Math 577 Spring 2018

HW3 Solutions

#1) Let $J: R \to R$ $J(x)=x^4$. J is twice differentiable, strictly convex but not elliptic, since

$$(f'(x)-f'(y), x-y)=4(x^2+xy+y^2)(x-y)^2$$ and there is no $\alpha>0$

such that $4(x^2+xy+y^2)(y-x)^2 \geq \alpha(y-x)^2$ $\forall x,y \in R$.

#2) Since $\sum_{i=1}^n u_i > 0$, define new variables $\eta_i = u_i / \sum_{i=1}^n u_i$, $i=1,\dots,n$.

$\Rightarrow J(u)=J(\eta)=\sum_{i=1}^n \beta_i \eta_i$. We want to find the extrema of $J(\eta)$ subject to $\eta \in U \equiv \{\eta \in R^n; \eta_i \geq 0, \sum_{i=1}^n \eta_i = 1\}$

Let $\beta_M = \max_i\{\beta_i\}$ and $\beta_m = \min_i\{\beta_i\}$.

We will show that

⊛ $\sup_{\eta \in U} J(\eta) = J(e_M) = \beta_M$ and $\inf_{\eta \in U} J(\eta) = J(e_m) = \beta_m$. (**)

Indeed, $J'(\eta)h = \sum_{i=1}^n \beta_i h_i \Rightarrow J'(e_M)(\eta - e_M) = \sum_{i=1}^n \beta_i(\eta_i - (e_M)_i)$

$$= \sum_{i=1}^n \beta_i \eta_i - \beta_M \leq \beta_M \underbrace{\sum_{i=1}^n \eta_i}_{=1} - \beta_M \leq 0$$

This establishes ⊛. Similarly, $J'(e_m)(\eta - e_m) \geq 0$ which establishes (**).

#3) First, $x^TAx=1 \Rightarrow x^TAx \leq 1$. Hence

$$\sup_{x^TAx \leq 1} b^Tx \geq \sup_{x^TAx=1} b^Tx.$$

Also note that the two set $\{x \in R^n; x^TAx \leq 1\}$ and $\{x \in R^n; x^TAx = 1\}$ are closed and bounded $\Rightarrow$ compact. Since $x \longmapsto b^Tx$ is continuous, both suprema are attained and are finite.

We will now show that $\sup_{x^TAx\le 1} b^Tx$ cannot be attained in the interior, i.e. for x such that $x^TAx<1$. Suppose otherwise.

$$\text{Let } \rho = \sup_{x^TAx\le 1} b^Tx = b^Tz \quad \text{with } z^TAz<1.$$

Consider $x=z+\epsilon b$ for $\epsilon>0$ to be chosen small

$$x^TAx = (z+\epsilon b)^TA(z+\epsilon b) = z^TAz + 2\epsilon b^TAz + \epsilon^2 b^TAb.$$

Since $z^TAz<1$, we can choose ϵ small enough so that

$x^TAx\le 1$. However $b^Tx = b^T(z+\epsilon b) = b^Tz+\epsilon\|b\|_2^2$
$= \rho+\epsilon\|b\|_2^2>\rho$.

This is a contradiction.
This shows that the two suprema (maxima) are equal, i.e. the two pbs. are equivalent.

To calculate the maximum value, we use Lagrange multipliers:

Note λ cannot be zero, for otherwise $b=0$.

$$\Rightarrow \quad b+\lambda(2Ax)=0 \Rightarrow x = -\frac{1}{2\lambda}A^{-1}b.$$

Now

$$1 = x^TAx = \frac{1}{4\lambda^2} b^TA^{-1}AA^{-1}b = \frac{1}{4\lambda^2} b^TA^{-1}b \Rightarrow \lambda = \pm\frac{1}{2}\sqrt{b^TA^{-1}b}$$

$$\Rightarrow x = \pm\frac{A^{-1}b}{\sqrt{b^TA^{-1}b}} \quad \Rightarrow b^Tx = \pm\sqrt{b^TA^{-1}b}$$

$$\Rightarrow \sup_{x^TAx=1} b^Tx = \sqrt{b^TA^{-1}b} \quad \text{and} \quad \inf_{x^TAx=1} b^Tx = -\sqrt{b^TA^{-1}b}.$$

#4) Let $b\in R^n$, $b\neq 0$. Suppose $x^TAx \ge x^TBx \quad \forall x\in R^n$.

Then
$$\begin{array}{ccc} \sup_{x^TAx\le 1} b^Tx & \le & \sup_{x^TBx\le 1} b^Tx \quad \leftarrow x^TAx\le 1 \Rightarrow x^TBx\le 1 \\ \| & & \| \\ \sqrt{b^TA^{-1}b} & \le & \sqrt{b^TB^{-1}b} \Rightarrow b^TA^{-1}b \le b^TB^{-1}b. \end{array}$$

Since the last inequality holds for any b, we have $B^{-1}\ge A^{-1}$.

#5) Let V be the set of all polynomials of degree $\leq n$ defined on $[0,b]$. We equip it with the norm

$$\|p\|_{L^\infty(0,b)} = \sup_{0\leq x\leq b} |p(x)|.$$

V, $\|\cdot\|_{L^\infty(0,b)}$ is a Banach space since it is finite dimensional. In fact V is isomorphic to R^{n+1}.

The map $\|\cdot\|_{L^\infty(a,b)} : V \to R$ is coercive and continuous. Let $U = \{p \in V,\ p(0)=1\}$.

U is convex, but we need to show it is closed in order to use Thm. 8.2-1. Let $\{P_k\}$ be a sequence of elements of U that converge to some $p \in V$.

$$\|p - p^{(k)}\|_{L^\infty(0,b)} \to 0 \Rightarrow p(0) = \lim_{k\to\infty} p^{(k)}(0) = 1. \checkmark$$

Hence the problem: Find $\tilde{p} \in U$, $\|\tilde{p}\|_{L^\infty(a,b)} = \inf_{p\in U} \|p\|_{L^\infty(0,b)}$ has a solution.

Note that $\|\tilde{p}\|_{L^\infty(a,b)} > 0$ for otherwise $\tilde{p} \equiv 0$ on (a,b) $\Rightarrow \tilde{p} \equiv 0\ \forall x$ since polynomials are analytic.

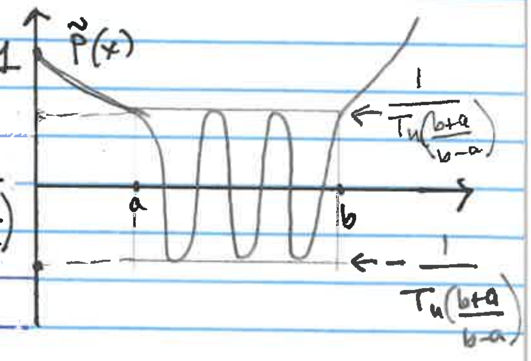
We will show that

$$\tilde{p}(x) = \frac{1}{T_n\left(\frac{b+a}{b-a}\right)} T_n\left(\frac{b+a-2x}{b-a}\right)$$ is the unique solution.

Let $x_k = \frac{b+a}{2} - \frac{b-a}{2}\cos\frac{k\pi}{n}$, $k=0,\dots,n$.

It can be shown that $\|\tilde{p}\|_{L^\infty(a,b)} = \frac{1}{T_n\left(\frac{b+a}{b-a}\right)}$ and

$$\tilde{p}(x_k) = (-1)^k \frac{1}{T_n\left(\frac{b+a}{b-a}\right)},\quad k=0,\dots,n$$



Now suppose there exists $q \in U$ such that

$$\|q\|_{L^\infty(a,b)} < \|\tilde{p}\|_{L^\infty(a,b)} = \frac{1}{T_n\left(\frac{b+a}{b-a}\right)}.$$

We can see that $\tilde{p}(x_k) - q(x_k) > 0$ k even

$\tilde{p}(x_k) - q(x_k) < 0$ k odd.

It follows from the intermediate value Theorem that $\exists\ \xi_1, \dots, \xi_n$ with

$$a = x_0 < \xi_1 < x_1 < \xi_2 < \dots < \xi_n < x_n = b$$

such that $\tilde{p}(\xi_i) - q(\xi_i) = 0 \quad i = 1, \dots, n.$

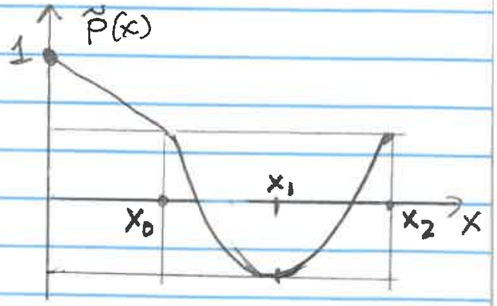
we also have

$$\tilde{p}(0) - q(0) = 0 .$$

Hence $\tilde{p}(x) - q(x)$ is a polynomial of degree $\leq n$ and vanishes at the $n+1$ distinct pts. $0 < \xi_1 < \dots < \xi_n$. This is a contradiction and shows that $\tilde{p}(x)$ is a solution to the minimization problem.

The proof of uniqueness is more complicated due to the loss of the strict inequalities above.

We look at the speial case of $n=2$.



let $e(x) = \tilde{p}(x) - p(x)$

where p is another minimizer.

we have $e(0) = 0$

$\circledast$ $e(x_0) \geq 0, \ e(x_1) \leq 0, \ e(x_2) \geq 0$

If all ineqs. in $\circledast$ are strict, then above argument shows $e = 0$.

If equality in $\circledast$ holds at more than one point, then $e \equiv 0$ $(n=2)$ which shows $p = \tilde{p}$.

Case 1 $e(x_0) = 0 \Rightarrow e(x_1) < 0, \ e(x_2) > 0 \Rightarrow \exists\ \xi \in (x_1, x_2)$ such that $e(\xi) = 0$. This combined with $e(0) = e(x_0) = 0 \Rightarrow p = \tilde{p}$

Case 2 $e(x_1) = 0 \Rightarrow e'(x_1) = 0$ since x_1 is a minimum of p and x_1 is interior. Again, $e \equiv 0$.

Case 3 $e(x_2) = 0 \Rightarrow e(x_0) > 0, \ e(x_1) < 0 \Rightarrow \exists\ \xi \in (x_0, x_1)$ s.t. $e(\xi) = 0$. This plus $e(0) = e(x_2) = 0 \Rightarrow e \equiv 0$.

#6) (a) $\int_{R^N} (\alpha f(v) + (1-\alpha) g(v))\, dv = \alpha \int_{R^N} f(v)\, dv + (1-\alpha) \int_{R^N} g(v)\, dv$

$$= \alpha \rho + (1-\alpha)\rho = \rho \quad ✓$$

The proof for the other two constraints is similar.

(b) $H'(f)h = \lim_{t \to 0} \frac{1}{t} \left[\int_{R^N} (f+th) \log (f+th)\, dv - \int_{R^N} f \log f \, dv \right]$.

Now

$$(f+th)\log(f+th) - f \log f = th \log(f+th) + f(\log(f+th) - \log f)$$

$$= th \log(f+th) + f \log\left(1 + t\frac{h}{f}\right).$$

For t small, $\log\left(1 + t\frac{h}{f}\right) = t\frac{h}{f} + O(t^2)$. Hence

$$\boxed{H'(f)h = \int_{R^N} (1 + \log f)\, h \, dv}.$$

(c) It is convenient to use the 2nd derivative characterization:

$$H''(f)(h,k) = \int_{R^N} \frac{hk}{f}\, dv$$

$$H''(f)(f-g, f-g) = \int_{R^N} \frac{(f-g)^2}{f}\, dv > 0 \qquad \forall f \neq g$$

Hence H is strictly convex.
H is also coercive, say by using the $L^1(R^N)$ norm.

Hence a minimum exists and is unique by strict convexity.

(d)(i) First, we need to verify that M satisfies the constraints. Indeed tedious calculations using polar coordinates for $N=2$ and spherical coordinates in 3D show that

$$\int_{R^N} M(v)\, dv = \rho, \quad \int_{R^N} v M(v)\, dv = \rho u \quad \text{and} \quad \int_{R^N} |v|^2 M(v)\, dv = \rho |u|^2 + TN! \quad ✓$$

(ii) we need to verify the "angle condition"

$$\int_{R^N} (1+\log M)(g-M)\,dv \geq 0 \qquad \forall g \in U.$$

Now $1+\log M = 1+\log\rho \ \frac{N}{2}\log(2\pi T) - \frac{|v-u|^2}{2T}$

$$= 1+\log\rho - \frac{N}{2}\log(2\pi T) - \frac{|v|^2}{2T} + \frac{v\cdot u}{T} - \frac{|u|^2}{2T}.$$

Note that $1+\log\rho - \frac{N}{2}\log(2\pi T) - \frac{|u|^2}{2T}$ are constant whereas $|v|^2$ and $v\cdot u$ appear as coefficients in the constraints. Hence and since both M and g satisfy these

$$\int_{R^N} (1+\log M)(g-M)\,dv = 0.$$

This is not surprising since the constraints are <u>linear</u> in f.

#7 (a)

$$J(v) = \int_0^1 (|v'|^2 + v^2(x))\,dx - 2h\int_0^1 |v'|\,dx + h^2$$

$$= \|v\|_1^2 - 2h\int_0^1 |v'|\,dx + h^2.$$

Using Cauchy-Schwarz, $\int_0^1 |v'|\,dx \leq \left(\int_0^1 |v'|^2 dx\right)^{1/2} = \|v'\|_{L^2(0,1)} \leq \|v\|_1.$ Hence

$$J(v) \geq \|v\|_1^2 - 2h\|v\|_1 + h^2. \qquad \otimes$$

Now

$$ab \leq \frac{a^2}{2}\epsilon + \frac{b^2}{2\epsilon} \quad \forall \varepsilon > 0 \ \underset{\epsilon=2}{\Rightarrow} \ ab \leq a^2 + \frac{b^2}{4}.$$

$a=h$, $b=\|v\|_1 \Rightarrow 2h\|v\|_1 \leq 2h^2 + \frac{\|v\|_1^2}{2}$. Hence

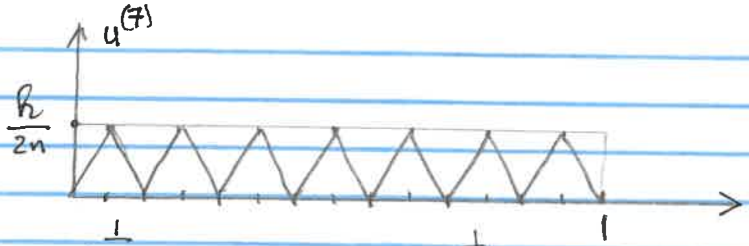
$$J(v) \geq \|v\|_1^2 - 2h^2 - \frac{\|v\|_1^2}{2} + h^2 = \frac{\|v\|_1^2}{2} - h^2$$

<u>Note</u> From $\circledast$ we also obtain $J(v) \geq (\|v\|_1 - h)^2$. $\circledast$

Actually this is necessary in showing that $\inf_{v\in V} J(v) = 0$

(b) It is easy to see

That $|v'| = h$ a.e.



$$\Rightarrow J(u^n) = \int_0^1 (u^n(x))^2 dx = n \int_0^{\frac{1}{2n}} \left(h\left(x-\frac{0}{n}\right)\right)^2 dx + n \int_{\frac{1}{2n}}^{\frac{1}{n}} \left(h\left(\frac{1}{n}-x\right)\right)^2 dx$$

$$= \frac{h^2}{12n^2}. \quad \Rightarrow \lim_{n\to\infty} J(u^n) = 0.$$

Also, from ⊛ we see That $\inf_{v\in V} J(v) = 0.$ ✓
$J(v) \geq 0$

(c) Suppose u is a minimizer. Then $J(u) = 0$.

However This would imply That $u = 0$ and $|u'| = h$ a.e. which is not possible in $H^1(0,1)$. ✓

(d) We invoke Thm. 8.2-2 and its extension given in The Remark following it.

$U = V = H^1(0,1)$ nonempty, convex Hilbert space

J continuous, coercive.

The only condition That is missing is The convexity of J.

So it must be The reason for The nonexistence a minimizer.

#8 (a) First note that for $u \in U$, the series $\sum_i \alpha_i u_i$ is absolutely convergent. Let $u, v \in U$, $\lambda_1, \lambda_2 \in \mathbb{R}$. The series $\sum_i (\lambda_1 u_i + \lambda_2 v_i)\alpha_i$ is absolutely convergent. Hence

$$\sum_i (\lambda_1 u_i + \lambda_2 v_i)\alpha_i = \lambda_1 \sum_i \alpha_i u_i + \lambda_2 \sum_i \alpha_i v_i = 0.$$

Hence U is a subspace of V. Now let $\{u^{(n)}\}_{n \geq 0}$ be a sequence of elements of U which converges to some $u \in V$. We have

$$\sum_i \alpha_i u_i = \sum_i \alpha_i \left((u_i - u_i^{(n)}) + u_i^{(n)} \right)$$

$$= \sum_i \alpha_i (u_i - u_i^{(n)}) + \cancelto{0}{\sum_i \alpha_i u_i^{(n)}} = \sum_i \alpha_i (u_i - u_i^{(n)}).$$

Now

$$\left| \sum_i \alpha_i u_i \right| \leq \sup_i |u_i - u_i^{(n)}| \sum_i \alpha_i \longrightarrow 0 \quad \text{as } n \to \infty$$

since $\sum_i \alpha_i < \infty$ and $\sup_i |u_i - u_i^{(n)}| \to 0$ as $n \to \infty$.

(b) The infimum exists so let $\rho = \inf_{v \in U} \|w - v\|$.

Assume that $\exists\ u \in U$ such that $\|w - u\| = \inf_{v \in U} \|w - v\| = \rho$.

Since $w_i \to 0$ and $u_i \to 0$, there exists an integer N such that

$$\rho = \|w - u\| = \max_{i \leq N} |w_i - u_i|.$$

We also choose N large enough so that $\sup_{i > N} |w_i - u_i| < \frac{\rho}{2}$.

Let $N_\rho = \{ i \leq N \text{ such that } |w_i - u_i| = \rho \}$. Clearly $\mathcal{N}_\rho \neq \emptyset$.

Also, since infinitely many α's are nonzero, $\exists\ \mu > N$ such that $\alpha_\mu > 0$.

Also let $\mathcal{N}_\rho' = \{ i > N,\ i \neq \mu \}$.

We will construct a $z \in U$ such that $\|w - z\| < \rho$, thus contradicting the existence of ~~Ao~~ a minimizer u.

let $z \in V$ be defined by

$$z_i = \begin{cases} u_i + \epsilon_i & i \in N_\rho \quad (|w_i - u_i| = \rho) \\ u_i & i \leq N,\ i \notin N_\rho \quad (|w_i - u_i| < \rho) \\ u_\mu + \epsilon_\mu & i = \mu \\ u_i & i > N,\ i \neq \mu. \end{cases}$$

The ϵ_i's are real numbers that will be chosen suitably so that $z \in U$. Now

① $$\sup_i |w_i - z_i| = \max\left\{ \max_{i \in N_\rho} |w_i - z_i|,\ \max_{\substack{i \leq N \\ i \notin N_\rho}} |w_i - u_i|,\ |w_\mu - z_\mu|,\ \sup_{i > N,\, i \neq \mu} |w_i - u_i| \right\}$$

② $$\max_{i \leq N,\, i \notin N_\rho} |w_i - z_i| = \max_{i \leq N,\, i \notin N_\rho} |w_i - u_i| < \rho$$

③ $$|w_\mu - z_\mu| = |w_\mu - u_\mu| + |\epsilon_\mu| < \rho/2 + |\epsilon_\mu|$$

④ $$\sup_{i > N,\, i \neq \mu} |w_i - z_i| = \sup_{i > N,\, i \neq \mu} |w_i - u_i| < \rho/2.$$

Since $\sum_i \alpha_i u_i = 0$, $\sum_i \alpha_i z_i = \sum_{i \in N_\rho} \alpha_i \epsilon_i + \alpha_\mu \epsilon_\mu$.

Now choose ϵ_i, $i \in N_\rho$ so that $\max_{i \in N_\rho} |w_i - z_i| = \max_{i \in N_\rho} |w_i - u_i - \epsilon_i| < \rho$ ⑤

This can be done for ϵ suff. small. ⑤ will also hold for $\epsilon \epsilon_i$, $i \in N_\rho$, $0 < \epsilon < 1$. Next choose ϵ_μ so that

$$\sum_i \alpha_i z_i = 0 \Leftarrow \epsilon_\mu = - \sum_{i \in N_\rho} \alpha_i \epsilon \epsilon_i / \alpha_\mu .$$

Now with ϵ chosen small enough, $|\epsilon_\mu| < \rho/2$.

This and ③ $\Rightarrow |w_\mu - z_\mu| < \rho$.

putting all pieces together, we have $z \in U$ and

$\|w - z\| < \rho = \|w - u\|$.