e\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} \langle\{\partial_n e_H\}\,,[e_H]\rangle_e\,.$$

As for the terms in (4.4) that contain γ, the problem is to contend with the weights h_e^{-1} of the new edges. Again, there are no contributions from the completely new edges. Thus

$$\gamma \sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|\,[e_H]\,|_e^2 \le \gamma \sum_{e\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} \nu(e) h_e^{-1}|\,[e_H]\,|_e^2$$

where for $e \in \mathcal{E}_H^I \cup \mathcal{E}_H^D$, $\nu(e) = \max\{\frac{h_e}{h_{e'}} \mid e' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D, e' \in e\} \ge 2$ is a number that measures the fineness of $\mathcal{T}_h$ with respect to $\mathcal{T}_H$. Assuming that $\nu(e)$ is uniformly bounded, i.e. $\mathcal{T}_h$ is not too fine relative to $\mathcal{T}_H$ we finally obtain (4.3). ∎

We next tackle the lack of coercivity of $a_h^\gamma(.,.)$ on the energy space E_h by showing that, nevertheless, as far as e_h is concerned, $a_h^\gamma(.,.)$ behaves like a norm!

Proposition 4.2. *There exists a constant γ_2 depending only on r and θ_0 such that if $\gamma \geq \gamma_2$, then for some constant $C_1 > 0$ depending only on r and θ_0 there holds*

$$a_h^\gamma(e_h, e_h) \geq \frac{1}{2} \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2 + C_1\gamma^2 \sum_{e \in \mathcal{E}_h^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 + C_1\gamma^2 \sum_{e \in \mathcal{E}_h^D} h_e^{-1} |g_D - u_h^\gamma|_e^2. \tag{4.5}$$

Proof. We have

$$a_h^\gamma(e_h, e_h) = \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2 - \sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \Big(2 \left\langle \{\partial_n e_h\}, [e_h] \right\rangle_e - \gamma h_e^{-1} |\,[e_h]\,|_e^2 \Big). \tag{4.6}$$

Moreover, for all $\chi \in V_h^r \cap H^1(\Omega)$ satisfying $\chi\big|_{\Gamma_D} = g_D$, we have

$$\sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left\langle \{\partial_n e_h\}, [e_h] \right\rangle_e = \sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left\langle \{\partial_n e_h\}, [\chi - u_h^\gamma] \right\rangle_e.$$

On the other hand by virtue of the orthogonality identity (3.2)

$$0 = a_h^\gamma(e_h, \chi - u_h^\gamma) = \sum_{K \in \mathcal{T}_h} (\nabla e_h, \nabla(\chi - u_h^\gamma))_K - \sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \Big(\left\langle \{\partial_n (\chi - u_h^\gamma)\}, [e_h] \right\rangle_e + \left\langle \{\partial_n e_h\}, [\chi - u_h^\gamma] \right\rangle_e - \gamma h_e^{-1} \left\langle [e_h], [\chi - u_h^\gamma] \right\rangle_e \Big).$$

Thus

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left\langle \{\partial_n e_h\}, [e_h] \right\rangle_e = & \sum_{K \in \mathcal{T}_h} (\nabla e_h, \nabla(\chi - u_h^\gamma))_K + \sum_{e \in \mathcal{E}_h^I} \Big(\left\langle \{\partial_n (\chi - u_h^\gamma)\}, [u_h^\gamma] \right\rangle_e + \gamma h_e^{-1} |\,[u_h^\gamma]\,|_e^2 \Big) \\ & - \sum_{e \in \mathcal{E}_h^D} \Big(\left\langle \{\partial_n (\chi - u_h^\gamma)\}, g_D - u_h^\gamma \right\rangle_e - \gamma h_e^{-1} |g_D - u_h^\gamma|_e^2 \Big). \end{aligned} \tag{4.7}$$

We now choose χ in (4.7) as in Theorem 2.1 (ii). Also, using the trace and inverse inequalities, for any $\epsilon > 0$ we obtain

$$\begin{aligned} \left| \sum_{e \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left\langle \{\partial_n e_h\}, [e_h] \right\rangle_e \right| \leq & \; c\epsilon \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2 + \Big(\gamma + \frac{c}{\epsilon}\Big) \sum_{e \in \mathcal{E}_h^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 \\ & + \Big(\gamma + \frac{c}{\epsilon}\Big) \sum_{e \in \mathcal{E}_h^D} h_e^{-1} |g_D - u_h^\gamma|_e^2. \end{aligned} \tag{4.8}$$

Using this in (4.6), we obtain

$$a_h^\gamma(e_h, e_h) \geq (1 - 2c\epsilon) \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2 - \Big(\gamma + \frac{2c}{\epsilon}\Big) \left(\sum_{e \in \mathcal{E}_h^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 + \sum_{e \in \mathcal{E}_h^D} h_e^{-1} |g_D - u_h^\gamma|_e^2 \right). \tag{4.9}$$

Now note that the last two sums in (4.9) are dominated by $\frac{c}{\gamma^2} \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2$ as shown by the a posteriori estimate (3.12). Hence, choosing $\epsilon = \frac{1}{8c}$ and then using (3.12), for γ sufficiently large we obtain the desired result ∎

Remark 4.1. *The proof of Proposition 4.2 also yields*

$$a_h^\gamma(e_h, e_h) \leq \Big(1 + \frac{1}{\gamma}\Big) \sum_{K \in \mathcal{T}_h} \|\nabla e_h\|_K^2 + C_2\gamma \sum_{e \in \mathcal{E}_h^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 + C_2\gamma \sum_{e \in \mathcal{E}_h^D} h_e^{-1} |g_D - u_h^\gamma|_e^2. \tag{4.10}$$

We now begin the proof of (4.1). Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$. Since $e_H \in E_H \subseteq E_h$, we integrate $\sum_{K\in\mathcal{T}_h}(\nabla e_H, \nabla v)_K$ by parts to obtain

$$\sum_{K\in\mathcal{T}_h}(\nabla e_H, \nabla v)_K = \sum_{K\in\mathcal{T}_h}(f + \Delta u_H^\gamma, v)_K + \sum_{e\in\mathcal{E}_h^I}\Big(\langle\{\partial_n e_H\}, [v]\rangle_e + \langle\{v\}, [\partial_n e_H]\rangle_e\Big)$$

$$(4.11) \qquad + \sum_{e\in\mathcal{E}_h^D}\langle\partial_n e_H, v\rangle_e + \sum_{e\in\mathcal{E}_h^N}\langle g_N - \partial_n u_H^\gamma, v\rangle_e \qquad \forall v \in E_h.$$

It then follows from (4.11) and the definition of $a_h^\gamma(\cdot,\cdot)$ that

$$\sum_{K\in\mathcal{T}_h}(f + \Delta u_H^\gamma, v)_K - \sum_{e\in\mathcal{E}_h^I}\langle[\partial_n u_H^\gamma], \{v\}\rangle_e + \sum_{e\in\mathcal{E}_h^N}\langle g_N - \partial_n u_H^\gamma, v\rangle_e = a_h^\gamma(e_H, v)$$

$$(4.12) \qquad + \sum_{e\in\mathcal{E}_h^I}\Big(\langle\{\partial_n v\}, [e_H]\rangle_e - \gamma h_e^{-1}\langle[e_H], [v]\rangle_e\Big)$$

$$+ \sum_{e\in\mathcal{E}_h^D}\Big(\langle\partial_n v, e_H\rangle_e - \gamma h_e^{-1}\langle e_H, v\rangle_e\Big) \quad \forall v \in E_h.$$

At this point we write $a_h^\gamma(e_H, v) = a_h^\gamma(u_h^\gamma - u_H^\gamma, v) + a_h^\gamma(e_h, v)$ and note that $a_h^\gamma(e_h, v) = 0\ \forall v \in V_h^r$. Also, it turns out that it is crucial to eliminate the troublesome terms containing γ. These considerations lead us to use test functions from the subspaces $V_h^r \cap H^1_{0,\Gamma_D}$ of E_h encountered in the proof of Theorem 3.2. We then have the key identity

$$\sum_{K\in\mathcal{T}_h}(f + \Delta u_H^\gamma, v)_K - \sum_{e\in\mathcal{E}_h^I}\langle[\partial_n u_H^\gamma], \{v\}\rangle_e + \sum_{e\in\mathcal{E}_h^N}\langle g_N - \partial_n u_H^\gamma, v\rangle_e = a_h^\gamma(u_h^\gamma - u_H^\gamma, v)$$

$$(4.13) \qquad - \sum_{e\in\mathcal{E}_h^I}\langle\{\partial_n v\}, [u_H^\gamma]\rangle_e + \sum_{e\in\mathcal{E}_h^D}\langle\partial_n v, g_D - u_H^\gamma\rangle_e$$

$$= \sum_{K\in\mathcal{T}_h}(\nabla(u_h^\gamma - u_H^\gamma), \nabla v)_K - \sum_{e\in\mathcal{E}_h^I}\langle\{\partial_n v\}, [u_h^\gamma]\rangle_e + \sum_{e\in\mathcal{E}_h^D}\langle\partial_n v, g_D - u_h^\gamma\rangle_e,$$

$\forall v \in V_h^r \cap H^1_{0,\Gamma_D}$. The principal thrust of the proof of convergence is to use (4.13) to bound the terms $h_K^2\|f + \Delta u_H^\gamma\|_K^2$, $h_e|\,[\partial_n u_H^\gamma]\,|_e^2$ and $h_e|g_N - \partial_n u_H^\gamma|_e^2$ by an appropriate functional of $u_h^\gamma - u_H^\gamma$. This estimation is accomplished by on one hand marking certain triangles and edges of $\mathcal{T}_H$ for refinement (Marking strategy) and on the other hand by insuring that the test function space $V_h^r \cap H^1_{0,\Gamma_D}$ is large enough to yield the desired estimates. Consequently, the refinement must be done according to some specific rules (Refinement strategy)

We next describe our marking strategy which is modeled after the one in Dörfler [14].

Marking Strategy

For some number $\theta \in (0, 1)$, let $\mathcal{R}_H^K, \mathcal{R}_H^I$ and $\mathcal{R}_H^N$ be any subsets of $\mathcal{T}_H, \mathcal{E}_H^I$ and $\mathcal{E}_H^N$ respectively such that

$$\sum_{K\in\mathcal{R}_H^K} h_K^2\|f + \Delta u_H^\gamma\|_K^2 \geq \theta \sum_{K\in\mathcal{T}_H} h_K^2\|f + \Delta u_H^\gamma\|_K^2$$

$$\sum_{e\in\mathcal{R}_H^I} h_e|\,[\partial_n u_H^\gamma]\,|_e^2 \geq \theta \sum_{e\in\mathcal{E}_H^I} h_e|\,[\partial_n u_H^\gamma]\,|_e^2$$

$$\sum_{e\in\mathcal{R}_H^N} h_e|g_N - \partial_n u_H^\gamma|_e^2 \geq \theta \sum_{e\in\mathcal{E}_H^N} h_e|g_N - \partial_n u_H^\gamma|_e^2.$$

With $E_{\mathcal{R}}$ and E denoting the sums on the left and right sides respectively, we have

$$E_{\mathcal{R}} \geq \theta E \tag{4.14}$$

Refinement Strategy

(I) A marked triangle $K \in \mathcal{R}_H^K$ will be cut into a number of equivalent triangles. This number depends on r as shown in Fig. 4.1

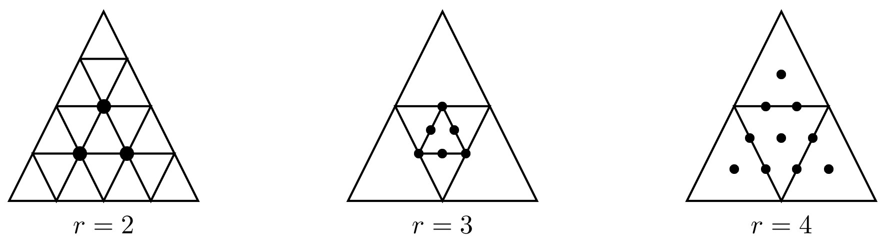

Fig. 4.1.

(II) Let $e = \partial K^+ \cap \partial K^- \in \mathcal{R}_H^I$ be a marked interior edge. Then one or both of K^+ and K^- will be cut in a manner depending on whether e is a full edge of both K^+ and K^-, or not (see Fig. 4.2).

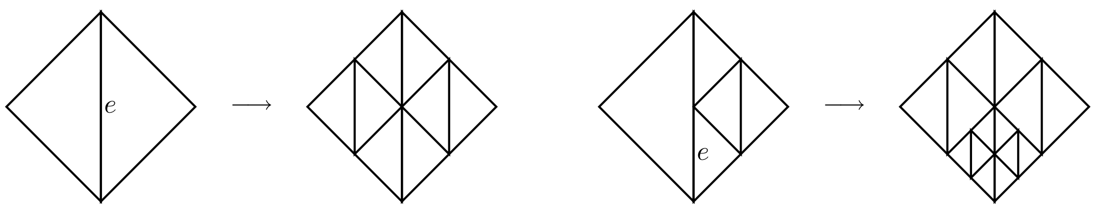

Fig. 4.2.

(III) Let $e = \partial K \cap \Gamma_N$ be a marked edge in $\mathcal{R}_H^N$. Then K will be cut into 4 equivalent triangles.

Remark 4.2. *(i) There may be some overlap between requirements (I), (II) and (III).*

(ii) Additional requirments may also be imposed. For instance, one may wish to curtail the number of hanging nodes after refinement. Indeed, to simplify the programmming we impose a maximum of one hanging node per interior edge. The combination of (I), (II) and such rules may lead to a finer mesh. This is acceptable since to a finer mesh there will correspond larger spaces V_h^r and $V_h^r \cap H^1_{0,\Gamma_D}$.

Estimation of $h_K^2 \|f + \Delta u_H^\gamma\|_K^2$

For $K \in \mathcal{R}_H^K$ consider the partition $\mathcal{T}_K$ shown in Fig. 4.1 corresponding to a given r with the understanding that the eventual refinement of K may be finer than $\mathcal{T}_K$. We introduce the finite dimensional spaces S_K given by

$$S_K = \{v \in C^0(K), v\big|_{K'} \in P_{r-1}(K')\ \forall K' \in \mathcal{T}_K,\ v = 0 \text{ on } \partial K\}.$$

It is clear that S_K is a subspace of $V_h^r \cap H^1_{0,\Gamma_D}$. Also, it is easily seen that a function in S_K is uniquely determined by its values at the nodes shown in Fig. 4.1. Thus $\dim(S_K) \leq d := r(r+1)/2 = \dim(P_{r-1}(K))$. Furthermore, for each r, a basis $\{\phi_i\}_{i=1}^d$ for S_K can be constructed by "gluing" together Lagrangian type functions corresponding to the individual triangles in the partition $\mathcal{T}_K$. Indeed, it is not hard to show the functions $\{\phi_i\}_{i=1}^d$ are linearly independent.

Now letting $\{\psi_i\}_{i=1}^d$ be the usual Lagrangian basis for $P_{r-1}(K)$ coresponding to the nodes shown in Fig. 4.3, we form the "Grammian" matrix G given by $G_{ij} = (\phi_j, \psi_i)_K,\ i, j = 1, \ldots, d$. We have

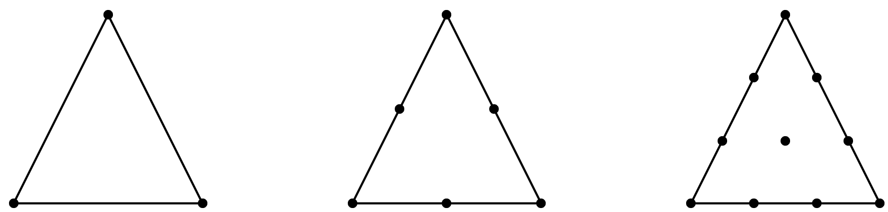

FIG. 4.3.

LEMMA 4.1. *G is nonsingular*

Proof. We will only consider the case $r = 2$; the remaining cases may be handled in a similar manner or verified by direct (and tedious) calculation. With ν_1, ν_2, ν_3 denoting the three nodes shown in Fig. 4.1, let $\mathbf{v^2}, \mathbf{v^3}$ be the vectors emanating from ν_1 and terminating at ν_2 and ν_3 respectively. Let also ϕ_1, ϕ_2, ϕ_3 be the (pyramidal basis functions corresponding to the nodes ν_1, ν_2, ν_3 and denote their supports by S^1, S^2, S^3. Clearly

$$\phi_2(x, y) = \phi_1(x - v_1^2, y - v_2^2),\ \forall(x, y) \in S^2 \text{ and } \phi_3(x, y) = \phi_1(x - v_1^3, y - v_2^3), \quad \forall(x, y) \in S^3.$$

Suppose there exists $\psi = ax + by + c \in P_1(K)$ such that $(\phi_j, \psi) = 0,\ j = 1, 2, 3$. We will show that $a = b = c = 0$ thus implying the linear independence of the rows of G.

$$\begin{aligned}
0 = (\psi, \phi_2)_K &= \int_{S^2} \psi(x, y)\phi_2(x, y)dxdy = \int_{S^2} \psi(x, y)\phi_1(x - v_1^2, y - v_2^2)dxdy \\
&= \int_{S^1} \psi(x + v_1^2, y + v_2^2)\phi_1(x, y)dxdy \\
&= \int_{S^1} \psi(x, y)\phi_1(x, y)dxdy + (av_1^2 + bv_2^2)\int_{S^1} \phi_1(x, y)dxdy.
\end{aligned}$$

Now $\int_{S^1} \psi(x, y)\phi_1(x, y)dxdy = (\psi, \phi_1)_K = 0$. On the other hand, ϕ_1 is nonnegative and nonzero, thus we conclude from the above that $av_1^2 + bv_2^2 = 0$. In a similar way, we obtain $av_1^3 + bv_2^3 = 0$. Since the vectors $\mathbf{v^2}, \mathbf{v^3}$ are linearly independent, it follows that $a = b = 0$. Now that this has been shown the fact that $c = 0$ readily follows from $(\psi, \phi_1)_K = 0$. ∎

COROLLARY 4.1. *Let $P : P_{r-1}(K) \to S_K$ denote the operator given by $(Pv, \chi)_K = (v, \chi)_K$, $\forall \chi \in S_K$. Then $\|P \cdot\|_K$ is a norm equivalent to $\|\cdot\|_K$ on $P_{r-1}(K)$ with constants that are independent of h_K.*

Proof. We only need to check the positivity of $\|P \cdot\|_K$ to see that it is a norm. Indeed, suppose $Pv = 0$ for some $v \in P_{r-1}(K)$. It then follows that $(v, \phi)_K = 0\, \forall \phi \in S_K$. Since G is nonsingular, it follows that $v = 0$. The equivalence of the norms is a consequence of finite dimensionality. The fact that the constants involved are $O(1)$ follows from a scaling argument. ∎

To estimate $f + \Delta u_H^\gamma$ we take $v = P(f + \Delta u_H^\gamma)$ in (4.13). We get

$$\begin{aligned}
\|P(f + \Delta u_H^\gamma)\|_K^2 &= (f + \Delta u_H^\gamma, P(f + \Delta u_H^\gamma))_K = \sum_{K' \in \mathcal{T}_{h,K}} (\nabla(u_h^\gamma - u_H^\gamma), \nabla P(f + \Delta u_H^\gamma))_{K'} \\
&- \sum_{e \in \mathcal{E}_{h,K}^I} \langle\{\partial_n P(f + \Delta u_H^\gamma)\}, [u_h^\gamma]\rangle_e + \sum_{e \in \mathcal{E}_h^D \cap \partial K} \langle\partial_n P(f + \Delta u_H^\gamma), g_D - u_h^\gamma)\rangle_e, \qquad (4.15)
\end{aligned}$$

where $\mathcal{T}_{h,K} = \{K' \in \mathcal{T}_h, K' \subseteq K\}$ and $\mathcal{E}_{h,K}^I = \{e \in \mathcal{E}_h^I,\ e \subseteq K\}$.

Now using the trace and inverse inequalities for any $\epsilon > 0$ we have

$$\sum_{K' \in \mathcal{T}_{h,K}} (\nabla(u_h^\gamma - u_H^\gamma), \nabla P(f + \Delta u_H^\gamma))_{K'} \le c\epsilon\|f + \Delta u_H^\gamma\|_K^2 + \frac{c}{\epsilon}\sum_{K' \in \mathcal{T}_{h,K}} h_{K'}^{-2}\|\nabla(u_h^\gamma - u_H^\gamma)\|_{K'}^2. \qquad (4.16)$$

Moreover,

$$\sum_{e\in\mathcal{E}^I_{h,K}} \langle\{\partial_n P(f+\Delta u^\gamma_H)\}\,,[u^\gamma_h]\rangle_e \le c\epsilon\|f+\Delta u^\gamma_H\|^2_K + \frac{c}{\epsilon}\sum_{e\in\mathcal{E}^I_{h,K}} h_e^{-3}|\,[u^\gamma_h]\,|^2_e, \tag{4.17}$$

and

$$\sum_{e\in\mathcal{E}^D_h\cap\partial K} |\,\langle\partial_n P(f+\Delta u^\gamma_H), g_D-u^\gamma_h\rangle_e\,| \le c\epsilon\|f+\Delta u^\gamma_H\|^2_K + \frac{c}{\epsilon}\sum_{e\in\mathcal{E}^D_h\cap\partial K} h_e^{-3}|g_D-u^\gamma_h|^2_e. \tag{4.18}$$

Now using (4.16),(4.17) and (4.18) with a small ϵ in (4.15), it follows from Corollary 4.1 that

$$\begin{aligned} h^2_K\|f+\Delta u^\gamma_H\|^2_K \le c\sum_{K'\in\mathcal{T}_{h,K}}\|\nabla(u^\gamma_h-u^\gamma_H)\|^2_{K'} + c\sum_{e\in\mathcal{E}^I_{h,K}} h_e^{-1}|\,[u^\gamma_h]\,|^2_e \\ + \sum_{e\in\mathcal{E}^D_h\cap\partial K} h_e^{-1}|g_D-u^\gamma_h|^2_e \end{aligned} \tag{4.19}$$

<u>Estimation of $h_e|\,[\partial_n u^\gamma_H]\,|^2_e$</u>

Let $e\in\mathcal{R}^I_H$ be a marked edge. It follows from the refinement strategy (see Fig. 4.2) that e is a full edge of both K^+ and K^-, where K^+, K^- may belong to $\mathcal{T}_H$ or one of them at most may have been formed after refinement. We construct a test function $v\in V^r_h\cap H^1_{0,\Gamma_D}$ as follows

(i) Let $\tilde{v}$ be the extension of $[\partial_n u^\gamma_H]\,|_e$ to $\tilde{K}:=K^+\cup K^-$ by constants along lines normal to e.
(ii) Let ℓ be the continuous piecewise linear function whose support is the shaded region in Fig. 4.4(b) and which assumes the value 1 at the midpoint of e.

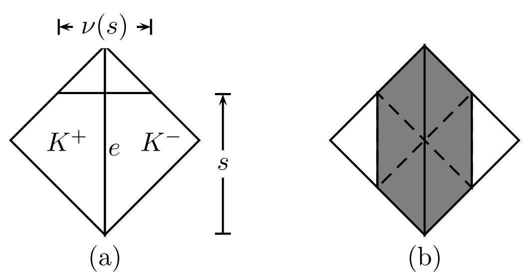

FIG. 4.4.

We take $v=\tilde{v}\ell$. Note that v belongs to $V^r_h\cap H^1_{0,\Gamma_D}$. Using this v in (4.13), we obtain

$$\begin{aligned} \langle[\partial_n u^\gamma_H]\,,\{v\}\rangle_e = \sum_{K'\in\mathcal{T}_{h,\tilde{K}}}\Big((f+\Delta u^\gamma_H, v)_{K'} - (\nabla(u^\gamma_h-u^\gamma_H),\nabla v)_{K'}\Big) \\ + \sum_{e\in\mathcal{E}^I_{h,\tilde{K}}}\langle\{\partial_n v\}\,,[u^\gamma_h]\rangle_e - \sum_{e\in\mathcal{E}^D_h\cap\partial\tilde{K}}\langle\partial_n v, g_D-u^\gamma_h\rangle_e\,, \end{aligned} \tag{4.20}$$

where $\mathcal{T}_{h,\tilde{K}}=\{K'\in\mathcal{T}_h, K'\subseteq\tilde{K}\}$ and $\mathcal{E}^I_{h,\tilde{K}}=\{e\in\mathcal{E}^I_h,\ e\subseteq\tilde{K}\}$. Now note that

$$\langle[\partial_n u^\gamma_H]\,,\{v\}\rangle_e = \int_e |\,[\partial_n u^\gamma_H]\,|^2\ell(s)ds. \tag{4.21}$$

With ℓ acting as a weight function, we have

$$h_e \int_e |\,[\partial_n u_H^\gamma]\,|^2 \ell(s) ds \geq c h_e |\,[\partial_n u_H^\gamma]\,|_e^2 \tag{4.22}$$

where c is independent of h_e. Moreover, since $0 \leq \ell \leq 1$ and $\tilde{v}$ is constant along lines normal to e,

$$\|v\|_{\tilde{K}}^2 \leq \|\tilde{v}\|_{\tilde{K}}^2 = \int_e |\,[\partial_n u_H^\gamma]\,|^2 \nu(s) ds \leq c h_e |\,[\partial_n u_H^\gamma]\,|_e^2, \tag{4.23}$$

where ν is as in Fig. 4.4(a). Now using the trace and inverse inequalities in (4.20), for any $\epsilon > 0$ we obtain

$$\begin{aligned} h_e |\,[\partial_n u_H^\gamma]\,|_e^2 \leq c\epsilon \|v\|_{\tilde{K}}^2 + \frac{c}{\epsilon}\Bigg(\sum_{K' \in \mathcal{T}_{h,\tilde{K}}} \Big(h_{K'}^2 \|f + \Delta u_H^\gamma\|_{K'}^2 + \|\nabla(u_h^\gamma - u_H^\gamma)\|_{K'}^2 \Big) \\ + \sum_{e \in \mathcal{E}_{h,\tilde{K}}^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 + \sum_{e \in \mathcal{E}_h^D \cap \partial\tilde{K}} h_e^{-1} |g_D - u_h^\gamma|_e^2 \Bigg). \end{aligned} \tag{4.24}$$

Hence using (4.21)- (4.23) in (4.24), and choosing ϵ sufficiently small, we arrive at

$$\begin{aligned} h_e |\,[\partial_n u_H^\gamma]\,|_e^2 \leq c \sum_{K' \in \mathcal{T}_{h,\tilde{K}}} \Big(h_{K'}^2 \|f + \Delta u_H^\gamma\|_{K'}^2 + \|\nabla(u_h^\gamma - u_H^\gamma)\|_{K'}^2 \Big) \\ + c \sum_{e \in \mathcal{E}_{h,\tilde{K}}^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 + c \sum_{e \in \mathcal{E}_h^D \cap \partial\tilde{K}} h_e^{-1} |g_D - u_h^\gamma|_e^2. \end{aligned} \tag{4.25}$$

<u>Estimation of $h_e|g_N - \partial_n u_H^\gamma|_e^2$</u>

<u>We define</u> a test function $v \in V_h^r \cap H_{0,\Gamma_D}^1$ as follows: Let $\tilde{v}$ be the extension of $g_N - \partial_n u_H^\gamma$ by constants along lines normal to e and let ℓ be the continuous piecewise linear function that vanishes outside of K and that assumes the value 1 a the midpoint of e. We let $v = \tilde{v}\ell$ and note that since $g_N - \partial_n u_H^\gamma$ is a polynomial of degree less than or equal to $r-2$, $v \in V_h^r \cap H_{0,\Gamma_D}^1$. Using this v in (4.13), we obtain

$$\langle g_N - \partial_n u_H^\gamma, v\rangle_e = -(f + \Delta u_H^\gamma, v)_K + \sum_{K' \in \mathcal{T}_{h,K}} (\nabla(u_h^\gamma - u_H^\gamma), \nabla v)_{K'} - \sum_{e \in \mathcal{E}_{h,K}^I} \langle \{\partial_n v\}, [u_h^\gamma]\rangle_e \tag{4.26}$$

Using the trace and inverse inequalities, we obtain

$$\begin{aligned} h_e \langle g_N - \partial_n u_H^\gamma, v\rangle_e \leq c\epsilon \|v\|_K^2 + \frac{c}{\epsilon}\Bigg(h_K^2 \|f + \Delta u_H^\gamma\|_K^2 + \sum_{K' \in \mathcal{T}_{h,K}} \|\nabla(u_h^\gamma - u_H^\gamma)\|_{K'}^2 \\ + \sum_{e \in \mathcal{E}_{h,K}^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2 \Bigg). \end{aligned} \tag{4.27}$$

As in steps (4.21), (4.22) and (4.23), we have

$$h_e |g_N - \partial_n u_H^\gamma|_e^2 \leq c h_e \langle g_N - \partial_n u_H^\gamma, v\rangle_e \text{ and } \|v\|_K^2 \leq c h_e |g_N - \partial_n u_H^\gamma|_e^2. \tag{4.28}$$

Thus, from (4.27) it follows that

$$h_e |g_N - \partial_n u_H^\gamma|_e^2 \leq c h_K^2 \|f + \Delta u_H^\gamma\|_K^2 + c \sum_{K' \in \mathcal{T}_{h,K}} \|\nabla(u_h^\gamma - u_H^\gamma)\|_{K'}^2 + c \sum_{e \in \mathcal{E}_{h,K}^I} h_e^{-1} |\,[u_h^\gamma]\,|_e^2. \tag{4.29}$$

We are now ready to state and prove the main result of this paper.

THEOREM 4.1. *Let u_h^γ and u_H^γ denote the DG solutions in V_h^r and V_H^r respectively and e_H and e_h the corresponding errors. Assume that*

(i) The data of the BVP (1.1)-(1.3) is such that $f \in P_{r-1}(\Omega)$, $g_D \in P_{r-1}(\Gamma_D)$ and $g_N \in P_{r-2}(\Gamma_N)$.

(ii) $\mathcal{T}_h$ is not too fine with respect to $\mathcal{T}_H$.

(iii) For some $\theta \in (0,1)$ the marking of triangles and edeges of the mesh $\mathcal{T}_H$ and their refinement is done according to the rules specified above.

Then, there exists γ_3 depending only on r, θ_0 and θ such that for all $\gamma \geq \gamma_3$, (4.1) holds with ρ given by (4.36).

Proof.

First, using the trace and inverse inequalities, we have for $\gamma \geq \gamma_4(r, \theta_0)$

$$a_h^\gamma(u_h^\gamma - u_H^\gamma, u_h^\gamma - u_H^\gamma) \geq \frac{1}{2}\sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_H^\gamma)\|_K^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} h_e^{-1}|\,[u_h^\gamma - u_H^\gamma]\,|_e^2. \tag{4.30}$$

On the other hand, from (4.19), (4.25) and (4.29), it follows that for some constant $C_3 > 0$ depending only on r and θ_0

$$E_{\mathcal{R}} \leq C_3\Big(\sum_{K\in\mathcal{T}_h} \|\nabla(u_h^\gamma - u_H^\gamma)\|_K^2 + \sum_{e\in\mathcal{E}_h^I} h_e^{-1}|\,[u_h^\gamma]\,|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{-1}|g_D - u_h^\gamma|_e^2\Big). \tag{4.31}$$

Next, using (4.2), (4.3), (4.30) and (4.31), we obtain

$$\begin{aligned} a_H^\gamma(e_H, e_H) + c\gamma\sum_{e\in\mathcal{E}_H^I} h_e^{-1}|\,[u_H^\gamma]\,|_e^2 + c\gamma\sum_{e\in\mathcal{E}_H^D} h_e^{-1}|g_D - u_H^\gamma|_e^2 &\geq a_h^\gamma(e_H, e_H)\\ &= a_h^\gamma(e_h, e_h) + a_h^\gamma(u_h^\gamma - u_H^\gamma, u_h^\gamma - u_H^\gamma)\\ &\geq a_h^\gamma(e_h, e_h) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\|\nabla(u_h^\gamma - u_H^\gamma)\|_K^2\\ &\geq a_h^\gamma(e_h, e_h) + \frac{E_{\mathcal{R}}}{2C_3} - \frac{1}{2}\Big(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}|\,[u_h^\gamma]\,|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{-1}|g_D - u_h^\gamma|_e^2\Big). \end{aligned} \tag{4.32}$$

Now from (3.20) it follows for $\gamma \geq \gamma_5(r, \theta_0, \theta)$ that

$$c\gamma\Big(\sum_{e\in\mathcal{E}_H^I} h_e^{-1}|\,[u_H^\gamma]\,|_e^2 + \sum_{e\in\mathcal{E}_H^D} h_e^{-1}|g_D - u_H^\gamma|_e^2\Big) \leq \frac{cE}{\gamma} \leq \frac{\theta E}{4C_3}. \tag{4.33}$$

Also from (4.5) we have

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}|\,[u_h^\gamma]\,|_e^2 + \sum_{e\in\mathcal{E}_h^D} h_e^{-1}|g_D - u_h^\gamma|_e^2 \leq \frac{1}{C_1\gamma^2} a_h^\gamma(e_h, e_h). \tag{4.34}$$

Thus, using (4.33) and (4.34) in (4.32) we obtain

$$a_H^\gamma(e_H, e_H) \geq \Big(1 - \frac{1}{2C_1\gamma^2}\Big)a_h^\gamma(e_h, e_h) + \frac{\theta E}{4c_3}. \tag{4.35}$$

We choose γ large so that $1 - \frac{1}{2C_1\gamma^2} > 0$. On the other hand, using (4.10) with H instead of h, (recall that this result holds for a generic mesh) it follows from (3.20) and (3.3) that for some constant $C_4 > 0$ depending only on r and θ_0 one has

$$E \geq C_4 a_H^\gamma(e_H, e_H).$$

Using this in (4.35), it follows that

$$\Big(1-\frac{\theta C_4}{4C_3}\Big)a_H^{\gamma}(e_H,e_H)\ge\Big(1-\frac{1}{2C_1\gamma^2}\Big)a_h^{\gamma}(e_h,e_h).$$

If $\frac{\theta C_4}{4C_3}\ge 1$ then this means that $a_h^{\gamma}(e_h,e_h)=0$. If on the other hand $0<\frac{\theta C_4}{4C_3}<1$, then we obtain (4.1) with ρ given by

$$\rho=\frac{1-\frac{\theta C_4}{4C_3}}{1-\frac{1}{2C_1\gamma^2}}. \tag{4.36}$$

The conclusion of the theorem now follows for γ sufficiently large. □

REMARK 4.3. *The conditions on the data of the BVP* (1.1)-(1.3) *are restrictive and are the price paid to simplify the proofs. We believe that they can be relaxed or dispensed with by introducing appropriate projections of the data functions. See e.g. [15] and [20]. The generalization of our results including an accounting for the effects of quadrature errors is being pursued.*

5. Numerical Experiments. In this section we present the results of some numerical experiments to exhibit the performance of the adaptive strategy outlined in section 4. We used the Baker version of the method since the forms $\{\partial_n v\}_e=\nabla v^+\cdot\mathbf{n}^+\big|_e$ are easier to implement.

As a representative of a problem with a smooth solution we chose

$$-\Delta u=2\pi^2\sin\pi x\sin\pi y,\quad \text{in } \Omega=[0,1]^2,\quad u=0\quad\text{on } \partial\Omega, \tag{P1}$$

with $u=\sin\pi x\sin\pi y$. The next problem has the smooth but oscillatory solution $u=\sin 8\pi x\sin 8\pi y$

$$-\Delta u=128\pi^2\sin 8\pi x\sin 8\pi y,\quad \text{in } \Omega=[0,1]^2,\quad u=0\quad\text{on } \partial\Omega. \tag{P2}$$

Finally, as an example of a solution with a singularity we considered the problem

$$-\Delta u=0,\quad \text{in } \Omega,\quad u=g_D\quad\text{on } \Gamma_D=\partial\Omega. \tag{P3}$$

where Ω is the polygon with vertices (0,0),(-1,-1),(1,-1),(1,1),(-1,1),(0,0) and has a reentrant corner at (0,0). The datum g_D is adjusted so that the solution is $u=r^{2/3}\sin\frac{2\theta}{3}$ in polar coordinates.

We generated an adaptive code written in the C language and ran the experiments on a workstation with an Intel Pentium 4 chip rated at 3.06 GHz. The linear systems were solved by Multigrid with point Gauss-Seidel smoothing as a preconditioner for the Conjugate Gradient method. To assess the performance of the estimator and the adaptive algorithm, we monitored the following 3 quantities

$$a_h^{\gamma}(e,e)=\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2-2\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\langle\{\partial_n e\},[e]\rangle_e+\gamma(r-1)^2\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}h_e^{-1}|[e]|_e^2$$

$$\|e\|_{1,h}=\Bigg(\sum_{K\in\mathcal{T}_h}\|\nabla e\|_K^2+\sum_{e\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\Big(h_e\,|\{\partial_n e\}|_e^2+\gamma(r-1)^2h_e^{-1}|[e]|_e^2\Big)\Bigg)^{1/2}$$

$$\eta=\Bigg(\sum_{K\in\mathcal{T}_h}h_K^2\|f+\Delta u_h^{\gamma}\|_K^2+\sum_{e\in\mathcal{E}_h^I}h_e\big|\big[\partial_n u_h^{\gamma}\big]\big|_e^2+\sum_{e\in\mathcal{E}_h^N}h_e\big|g_N-\partial_n u_h^{\gamma}\big|_e^2$$
$$+\gamma^2(r-1)^4\sum_{e\in\mathcal{E}_h^I}h_e^{-1}\big|[u_h^{\gamma}]\big|_e^2+\gamma^2(r-1)^4\sum_{e\in\mathcal{E}_h^D}h_e^{-1}\big|g_D-u_h^{\gamma}\big|_e^2\Bigg)^{1/2}.$$

These quantities are modified versions of the bilinear form, the energy norm and the residual error estimator. Since the coercivity threshold is known to increase quadratically as a function of the degree $r-1$, we replaced γ by $\gamma(r-1)^2$. This way, the calculations could be performed without the need for adjusting γ with r. Following the same reasoning, we attached $\gamma^2(r-1)^4$ to the jump terms of the residual estimator since γ^2 accompanied these terms both in the upper and lower bounds.

The first set of experiments concerned a study of the effectivity index $\eta/\|e\|_{1,h}$ as a function of the degrees of freedom (dof's). Figure 5.1 shows the effectivity indices for all three test problems. Starting with an initial mesh of 16 triangles (96 dof's), the mesh was refined uniformly until a maximum of about 10^6. In particular, the indices behaved rather well with values close to 1 (more so for (P1) and (P2) than for (P3)) and the index for (P2) took longer to stabilize given the oscillatory nature of the solution. Similar behaviour was observed for $r=2,4,5$.

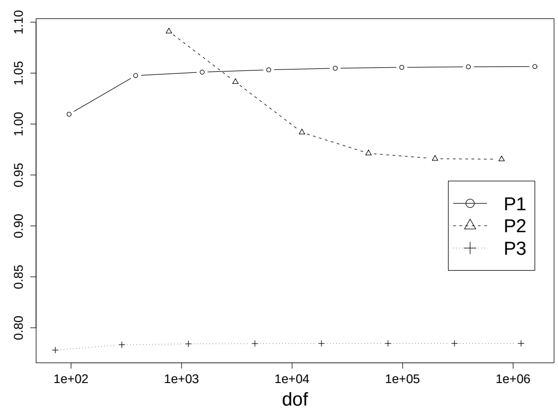

FIG. 5.1. *Effectivity Indices;* $r=3$, $\gamma=6.3$.

We also wanted to study the effect of γ on the effectivity indices. Figure 5.2 shows the results of experiments concerning test problems (P2) and (P3), $r=3$ and values of γ from 5 to 1000. While such an effect does indeed exist, it is nevertheless quite mild as evidenced by the narrow range of the changes in the effectivity indices. We also note that the effectivity indices seem to be convergent as γ increases. Similar results were obtained for (P1) and $r=2,4,5$.

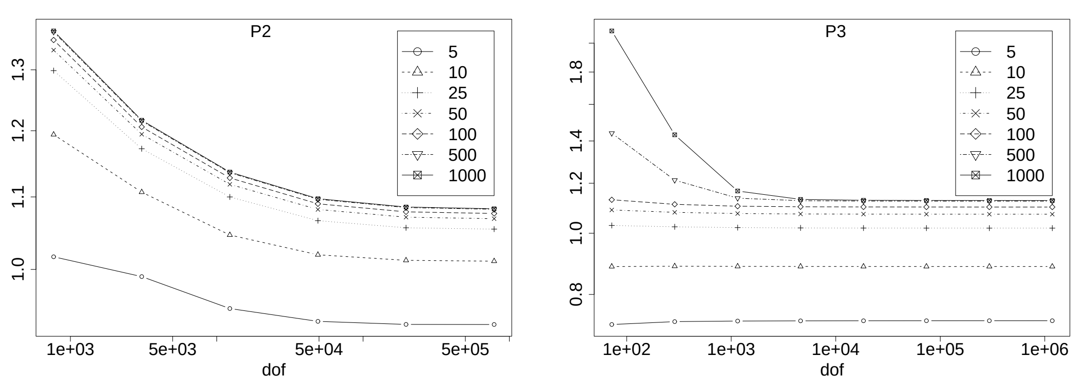

FIG. 5.2. *Dependence of effectivity indices on* γ*;* $r=3$.

The remaining experiments were devoted to the validation of the convergence characteristics of the adaptive algorithm. We ran several experiments with $r=2,3,4$, all 3 test problems and several values of θ and γ. In all cases all 3 quantities $a_h^\gamma(e,e)$, $\|e\|_{1,h}$ and η decreased monotonically. We should also mention that in order to simplify the program, we cut the marked triangles into 4 triangles only, in variance with the patterns shown in Figure 4.1. The plots in Figure 5.3 show the excellent agreement between the error $\|e\|_{1,h}$ and the estimator η. On the other hand, the bilinear form $a_h^\gamma(e,e)$ seems to follow a very similar but parallel trajectory, evidence of its equivalence to the other two. The two plots of Figure 5.4 show the corresponding final meshes for P2 and P3 respectively.

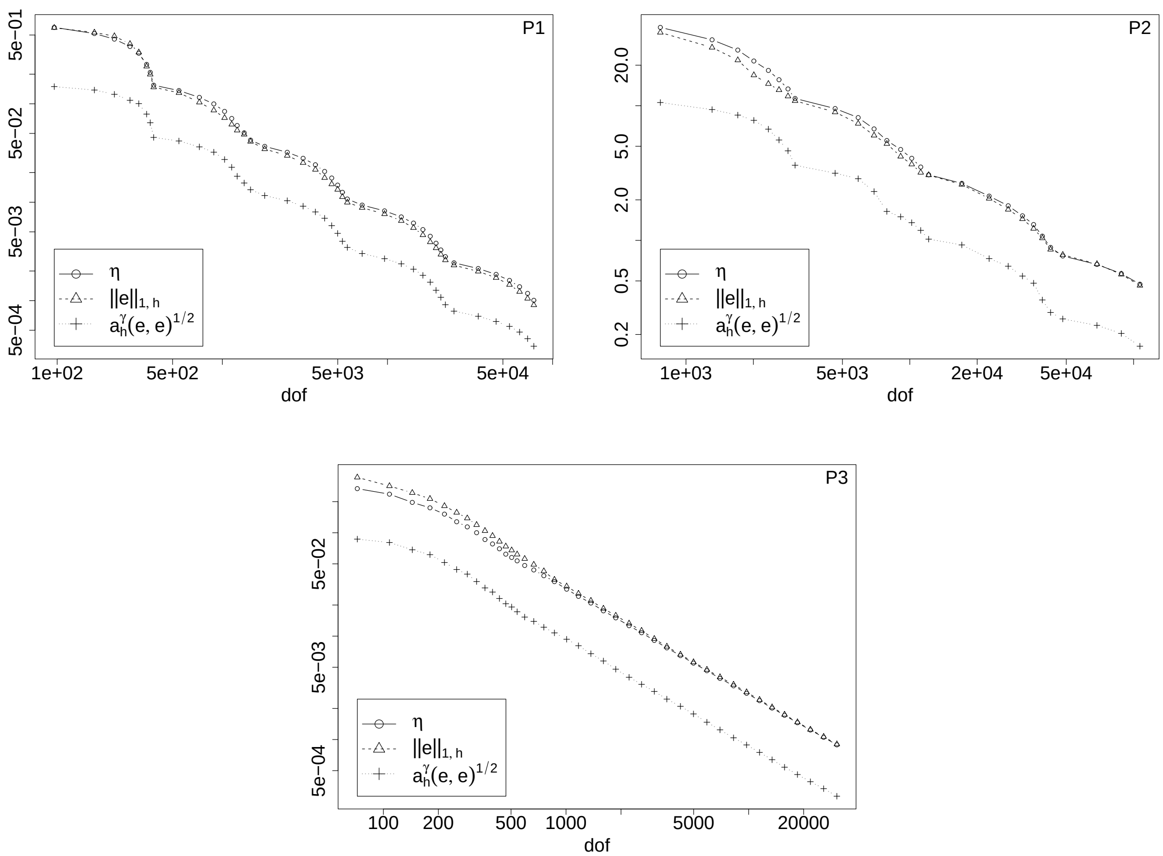

Fig. 5.3. $r = 3,\ \gamma = 6.3,\ \theta = 0.5$

Next, we wanted to study the effect of the choice of θ on the performance of the adaptive algorithm. Indeed, the experiments indicate that while convergence is not in doubt, the patterns of refinement are strongly influenced by this choice as evidenced by the final triangle count and more importantly the CPU time. Postponing a detailed study of this important issue to a future work, we nevertheless maintain that if we accept the criterion that the most efficient algorithm is the one with the least execution time, then larger values of θ should be preferred. Tables 5.1 - 5.3 show respectivley the CPU time, the number of iterations to convergence and the triangle count in the final mesh for the 3 test problems and $\theta = 0.3, 0.5, 0.9$. While smaller values of θ lead to a smaller number of triangles, they are up to 6 times costlier in CPU time. This is due to the fact that at every cycle, relatively few triangles and edges are refined resulting in a large, one could say unacceptable, number of cycles.

Tables 5.4, 5.5 and 5.6 encapsulate the results of an attempt to study the effect of θ on both the number of refinement levels and the distribution of triangles over the levels; in a sense they provide a spectral analysis of the mesh hierarchy. For test problem P1, the value $\theta = 0.9$ caused a shift of the refinement to higher level with a substantial number of triangles on level 6 (Table 5.4). On the other hand, Table 5.6 shows the opposite behaviour for test problem P3 whereby the smaller values of $\theta = 0.3, 0.5$

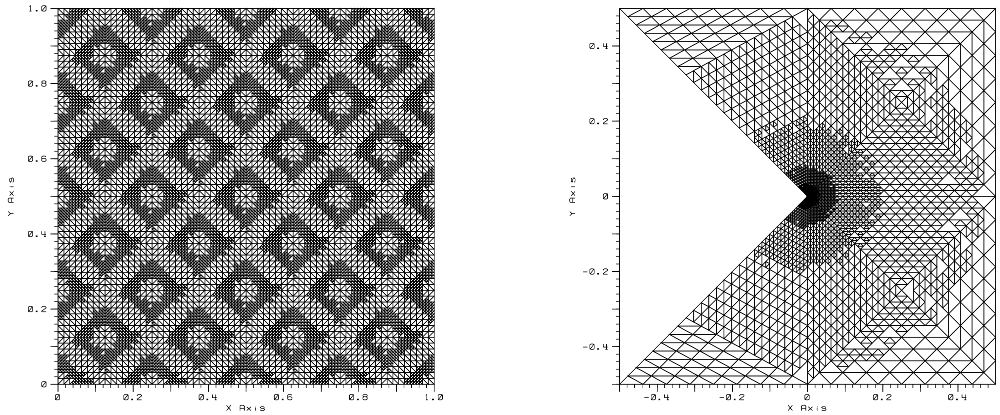

FIG. 5.4. *Final meshes for P2, P3; $r = 3$, $\gamma = 6.3$, $\theta = 0.5$*

θ	P1	P2	P3
0.3	65	95	35
0.5	24	34	17
0.9	11	13	11

TABLE 5.1
Total CPU time (sec)

θ	P1	P2	P3
0.3	108	75	89
0.5	40	26	39
0.9	9	6	14

TABLE 5.2
Adaptive Iterations

θ	P1	P2	P3
0.3	12853	16943	4458
0.5	12823	17780	5064
0.9	20170	22394	8208

TABLE 5.3
Total Triangles in Final Mesh

Level / θ	0.3	0.5	0.9
0-3	0	0	0
4	1177	1187	36
5	11676	11636	14942
6	-	-	5192

TABLE 5.4
P1 ($|\mathcal{T}_0| = 16$) Level Leaf Triangle Distribution

Level / θ	0.3	0.5	0.9
0-2	0	0	0
3	5275	4996	3458
4	11668	12784	18936

TABLE 5.5
P2 ($|\mathcal{T}_0| = 128$) Level Leaf Triangle Distribution

created 5 additional levels albeit with relatively small number of additional triangles.

The limited scope of these experiments provide a validation of the theoretical results of the paper. They also point to the importance of further exploration of the mechanisms of marking and refinement. In particular, a static choice of θ is far from being satisfactory and must be replaced by a more dynamic (adaptive!) mechanism.

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Level / θ	0.3	0.5	0.9
0-2	0	0	0
3	374	314	57
4	1311	1496	2256
5	873	1065	1955
6	625	715	1316
7	409	483	910
8	275	325	587
9	192	208	416
10	120	148	281
11	78	92	186
12	60	66	110
13	42	50	94
14	18	18	40
15	18	18	-
16	18	18	-
17	18	18	-
18	23	22	-
19	4	8	-

TABLE 5.6
P3 ($|\mathcal{T}_0| = 12$) Level Leaf Triangle Distribution

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